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BOUNDARY VALUE AND EXTREMUM PROBLEMS FOR GENERALIZED OBERBECK–BOUSSINESQ MODEL

R.V. BRIZITSKII, ZH.YU. SARITSKAYA, R.R. KRAVCHUK

Abstract. Boundary value and extremum problems for a generalized Oberbeck–Boussinesq model are considered under the assumption that the reaction coefficient depends nonlinearly on the substance's concentration. In the case when reaction coefficient and cost functionals are Fréchet differentiable, an optimality system for the extremum problem is obtained. For the quadratic reaction coefficient a local uniqueness of the optimal solution is proved.

Keywords: nonlinear mass transfer model, generalized Oberbeck–Boussinesq model, extremum problem, control problem, optimality system, local uniqueness.

1. Introduction

For a long time an interests for the studying of inverse problems for linear and nonlinear models of heat-and-mass transfer does not fade. These mentioned problems consist in recovering of unknown densities of boundary or distributed sources of coefficients in differential equations of the model or in boundary conditions with the help of additional information about the system's state, which is described by a model. One of such methods of considering inverse problems is the optimisation method which implies the reduction of inverse or identification problems to extremum problems (see more in [1]).

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The papers [2, 3] are dedicated to the study of the inverse problems for a linear model of heat-and-mass transfer which consists of a convection-diffusion-reaction equation with boundary conditions. Let us also further note the articles [4, 5, 6, 7, 8, 9, 10], in which inverse problems for nonlinear heat-and-mass transfer models in a classical approximation of Oberbeck–Boussinesq were considered. The papers [12, 13, 14, 15, 16, 17] are focused on boundary value and extremum problems for convection-diffusion-reaction equation, in which reaction coefficient depends nonlinearly on substance's concentration. In [18, 19] similar models of complex heat transfer were considered. From a range of papers dedicated to the study of boundary value and extremum problems for various models which generalise a classical approximation of Oberbeck–Boussinesq let us note [20, 21, 22]. About the research of more complicated rheological models and of models of multi-component viscous compressible fluids see, respectively, [23, 24] and [25, 26].

2. Solvability of boundary value problem

In a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ, the considered boundary value problem is

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = \mathbf{f} + \beta\mathbf{G}\varphi,\ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \tag{1}$$

$$-\lambda\Delta\varphi + \mathbf{u}\cdot\nabla\varphi + k(\varphi,\mathbf{x})\varphi = f \text{ in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{0},\ \ \varphi = 0 \text{ on } \Gamma. \tag{3}$$

Here $\mathbf{u}$ is a velocity vector, function φ represents the concentration of the pollutant, $p = P/\rho$, where P is pressure, $\rho = \text{const}$ is the fluid density, $\nu = \text{const} > 0$ is the constant kinematic viscosity, $\lambda = \text{const} > 0$ is the constant diffusion coefficient, β is the coefficient of mass expansion, $\mathbf{G} = -(0, 0, G)$ is the acceleration of gravity, $\mathbf{f}$ and f are volume densities of external forces or external sources of the substance, the function $k = k(\varphi, \mathbf{x})$ is the reaction coefficient, where $\mathbf{x} \in \Omega$ This problem (1)–(3) for given functions $\mathbf{f}, f, \beta$ and k will be called Problem 1 below.

In this paper the global solvability of Problem 1 is proved, sufficient condition for its solution's uniqueness are stated, maximum principle is proved and the correctness of the model (1)–(3) is discussed.

Furthermore, with the help of the optimisation approach the problem of restoration of resources' volume density f using the concentration φ, which was measured in a subdomain $Q \subset \Omega$, is reduced to the extremum problem. Its solvability is proved in a general case. When the reaction coefficient and cost functionals are Fréchet differentiable we get optimality systems for the extremum problem. For the quadratic reaction coefficient which leads to the substance's concentration being of 3rd order nonlinearity in the equation (2) we state sufficient conditions of local uniqueness of optimal solutions for particular extremum problems.

While studying the considered problems, we will use Sobolev functional spaces $H^s(D)$, $s \in \mathbb{R}$. Here D means either the domain Ω or some subset $Q \subset \Omega$, or the boundary Γ. By $\|\cdot\|_{s,Q}, |\cdot|_{s,Q}$ and $(\cdot,\cdot)_{s,Q}$ we will denote the norm, seminorm and the scalar product in $H^s(Q)$. The norms and scalar products in $L^2(Q)$ and $L^2(\Omega)$ will be denote corresponding by $\|\cdot\|_Q$ and $(\cdot,\cdot)_Q$, $\|\cdot\|_\Omega$ and $(\cdot,\cdot)_\Omega$. Let $L^p_+(\Omega) = \{k \in L^p(\Omega) : k \geq 0\}$, $p \geq 3/2$, and by $V = \{\mathbf{v} \in H^1_0(\Omega)^3 : \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\}$ we introduce the main functional space for the velocity vector.

Let us also present the product of spaces $H = H_0^1(\Omega)^3 \times H_0^1(\Omega)$ and functional space $H^* = H^{-1}(\Omega)^3 \times H^{-1}(\Omega)$ dual to H.

Let the following conditions hold:

(i) Ω is a bounded domain in the space $\mathbb{R}^3$ with boundary $\Gamma \in C^{0,1}$;

(ii) $\mathbf{f} \in L^2(\Omega)^3$, $f \in L^2(\Omega)$, $\mathbf{b} = \beta\mathbf{G} \in L^2(\Omega)^3$;

(iii) For any function $w \in H^1(\Omega)$ the embedding $k(w, \cdot) \in L_+^p(\Omega)$ is true for some $p \geq 3/2$, which does not depend on w, and on any sphere $B_r = \{w \in H^1(\Omega) : \|w\|_{1,\Omega} \leq r\}$ of radius r the following inequality takes place:

$$\|k(w_1, \cdot) - k(w_2, \cdot)\|_{L^p(\Omega)} \leq L\|w_1 - w_2\|_{L^4(\Omega)} \;\; \forall w_1, w_2 \in B_r.$$

Here L is the constant which depends on r, but does not depend on $w_1, w_2 \in B_r$.

Let us note that the condition (iii) describes an operator, acting from $H^1(\Omega)$ to $L^p(\Omega)$, where $p \geq 3/2$, which gives us an opportunity to take into consideration the dependence of the reaction coefficient on either the component φ of solution the $(\mathbf{u}, \varphi, p)$ of Problem 1 or on the spatial variable $\mathbf{x} \in \Omega$. For example, $\tilde{k}_1 = \varphi^2$ (or $\tilde{k}_1 = \varphi^2|\varphi|$) in subdomain $Q \subset \Omega$ or $\tilde{k}_1 = k_0(\mathbf{x})$ in $\Omega\setminus\overline{Q}$, where $k_0(\mathbf{x}) \in L_+^{3/2}(\Omega\setminus\overline{Q})$.

From a physical point of view, the coefficient $\tilde{k}_1$ corresponds to the situation, when the substance's decomposition rate is proportional to the square (or cube) of substance's concentration in a subdomain $Q \subset \Omega$ and outside Q, and the rate of the chemical reaction depends only on a spatial variable [14, 16].

Let us also remind that on the strength of the Sobolev embedding theorem the space $H^1(\Omega)$ is embedded into the space $L^s(\Omega)$ continuously at $s \leq 6$ and compactly at $s < 6$, with some constant C_s which depends on s and on Ω, and besides the following estimate is true:

$$\|\varphi\|_{L^s(\Omega)} \leq C_s\|\varphi\|_{1,\Omega} \;\; \forall \varphi \in H^1(\Omega). \tag{4}$$

We will use the following technical lemma (see [1]).

Lemma 1. *If conditions (i), (ii) hold then there are such positive constants* $C_0, C_1, \delta_0, \delta_1, \gamma_1, \gamma_2, \gamma_p, \beta_1$*, depending on* Ω *or on* Ω *and on p, respectively, and a positive constant* β_0*, depending on* $\|\mathbf{b}\|_\Omega$ *that for any function* $k_0 \in L_+^p(\Omega)$*, where* $p \geq 3/2$*,* $\mathbf{u} \in V$*,* $\mathbf{b} \in L^2(\Omega)^3$*, the following relations are correct:*

$$|(\nabla\mathbf{v}, \nabla\mathbf{w})| \leq C_0\|\mathbf{v}\|_{1,\Omega}\|\mathbf{w}\|_{1,\Omega} \;\; \forall \mathbf{v}, \mathbf{w} \in H^1(\Omega)^3,$$

$$|(\mathbf{b}h, \mathbf{v})| \leq \beta_0\|h\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega} \; \forall \mathbf{w}, \mathbf{v} \in H^1(\Omega)^3, h \in H^1(\Omega),$$

$$|((\mathbf{w}\cdot\nabla)\mathbf{v}, \mathbf{z})| \leq \gamma_1\|\mathbf{w}\|_{L^4(\Omega)^3}\|\mathbf{v}\|_{1,\Omega}\|\mathbf{z}\|_{1,\Omega} \;\; \forall \mathbf{w}, \mathbf{v}, \mathbf{z} \in H^1(\Omega)^3,$$

$$((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}) = -((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{v}), \; ((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{v}) = 0 \; \forall \mathbf{v}, \mathbf{w} \in H^1(\Omega)^3, \tag{5}$$

$$(\nabla\mathbf{v}, \nabla\mathbf{v}) \geq \delta_0\|\mathbf{v}\|_{1,\Omega}^2 \; \forall \mathbf{v} \in H_0^1(\Omega)^3, \tag{6}$$

$$\sup_{\mathbf{v} \in H_0^1(\Omega)^3, \mathbf{v} \neq \mathbf{0}} -(\mathrm{div}\mathbf{v}, p)/\|\mathbf{v}\|_{1,\Omega} \geq \beta_1\|p\|_\Omega \;\; \forall p \in L_0^2(\Omega), \tag{7}$$

$$|(\nabla h, \nabla\eta)| \leq C_1\|h\|_{1,\Omega}\|\eta\|_{1,\Omega}, \;\; |(k_0h, \eta)| \leq \gamma_p\|k_0\|_{L^p(\Omega)}\|h\|_{1,\Omega}\|\eta\|_{1,\Omega}, \tag{8}$$

$$|(\mathbf{w}\cdot\nabla h, \eta)| \leq \gamma_2\|\mathbf{w}\|_{L^4(\Omega)^3}\|h\|_{1,\Omega}\|\eta\|_{1,\Omega} \; \forall \mathbf{w} \in H^1(\Omega)^3, \; h, \eta \in H^1(\Omega);$$

$$(\mathbf{u}\cdot\nabla h, h) = 0, \;\; (\nabla h, \nabla h) \geq \delta_1\|h\|_{1,\Omega}^2,$$

$$\lambda(\nabla h, \nabla h) + (k_0h, h) \geq \lambda_*\|h\|_{1,\Omega}^2 \; \forall h \in H_0^1(\Omega), \; \lambda_* \equiv \delta_1\lambda.$$

From second estimate in (8) the inequality for the function $k(\varphi)$ that satisfies conditions (iii) follows:

(9)
$$|((k(\varphi_1)-k(\varphi_2))\varphi,\eta)|\le\gamma_p L\|\varphi_1-\varphi_2\|_{L^4(\Omega)}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega}\ \ \forall\varphi,\varphi_1,\varphi_2,\eta\in H_0^1(\Omega).$$

Let us multiply the first equation in (1) by a function $\mathbf{v}\in H_0^1(\Omega)^3$ and the equation (2) by a function $h\in H_0^1(\Omega)$ and integrate over Ω, using Green's formulae, we are obtaining the weak formulation of Problem 1.

(10) $$\nu(\nabla\mathbf{u},\nabla\mathbf{v})+((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v})-(p,\operatorname{div}\mathbf{v})=(\mathbf{f},\mathbf{v})+(\mathbf{b}\varphi,\mathbf{v})\ \ \forall\mathbf{v}\in H_0^1(\Omega)^3,$$

(11) $$\lambda(\nabla\varphi,\nabla h)+(k(\varphi)\varphi,h)+(\mathbf{u}\cdot\nabla\varphi,h)=(f,h)\ \ \forall h\in H_0^1(\Omega).$$

The triple $(\mathbf{u},\varphi,p)\in V\times H_0^1(\Omega)\times L_0^2(\Omega)$ which satisfies (10), (11) will be called a weak solution of Problem 1.

We consider the restriction of (10) on the space V:

(12) $$\nu(\nabla\mathbf{u},\nabla\mathbf{v})+((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v})=(\mathbf{f},\mathbf{v})+(\mathbf{b}\varphi,\mathbf{v})\ \ \forall\mathbf{v}\in V.$$

To prove the existence of the weak solution of Problem 1 it is enough to prove the existence of the solution $(\mathbf{u},\varphi)\in V\times H_0^1(\Omega)$ of the problem (11), (12). About the restoration of pressure see more in [1, p. 89]. In its turn the proof of the solvability of the problem (11), (12) will be constructed with the help of the fixed-point Shauder theorem (see [1]).

Let us set $\mathbf{z}=(\mathbf{w},\tau)\in H$ and $\mathbf{y}=(\mathbf{u},\varphi)\in H$ and construct a mapping $F:H\to H$, acting by formula $F(\mathbf{z})=\mathbf{y}$, where $\mathbf{y}=(\mathbf{u},\varphi)$ is the solution of the linear problem

(13) $$a_1(\mathbf{u},\mathbf{v})\equiv\nu(\nabla\mathbf{u},\nabla\mathbf{v})+((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v})=(\mathbf{f},\mathbf{v})+(\mathbf{b}\,\varphi,\mathbf{v})\ \ \forall\mathbf{v}\in V,$$

(14) $$a_2(\varphi,h)\equiv\lambda(\nabla\varphi,\nabla h)+(k(\tau)\varphi,h)+(\mathbf{w}\cdot\nabla\varphi,h)=(f,h)\ \ \forall h\in H_0^1(\Omega).$$

From the condition (iii) and lemma 1 it follows that the form $a_2:H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{R}$, is continuous and coercive with the constant $\lambda_*=\delta_1\lambda$. Then for all $\mathbf{w}\in H_0^1(\Omega)^3$ and $\tau\in H_0^1(\Omega)$ there exists a unique solution $\varphi\in H_0^1(\Omega)$ of the problem (14) for which the estimate holds

(15) $$\|\varphi\|_{1,\Omega}\le M_\varphi\equiv C_*\|f\|_\Omega,\ \ C_*=\lambda_*^{-1}.$$

By Lemma 1 the form $a_1:H_0^1(\Omega)^3\times H_0^1(\Omega)^3\to\mathbb{R}$ is continuous and coercive on $V\times V$ with the constant $\nu_*=\delta_0\nu$. Then for all $\mathbf{w}$ and τ there exists a unique solution $\mathbf{u}\in V$ of the problem (13).

We set $\mathbf{v}=\mathbf{u}$ in (13). From Lemma 1 the inequality follows

(16) $$\nu_*\|\mathbf{u}\|_{1,\Omega}^2\le\|\mathbf{f}\|_\Omega\|\mathbf{u}\|_{1,\Omega}+\beta_0\|\varphi\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega}.$$

From the estimate (16) and taking into account (15) we can derive the following estimate

(17) $$\|\mathbf{u}\|_{1,\Omega}\le M_{\mathbf{u}}=\nu_*^{-1}(\|\mathbf{f}\|_\Omega+\beta_0C_*\|f\|_\Omega).$$

Then there exists a solution $\mathbf{y}=(\mathbf{u},\varphi)\in H$ of the problem (13), (14), for which the estimate takes place

(18) $$\|\mathbf{y}\|_H\le M_{\mathbf{u}}+M_\varphi.$$

In the space H let us introduce a sphere $B_r=\{\mathbf{y}\in H:\|\mathbf{y}\|_H\le r\}$, where $r=M_\varphi+M_{\mathbf{u}}$. from the construction of B_r and from (18) it follows that the operator F which was defined above is mapping the sphere B_r into itself with any $\mathbf{z} = (\mathbf{w}, \tau) \in H$.

Let us prove that F is continuous and compact on B_r. Let $\mathbf{z}_n = (\mathbf{w}_n, \tau_n)$, $n = 1, 2, ...$ be an arbitrary sequence from B_r.

Due to the reflexivity of the spaces $H_0^1(\Omega)$ and $H_0^1(\Omega)^3$ and to the compactness of the embeddings $H^1(\Omega) \subset L^4(\Omega)$ and $H^1(\Omega)^3 \subset L^4(\Omega)^3$ there is a subsequence of the sequence $\{\mathbf{z}_n\} = \{(\mathbf{w}_n, \tau_n)\}$, which we will again denote by $\{\mathbf{z}_n\}$, and there is a function $\mathbf{z} = (\mathbf{w}, \tau) \in B_r$ such that $\mathbf{w}_n \to \mathbf{w}$ weakly in $H^1(\Omega)^3$ and strongly in $L^4(\Omega)^3$ at $n \to \infty$, $\tau_n \to \tau$ weakly in $H^1(\Omega)$ and strongly in $L^4(\Omega)$ at $n \to \infty$.

Let $\mathbf{y}_n = F(\mathbf{z}_n)$, $\mathbf{y} = F(\mathbf{z})$. These equations mean that $\mathbf{y} = (\mathbf{u}, \varphi) \in H$ is the solution of the problem (13), (14) and $\mathbf{y}_n = (\mathbf{u}_n, \varphi_n) \in H$ is the solution of the problem

$$\nu(\nabla \mathbf{u}_n, \nabla \mathbf{v}) + ((\mathbf{w}_n \cdot \nabla)\mathbf{u}_n, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + (\mathbf{b}\varphi_n, \mathbf{v}) \ \ \forall \mathbf{v} \in V, \tag{19}$$

$$\lambda(\nabla \varphi_n, \nabla h) + (k(\tau_n)\varphi_n, h) + (\mathbf{w}_n \cdot \nabla \varphi_n, h) = (f, h) \ \forall h \in H_0^1(\Omega), \tag{20}$$

that can be obtained from (13), (14) by the substitution of $\mathbf{z} = (\mathbf{w}, \tau)$ by $\mathbf{z}_n = (\mathbf{w}_n, \tau_n)$.

Let us show that $\mathbf{y}_n \to \mathbf{y}$ strongly in H or, equivalently, that $\varphi_n \to \varphi$ strongly in $H^1(\Omega)$ and $\mathbf{u}_n \to \mathbf{u}$ strongly in $H^1(\Omega)^3$ at $n \to \infty$.

For this let us subtract (13), (14) from (19), (20). Taking into account that

$$(k(\tau_n)\varphi_n, h) - (k(\tau)\varphi, h) = (k(\tau_n)(\varphi_n - \varphi), h) + ((k(\tau_n) - k(\tau))\varphi, h),$$

we get

$$\lambda(\nabla(\varphi_n - \varphi), \nabla h) + (k(\tau_n)(\varphi_n - \varphi), h) + (\mathbf{w}_n \cdot \nabla(\varphi_n - \varphi), h) =$$

$$= -((\mathbf{w}_n - \mathbf{w}) \cdot \nabla \varphi, h) - ((k(\tau_n) - k(\tau))\varphi, h) \ \ \forall h \in H_0^1(\Omega), \tag{21}$$

$$\nu(\nabla(\mathbf{u}_n - \mathbf{u}), \nabla \mathbf{v}) + ((\mathbf{w}_n \cdot \nabla)(\mathbf{u}_n - \mathbf{u}), \mathbf{v}) =$$

$$= (((\mathbf{w}_n - \mathbf{w}) \cdot \nabla)\mathbf{u}, \mathbf{v}) + (\mathbf{b}(\varphi_n - \varphi), \mathbf{v}) \ \ \forall \mathbf{v} \in V. \tag{22}$$

Using the estimate (9) at $\varphi_1 = \tau_n$, $\varphi_2 = \tau$, $\varphi = \varphi_n$ and the estimate $\|\varphi_n\|_{1,\Omega} \leq M_\varphi$ for $n = 1, 2...$ that follows from (15), we obtain

$$|((k(\tau_n) - k(\tau)\varphi_n, h)| \leq \gamma_p L M_\varphi \|\tau_n - \tau\|_{L^4(\Omega)} \|h\|_{1,\Omega} \to 0 \text{ at } n \to \infty \ \forall h \in H_0^1(\Omega). \tag{23}$$

Substituting $h = \varphi - \varphi_n$ in (21) and taking into account lemma 1 and (23) we conclude that $\|\varphi_n - \varphi\|_{1,\Omega} \to 0$ at $n \to \infty$. After denoting $\mathbf{v} = \mathbf{u} - \mathbf{u}_n$ in (22) we can get from this and from lemma 1 that $\|\mathbf{u} - \mathbf{u}_n\|_{1,\Omega} \to 0$ at $n \to \infty$.

Then the operator F is continuous and compact and from the fixed-point Shauder theorem it follows that F has a fixed point $\mathbf{y} = F(\mathbf{y}) \in H$ which is the solution of the system (11), (12).

Due to (7) for the pressure p and for any arbitrary (as much as necessary small) number $\delta > 0$ there exists such function $\mathbf{v}_0 \in H_0^1(\Omega)^3$, $\mathbf{v}_0 \neq \mathbf{0}$ that

$$-(\text{div}\mathbf{v}_0, p) \geq \beta_2 \|\mathbf{v}_0\|_{1,\Omega} \|p\|_\Omega, \ \beta_2 = (\beta_1 - \delta) > 0.$$

Let $\mathbf{v} = \mathbf{v}_0$ in (10), then taking into account this inequality and lemma 1 we obtain

$$\beta_2 \|\mathbf{v}_0\|_{1,\Omega} \|p\|_\Omega \leq \nu C_0 \|\mathbf{v}_0\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} + \gamma_1 \|\mathbf{v}_0\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega}^2 + \beta_0 \|\varphi\| \|\mathbf{v}_0\|_{1,\Omega}$$

Dividing by $\|\mathbf{v}_0\|_{1,\Omega} \neq 0$ and accounting for the estimates (15), (17), we derive that

$$\|p\|_\Omega \leq C_p = \beta_2^{-1}[(\nu + \gamma_1 M_\mathbf{u})M_\mathbf{u} + \|\mathbf{f}\|_\Omega + \beta_0 M_\varphi]. \tag{24}$$

Let us state sufficient conditions for the uniqueness of the solution of the problem (11), (12). Let $(\mathbf{u}_i, \varphi_i) \in V \times H_0^1(\Omega)$, $i = 1, 2$ be solutions of the problem (11), (12). It is obvious that their differences $\varphi = \varphi_1 - \varphi_2$ and $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ satisfy the relations

$$\lambda(\nabla\varphi, \nabla h) + (k(\varphi_1)(\varphi, h) + (\mathbf{u}_1 \cdot \nabla\varphi, h) =$$

$$= -(k(\varphi_1) - k(\varphi_2), \varphi_2 h) - (\mathbf{u} \cdot \nabla\varphi_2, h)\ \forall h \in H_0^1(\Omega), \tag{25}$$

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + ((\mathbf{u}_1 \cdot \nabla)\mathbf{u}, \mathbf{v}) = (\mathbf{b}\varphi, \mathbf{v}) - ((\mathbf{u} \cdot \nabla)\mathbf{u}_2, \mathbf{v})\ \forall \mathbf{v} \in V. \tag{26}$$

Denoting $h = \varphi$ in (25) and $\mathbf{v} = \mathbf{u}$ in (26), we conduct from lemma 1 and from the inequalities(9), (4) that

$$\lambda_* \|\varphi\|_{1,\Omega} \le \gamma_p C_4 L M_\varphi \|\varphi\|_{1,\Omega} + \gamma_2 M_\varphi \|\mathbf{u}\|_{1,\Omega}, \tag{27}$$

$$\nu_* \|\mathbf{u}\|_{1,\Omega} \le \beta_0 \|\varphi\|_{1,\Omega} + \gamma_1 M_{\mathbf{u}} \|\mathbf{u}\|_{1,\Omega}. \tag{28}$$

Let us introduce dimensionless counterparts of Reynolds number and diffusive Rayleigh number [1]:

$$\mathrm{Re} = (\gamma_1/\delta_0\nu) M_{\mathbf{u}},\ \ \mathrm{Ra} = (\gamma_2/\delta_0\nu)(\beta_0/\delta_1\lambda) M_\varphi. \tag{29}$$

Let the condition $\gamma_1 M_{\mathbf{u}} \le \nu_*/2$ or $\mathrm{Re} \le 1/2$ holds. Then from (28) we conduct that

$$\|\mathbf{u}\|_{1,\Omega} \le 2(\beta_0/\delta_0\nu)\|\varphi\|_{1,\Omega}. \tag{30}$$

Accounting (30), from (27) we derive the inequality

$$\|\varphi\|_{1,\Omega} \le \gamma_p (1/\lambda\delta_1) C_4 L M_\varphi \|\varphi\|_{1,\Omega} + 2(\gamma_2/\delta_0\nu)(\beta_0/\lambda\delta_1) M_\varphi =$$

$$= (\gamma_p (\delta_0\nu/\beta_0\gamma_2) C_4 L + 2)\mathrm{Ra}\|\varphi\|_{1,\Omega} \tag{31}$$

Let the following smallness conditions be satisfied:

$$\mathrm{Re} \le 1/2,\ \ (\gamma_p(\delta_0\nu/\beta_0\gamma_2) C_4 L + 2)\mathrm{Ra} < 1. \tag{32}$$

Then from the estimates (31) and (30) it can be obtained consistently that $\|\varphi\|_{1,\Omega} = 0$ and $\|\mathbf{u}\|_{1,\Omega} = 0$ or $\varphi_1 = \varphi_2$ and $\mathbf{u}_1 = \mathbf{u}_2$.

Subtracting (10) at $(\mathbf{u}_2, \varphi_2, p_2)$ from (10) at $(\mathbf{u}_1, \varphi_1, p_1)$ and taking into account that $\mathbf{u} = \mathbf{0}$ and $\varphi = 0$, we conclude that the difference $p = p_1 - p_2$ satisfies the equation

$$-(p, \operatorname{div} \mathbf{v}) = 0\ \ \forall \mathbf{v} \in H_0^1(\Omega). \tag{33}$$

Then due to (7) we obtain that $p = 0$ or $p_1 = p_2$.

Let us formulate obtained results as a following theorem.

Theorem 1. *If the conditions (i)–(iii) hold then there exists a weak solution* $(\mathbf{u}, \varphi, p) \in V \times H_0^1(\Omega) \times L_0^2(\Omega)$ *of Problem 1 and the estimates (15), (17) and (24) are true. If, besides, the condition (32) is met then the weak solution of Problem 1 is unique.*

Arguing as [27], let us state additional properties of the solution of Problem 1.

Let, besides (i)-(iii), the following conditions be satisfied:

(iv) $0 \le f \le f_{\max}$ a.e. in Ω, where $f_{\max} > 0$;

(v) the nonlinearity $k(\varphi)\varphi$ is monotonic in the following sense:

$$(k(\varphi_1)\varphi_1 - k(\varphi_2)\varphi_2, \varphi_1 - \varphi_2) \ge 0\ \ \forall \varphi_1, \varphi_2 \in H^1(\Omega).$$

Lemma 2. *When the conditions (iv), (v) hold, the component φ of the weak solution $(\mathbf{u},\varphi,p)$ of Problem 1 satisfies the inequality*

$$0\le\varphi\le M \text{ a.e. in } \overline{\Omega},$$

where M is a positive number, with which the equality $f_{\max}=k(M)M$ holds.

Remark 1. It is not difficult to check that the monotone nonlinearily of $k(\varphi)\varphi$ is caused by the power reaction coefficients, for example $k_1(\varphi)=\varphi^2$ and $k_2(\varphi)=\varphi^2|\varphi|$ as in [16, 17, 18], for which the parameter M can be computed easily.

Proof. Let $\tilde\varphi=\min\{\varphi,0\}$. It is clear that $\tilde\varphi\in H_0^1(\Omega)$ and the following relations hold:

$$(\nabla\varphi,\nabla\tilde\varphi)=(\nabla\tilde\varphi,\nabla\tilde\varphi),\ (k(\varphi)\varphi,\tilde\varphi)=(k(\varphi)\tilde\varphi,\tilde\varphi),\ (\mathbf{u}\cdot\nabla\varphi,\tilde\varphi)=(\mathbf{u}\cdot\nabla\tilde\varphi,\tilde\varphi)=0.$$

Taking this into account and denoting $h=\tilde\varphi$ in (11), we obtain the relation

$$\lambda(\nabla\tilde\varphi,\nabla\tilde\varphi)+(k(\varphi)\tilde\varphi,\tilde\varphi)=(f,\tilde\varphi). \tag{34}$$

As $\tilde\varphi\in H_0^1(\Omega)$ then according to lemma 1 and to (34) we derive the inequality

$$\lambda_*\|\tilde\varphi\|_{1,\Omega}^2\le(f,\tilde\varphi). \tag{35}$$

If $f>0$ a.e. in Ω then $(f,\tilde\varphi)<0$ and it follows from (35) that $\|\tilde\varphi\|_{1,\Omega}=0$. The latter means that from the assumption that in some subdomain $D\subset\Omega$ concentration $\varphi<0$ a.e. in D one can conclude that $\varphi=0$ a.e. in D.

Let us denote by $M>0$ some positive number and introduce a function $\psi=\max\{\varphi-M,0\}$. As it was done above, it is not hard to check that $\psi\in H_0^1(\Omega)$ and when substituting $h=\psi$ in (11) the equality is true

$$\lambda(\nabla\psi,\nabla\psi)+(k(\varphi)\varphi,\psi)=(f,\psi). \tag{36}$$

By Q_M we denote a subdomain in Ω, in which $\varphi>M$. It clear, that

$$(k(\varphi)\varphi,\psi)=(k(\varphi)\varphi,\psi)_{Q_M}=(k(\psi+M)(\psi+M),\psi)_{Q_M}.$$

By condition (v) the functions $\varphi_1=\psi+M$ and $\varphi_2=M$ from $H^1(\Omega)$ satisfy the relation

$$0\le(k(\psi+M)(\psi+M)-k(M)M,\psi)=(k(\psi+M)(\psi+M)-k(M)M,\psi)_{Q_M}. \tag{37}$$

Substracting $(k(M)M,\psi)$ from the both sides of (36) and due to lemma 1 and (37) we deduct

$$\lambda_*\|\psi\|_{1,\Omega}^2\le(f-k(M)M,\psi)_{Q_M}. \tag{38}$$

If $f_{\max}\le k(M)M$ then $\psi=0$ a.e. in Ω. ■

Remark 2. From the proof of lemma 2 it is not difficult to notice that as an alternative for the condition (v) we can use the boundedness of $k(\varphi,\cdot)$ from below, i.e. $k(\varphi,\cdot)\ge k_0>0$ a.e. in Ω. From the one side power reaction coefficients don't satisfy this condition but the conditions (iii) give an opportunity to consider, for example, $k(\varphi,\mathbf{x})=\varphi^2+\beta(\mathbf{x})$, where $\beta(\mathbf{x})\in L^2(\Omega)$: $\beta(\mathbf{x})\ge c_0>0$.

Let us also note the papers [18, 19] that contain similar results for related models.

3. Optimal control problem

In the framework of the optimisation approach the problem of the restoration of the sources' density f using an additional information about the solution of the Problem 1 can be reduced to the problem of distributed control (see for example [11, 17]).

To state the control problem let us divide the whole set of initial data of the Problem 1 into two groups: group of fixed data which includes functions $\mathbf{f}, \mathbf{b}$ and $k(\varphi)$, and group of controls consisting of function f. Here we assume that it can be changed in some set K.

We set $X = H_0^1(\Omega)^3 \times H_0^1(\Omega) \times L_0^2(\Omega)$, $Y = H^{-1}(\Omega)^3 \times H^{-1}(\Omega) \times L_0^2(\Omega)$, $\mathbf{x} = (\mathbf{u}, \varphi, p) \in X$ and introduce an operator $F = (F_1, F_2)$ by formula

$$\langle F_1(\mathbf{x}, f), (\mathbf{v}, h)\rangle = \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + \lambda(\nabla\varphi, \nabla h) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) - (p, \operatorname{div}\mathbf{v}) +$$

$$+ (k(\varphi)\varphi, h) + (\mathbf{u}\cdot\nabla\varphi, h) - (\mathbf{f}, \mathbf{v}) - (\mathbf{b}\varphi, \mathbf{v}) - (f, h),$$

$$\langle F_2(\mathbf{x}, f), r\rangle = -(\operatorname{div}\mathbf{u}, r)$$

and rewrite (11) in the form $F(\mathbf{x}, f) = 0$.

Let $I : X \to \mathbb{R}$ be a weakly semicontinuous below cost functional. We consider the following conditional minimization problem:

$$J(\mathbf{x}, f) \equiv (\mu_0/2)I(\mathbf{x}) + (\mu_1/2)\|f\|_\Omega^2 \to \inf,\ F(\mathbf{x}, f) = 0,\ (\mathbf{x}, f) \in X \times K. \quad (39)$$

Denote by $Z_{ad} = \{(\mathbf{x}, f) \in X \times K : F(\mathbf{x}, f) = 0, J(\mathbf{x}, f) < \infty\}$ the set of feasible pairs for the problem (39) and assume that the following conditions hold:

(j) $K \subset L^2(\Omega)$ is nonempty convex closed set,

(jj) $\mu_0 > 0$, $\mu_1 \geq 0$ and K is a bounded set, or $\mu_0 > 0$, $\mu_1 > 0$ and the functional I is bounded below.

The following cost functionals can be used in the capacity of the possible ones:

$$I_1(\varphi) = \|\varphi - \varphi^d\|_Q^2 = \int_\Omega |\varphi - \varphi^d|^2 d\mathbf{x},\ \ I_2(\varphi) = \|\varphi - \varphi^d\|_{1,Q}^2,$$

$$I_3(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}^d\|_Q^2,\ \ I_4(p) = \|p - p^d\|_Q^2. \quad (40)$$

Here function $\varphi^d \in L^2(Q)$ denotes some desired concentration field given in a subdomain $Q \subset \Omega$. Functions $\mathbf{u}^d$ and p^d have similar sense for the velocity field or pressure.

Theorem 2. *Let the conditions (i)–(iii) and (j), (jj) hold and $I : X \to \mathbb{R}$ is a weakly semicontinuous below functional and let $Z_{ad} \neq \emptyset$. Then there is at least one solution $(\mathbf{x}, f) \in X \times K$ of the control problem (39).*

Proof. Let $(\mathbf{x}_m, f_m) = (\mathbf{u}_m, \varphi_m, p_m, f_m) \in Z_{ad}$ be a minimizing sequence for which the following is true:

$$\lim_{m\to\infty} J(\mathbf{x}_m, f_m) = \inf_{(\mathbf{x},f)\in Z_{ad}} J(\mathbf{x}, f) \equiv J^*.$$

This and the condition of the theorem imply the estimate $\|f_m\|_\Omega \leq c_1$. From Theorem 1 it follows directly that $\|\mathbf{u}_m\|_{1,\Omega} \leq c_2$, $\|\varphi_m\|_{1,\Omega} \leq c_3$ and $\|p_m\|_\Omega \leq c_4$, where the constants $c_1, c_2, \ldots$ do not depend on m.

Then $\mathbf{u}^* \in V$, $\varphi^* \in H_0^1(\Omega)$, $p^* \in L_0^2(\Omega)$ and $f^* \in K$ are the weak limits of some subsequences of sequences $\{\mathbf{u}_m\}$, $\{\varphi_m\}$, $\{p_m\}$ and $\{f_m\}$. Corresponding subsequences will be also denoted by $\{\mathbf{u}_m\}$, $\{\varphi_m\}$, $\{p_m\}$ and $\{f_m\}$, respectively. With this in mind, it can be considered that

$$\mathbf{u}_m \to \mathbf{u}^* \in V \text{ weakly in } H^1(\Omega)^3 \text{ and strongly in } L^s(\Omega)^3,\ s < 6, \tag{41}$$

$$\varphi_m \to \varphi^* \in H_0^1(\Omega) \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^s(\Omega),\ s < 6, \tag{42}$$

$$p_m \to p^* \in L_0^2(\Omega),\ \ f_m \to f^* \in K \text{ weakly in } L^2(\Omega) \text{ at } m \to \infty. \tag{43}$$

The relation $F_2(\mathbf{x}^*, f^*) = 0$ is obvious. Let us show that $F_1(\mathbf{x}^*, f^*) = 0$, i.e.

$$\nu(\nabla\mathbf{u}^*, \nabla\mathbf{v}) + \lambda(\nabla\varphi^*, \nabla h) + ((\mathbf{u}^*\cdot\nabla)\mathbf{u}^*, \mathbf{v}) - (p^*, \operatorname{div}\mathbf{v}) + (k(\varphi^*)\varphi^*, h) +$$

$$+(\mathbf{u}^*\cdot\nabla\varphi^*, h) = (\mathbf{f}, \mathbf{v}) + (\mathbf{b}\varphi^*, \mathbf{v}) + (f^*, h)\ \forall(\mathbf{v}, h) \in H_0^1(\Omega)^3 \times H_0^1(\Omega). \tag{44}$$

We note that for all $m = 1, 2, ...$ the pair $(\mathbf{x}_m, f_m)$ satisfies to relation

$$\nu(\nabla\mathbf{u}_m, \nabla\mathbf{v}) + \lambda(\nabla\varphi_m, \nabla h) + ((\mathbf{u}_m\cdot\nabla)\mathbf{u}_m, \mathbf{v}) - (p_m, \operatorname{div}\mathbf{v}) + (k(\varphi_m)\varphi_m, h) +$$

$$+(\mathbf{u}_m\cdot\nabla\varphi_m, h) = (\mathbf{f}, \mathbf{v}) + (\mathbf{b}\varphi_m, \mathbf{v}) + (f_m, h)\ \ \forall(\mathbf{v}, h) \in H_0^1(\Omega)^3 \times H_0^1(\Omega). \tag{45}$$

Let us pass to the limit in (45) at $m \to \infty$. All linear summands in (45) turn into the corresponding ones in (66).

Let us consider separately the nonlinear summand $(k(\varphi_m)\varphi_m, h)$. From the condition (iii) it follows that $k(\varphi_m) \to k(\varphi^*)$ strongly in $L^{3/2}(\Omega)$ at $m \to \infty$. With the help of (42) it is not difficult to show that $\varphi_m h \to \varphi^* h$ weakly in $L^3(\Omega)$ for all $h \in H_0^1(\Omega)$. Then $k(\varphi_m)\varphi_m h \to k(\varphi^*)\varphi^* h$ strongly in $L^1(\Omega)$ or $(k(\varphi_m)\varphi_m, h) \to (k(\varphi^*)\varphi^*, h)$ at $m \to \infty$ for all $h \in H_0^1(\Omega)$.

The following equality holds:

$$((\mathbf{u}_m\cdot\nabla)\mathbf{u}_m, \mathbf{v}) - ((\mathbf{u}^*\cdot\nabla)\mathbf{u}^*, \mathbf{v}) = (((\mathbf{u}_m - \mathbf{u}^*)\cdot\nabla)\mathbf{u}_m, \mathbf{v}) + ((\mathbf{u}^*\cdot\nabla)(\mathbf{u}_m - \mathbf{u}^*), \mathbf{v})$$

From Lemma 1, (41) and due to the uniform boundedness of $\|\mathbf{u}_m\|_{1,\Omega}$ over m we obtain that

$$|(((\mathbf{u}_m - \mathbf{u}^*)\cdot\nabla)\mathbf{u}_m, \mathbf{v})| \le \gamma_1\|\mathbf{u}_m - \mathbf{u}^*\|_{L^4(\Omega)^3}\|\mathbf{u}_m\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega} \to 0 \text{ at } m \to \infty.$$

From (14) it follows that $((\mathbf{u}^*\cdot\nabla)(\mathbf{u}_m - \mathbf{u}^*), \mathbf{v}) = -((\mathbf{u}^*\cdot\nabla)\mathbf{v}, \mathbf{u}_m - \mathbf{u}^*)$. Arguing as above we obtain that

$$|((\mathbf{u}^*\cdot\nabla)\mathbf{v}, \mathbf{u}_m - \mathbf{u}^*)| \le \gamma_1\|\mathbf{u}^*\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}\|\mathbf{u}_m - \mathbf{u}^*\|_{L^4(\Omega)^3} \to 0 \text{ at } m \to \infty.$$

For the nonlinear summand $(\mathbf{u}_m\cdot\nabla\varphi_m, h)$ the following relation is satisfied:

$$(\mathbf{u}_m\cdot\nabla\varphi_m, h) - (\mathbf{u}^*\cdot\nabla\varphi^*, h) = ((\mathbf{u}_m - \mathbf{u}^*)\cdot\nabla\varphi_m, h) + (\mathbf{u}^*\cdot\nabla(\varphi_m - \varphi^*), h).$$

From Lemma 1, (41) and the estimate $\|\varphi_m\|_{1,\Omega} \le c_4$ we derive the following

$$|((\mathbf{u}_m - \mathbf{u}^*)\cdot\nabla\varphi_m, h)| \le \gamma_2\|\mathbf{u}_m - \mathbf{u}^*\|_{L^4(\Omega)^3}\|\varphi_m\|_{1,\Omega}\|h\|_{1,\Omega} \to 0 \text{ at } m \to \infty.$$

Due to the weak convergence $\varphi_m \to \varphi^*$ in $H^1(\Omega)$ (see (42)) we have

$$(\mathbf{u}^*\cdot\nabla(\varphi_m - \varphi^*), h) = (\nabla(\varphi_m - \varphi^*), h\,\mathbf{u}^*) \to 0 \text{ at } m \to \infty\ \forall h \in H_0^1(\Omega).$$

As the functional J is weakly semicontinuous on $X \times L^2(\Omega)$ then from the aforesaid it follows that $J(\mathbf{x}^*, f^*) = J^*$. ■

Remark 3. It is clear, that the functionals I_i, $i = 1, ..., 4$ from (40) satisfy the conditions of the Theorem 2.

4. **Optimality system**

Let additionally to (i)–(iii) the following conditions hold:
(iv) $k(\varphi)$ be continuously Fréchet differentiable function and
$(k(\varphi)\varphi)' = \beta(\varphi)\tau$ for all $\tau \in H_0^1(\Omega)$ and $\beta(\varphi) \in L_+^2(\Omega)$.

With the help of the Lagrange principle (see [29]), let us derive necessary optimality conditions for the problem (39). For this purpose let us denote by $X^* = V^* \times H^{-1}(\Omega) \times L_0^2(\Omega)$ and by $Y^* = H \times L_0^2(\Omega)$ spaces dual to X and Y.

It is not hard to show that if the condition (iv) is satisfied then the partial Fréshet derivative with the respect to $\mathbf{x}$ of the operator $F : X \to Y$ at any point $(\hat{\mathbf{x}}, \hat{f}) = (\hat{\mathbf{u}}, \hat{\varphi}, \hat{p}, \hat{f})$ is a linear continuous operator $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{f}) : X \to Y$, which brings in the correspondence with every element $(\mathbf{w}, h, r) \in X$ an element $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{f})(\mathbf{w}, h, r) = (\hat{y}_1, \hat{y}_2) \in Y$. Here the elements $\hat{y}_1 \in H^*$, $\hat{y}_2 \in L_0^2(\Omega)$, are defined by triples $(\hat{\mathbf{u}}, \hat{\varphi}, \hat{p})$ and $(\mathbf{w}, \tau, r)$ from relations

$$\langle \hat{y}_1, (\mathbf{v}, \tau)\rangle = \nu(\nabla \mathbf{w}, \nabla \mathbf{v}) + \lambda(\nabla \tau, \nabla h) + ((\mathbf{w} \cdot \nabla)\hat{\mathbf{u}}, \mathbf{v}) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \mathbf{v}) +$$

$$+(\beta(\hat{\varphi})\tau, h) + (\mathbf{w} \cdot \nabla \hat{\varphi}, h) + (\hat{\mathbf{u}} \cdot \nabla \tau, h) - (\operatorname{div} \mathbf{v}, r) - (\mathbf{b}\tau, \mathbf{v}) \;\; \forall (\mathbf{v}, \tau) \in H,$$

$$\langle \hat{y}_2, r\rangle = -(\operatorname{div} \mathbf{w}, r) \;\; \forall r \in L_0^2(\Omega). \tag{46}$$

By $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{f})^* : Y^* \to X^*$ we denote an operator which is adjoint to $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{f})$.

According the general theory of smooth-convex extremum problems [29], we introduce an element $\mathbf{y}^* = ((\xi, \theta), \sigma) \in Y^* = H \times L_0^2(\Omega)$, to which we will refer as to an adjoint state and define the Lagrangian $\mathcal{L} : X \times K \times \mathbb{R} \times Y^* \to \mathbb{R}$ by the formula

$$\mathcal{L}(\mathbf{x}, f, \lambda_0, \mathbf{y}^*) = \lambda_0 J(\mathbf{x}, f) + \langle \mathbf{y}^*, F(\mathbf{x}, f)\rangle_{Y^* \times Y} \equiv \lambda_0 J(\mathbf{x}, f) +$$

$$+\langle F_1(\mathbf{x}, f), (\xi, \theta)\rangle_{H^* \times H} + (F_2(\mathbf{x}, f), s). \tag{47}$$

The proof of the following theorem will be conducted according to the scheme from [1].

Theorem 3. *Under the assumptions (i)–(iv) and (j), (jj) let the element* $(\hat{\mathbf{x}}, \hat{f}) \in X \times K$ *be a local minimizer for the problem (39) and let the cost functional* I *be continuously Fréchet differentiable with respect to the state* $\mathbf{x}$ *at point* $\hat{\mathbf{x}}$*. Then there exists a nonzero Lagrange multiplier* $(\lambda_0, \mathbf{y}^*) = (\lambda_0, \xi, \theta, \sigma) \in \mathbb{R}^+ \times Y^*$ *such that the Euler–Lagrange equation takes place*

$$F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{f})^* \mathbf{y}^* = -\lambda_0 J'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{f}) \;\; \text{in } X^*,$$

which is equivalent to the relations

$$\nu(\nabla \mathbf{w}, \nabla \xi) + \lambda(\nabla \tau, \nabla \theta) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \xi) + ((\mathbf{w} \cdot \nabla)\hat{\mathbf{u}}, \xi) + (\beta(\hat{\varphi})\tau, \theta) - (\operatorname{div} \mathbf{w}, \sigma) +$$

$$+(\mathbf{w} \cdot \nabla \hat{\varphi}, \theta) + (\hat{\mathbf{u}} \cdot \nabla \tau, \theta) - (\mathbf{b}\tau, \xi) = -\lambda_0(\mu_0/2)(\langle I'_{\mathbf{u}}(\hat{\mathbf{x}}), \mathbf{w}\rangle + \langle I'_{\varphi}(\hat{\mathbf{x}}), \tau\rangle) \; \forall (\mathbf{w}, \tau) \in H, \tag{48}$$

$$(\operatorname{div} \xi, r) = \lambda_0(\mu_0/2)(I'_p(\hat{\mathbf{x}}), r) \;\; \forall r \in L_0^2(\Omega), \tag{49}$$

and the minimum principle $\mathcal{L}(\hat{\mathbf{x}}, \hat{f}, \lambda_0, \mathbf{y}^*) \le \mathcal{L}(\hat{\mathbf{x}}, f, \lambda_0, \mathbf{y}^*)$ *for all* $f \in K$ *holds, which is equivalent to the inequality*

$$\lambda_0 \mu_1(\hat{f}, f - \hat{f}) - (f - \hat{f}, \theta) \ge 0 \; \forall f \in K. \tag{50}$$

If, besides, (32) holds for all $f \in K$*, then any nontrivial Lagrange multiplier* $(\lambda_0, \mathbf{y}^*)$*, satisfying (48)–(50) is regular, i.e. it has the form* $(1, \mathbf{y}^*)$ *and is determined uniquely for a given pair* $(\hat{\mathbf{x}}, \hat{f})$*.*

Proof. According to [29], for the proof of the theorem 3 it's enough to show that the operator $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{f}) : X \to Y$ is a Fredholm operator. Due to (46), the operator $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{f}) : X \to Y$ can be introduced in the form $F'_{\mathbf{x}} = \Phi + \hat{\Phi} \equiv (\Phi_1, \Phi_2) + (\hat{\Phi}_1, 0)$. Here $\Phi_2(\mathbf{x}) = \operatorname{div} \mathbf{w}$, and operators Φ_1 and $\hat{\Phi}_1 : X \to H^*$ act by formulae

$$\langle \Phi_1(\mathbf{w}, \tau, r), (\mathbf{v}, h) \rangle = \nu(\nabla \mathbf{w}, \nabla \mathbf{v}) + \lambda(\nabla \tau, \nabla h) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \mathbf{v}) +$$

$$+ (\beta(\hat{\varphi})\tau, h) + (\hat{\mathbf{u}} \cdot \nabla \tau, h) - (\operatorname{div} \mathbf{v}, r) - (\mathbf{b}\tau, \mathbf{v}),$$

$$\langle \hat{\Phi}_1(\mathbf{w}, \tau, r), (\mathbf{v}, h) \rangle = ((\mathbf{w} \cdot \nabla)\hat{\mathbf{u}}, \mathbf{v}) + (\mathbf{w} \cdot \nabla \hat{\varphi}, h).$$

Let us show that the operator $\Phi : X \to Y$ is an isomorphism. For this purpose it is enough to prove that for any pair $(\mathbf{F}, s) \in H^* \times L^2_0(\Omega)$ there exists a unique solution $(\mathbf{w}, \tau, r) \in X$ of the linear problem

$$\nu(\nabla \mathbf{w}, \nabla \mathbf{v}) + \lambda(\nabla \tau, \nabla h) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \mathbf{v}) + (\beta(\hat{\varphi})\tau, h) +$$

$$+ (\hat{\mathbf{u}} \cdot \nabla \tau, h) - (\operatorname{div} \mathbf{v}, r) - (\mathbf{b}\tau, \mathbf{v}) = \langle \mathbf{F}, (\mathbf{v}, h) \rangle \ \ \forall (\mathbf{v}, h) \in H, \tag{51}$$

$$\operatorname{div} \mathbf{w} = s \text{ in } \Omega. \tag{52}$$

It is not hard to check that the problem (51), (52) is equivalent to the problem

$$\nu(\nabla \mathbf{w}, \nabla \mathbf{v}) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \mathbf{v}) - (\operatorname{div} \mathbf{v}, r) - (\mathbf{b}\tau, \mathbf{v}) =$$

$$= \langle \mathbf{f}, \mathbf{v} \rangle_{-1,\Omega} \ \forall \mathbf{v} \in H^1_0(\Omega), \ \operatorname{div} \mathbf{w} = s \text{ in } \Omega, \tag{53}$$

$$\lambda(\nabla \tau, \nabla h) + (\beta(\hat{\varphi})\tau, h) + (\hat{\mathbf{u}} \cdot \nabla \tau, h) = \langle f, h \rangle_{-1,\Omega} \ \ \forall h \in H^1_0(\Omega), \tag{54}$$

where $\langle \mathbf{F}, (\mathbf{v}, h) \rangle_{H^* \times H} = \langle \mathbf{f}, \mathbf{v} \rangle_{-1,\Omega} + \langle f, h \rangle_{-1,\Omega}$.

Let us denote by $V^\perp$ the orthogonal complement of V with respect to the inner product $(\nabla \cdot, \nabla \cdot)$. Since $s \in L^2_0(\Omega)$, there is a unique function $\mathbf{w}_0 \in V^\perp$ (see [28]) such that $\operatorname{div} \mathbf{w}_0 = s$ and $\|\mathbf{w}_0\|_{1,\Omega} \le \beta^{-1} \|s\|_\Omega$, where β is a constant from (7).

We consider a restriction of (53) on the space V:

$$\nu(\nabla \mathbf{w}, \nabla \mathbf{v}) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \mathbf{v}) - (\mathbf{b}\tau, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{-1,\Omega} \ \forall \mathbf{v} \in H^1_0(\Omega), \ \operatorname{div} \mathbf{w} = s \text{ in } \Omega. \tag{55}$$

We are looking for the function $\mathbf{w}$ in the form of $\mathbf{w} = \mathbf{w}_0 + \tilde{\mathbf{w}}$, where $\tilde{\mathbf{w}} \in V$ is an unknown function. Inserting $\mathbf{w} = \mathbf{w}_0 + \tilde{\mathbf{w}}$ in (55) we obtain

$$\nu(\nabla \tilde{\mathbf{w}}, \nabla \mathbf{v}) + ((\hat{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{w}}, \mathbf{v}) - (\mathbf{b}\tau, \mathbf{v}) =$$

$$= \langle \mathbf{f}, \mathbf{v} \rangle_{-1,\Omega} - \nu(\nabla \mathbf{w}_0, \nabla \mathbf{v}) - ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}_0, \mathbf{v}) \ \forall \mathbf{v} \in H^1_0(\Omega), \ \operatorname{div} \tilde{\mathbf{w}} = 0 \text{ in } \Omega, \tag{56}$$

The existence of the unique solution $\tau \in H^1_0(\Omega)$ of the problem (54) follows from the lemma 1, besides the following estimate holds

$$\|\tau\|_{1,\Omega} \le C_* \|f\|_\Omega. \tag{57}$$

Also due to the lemma 1 we can conclude that for any $\tau \in H^1_0(\Omega)$ there is a unique solution $\tilde{\mathbf{w}} \in V$ of the problem (56) and

$$\|\tilde{\mathbf{w}}\|_{1,\Omega} \le 1/(\delta_0 \nu)(\|\mathbf{f}\|_\Omega + \beta_0 C_* \|f\|_\Omega + \beta^{-1}(C_0 \nu + \gamma_0 M_{\mathbf{u}})\|s\|_\Omega).$$

Then the function $\mathbf{w} = \mathbf{w}_0 + \tilde{\mathbf{w}}$ is the solution of problem (55) for which the estimate holds

(58)
$$\|\mathbf{w}\|_{1,\Omega} \le M_{\mathbf{w}} := 1/(\delta_0\nu)(\|\mathbf{f}\|_\Omega + \beta_0 C_*\|f\|_\Omega + \beta^{-1}(C_0\nu + \gamma_0 M_{\mathbf{u}})\|s\|_\Omega) + \beta^{-1}\|s\|_\Omega.$$

Arguing as in [1] hence we conclude that the triple $(\mathbf{w}, \tau, r)$ is the solution of the problem (51). Analogously, due to (24) from (53) using (57) and (58) we derive the following estimate for r

$$\|r\|_\Omega \le \beta_2^{-1}(\nu C_0 + \gamma_0 M_{\mathbf{u}})M_{\mathbf{w}} + \beta_2^{-1}(\beta_0 C_*\|f\|_\Omega + \|\mathbf{f}\|_\Omega). \qquad (59)$$

Let $(\mathbf{w}_1, \tau_1, r_1)$ and $(\mathbf{w}_2, \tau_2, r_2)$ be solutions of the problem (51), (52). Then the differences $\mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2$, $\tau = \tau_1 - \tau_2$ and $r = r_1 - r_2$ satisfy the relations

$$\nu(\nabla\mathbf{w}, \nabla\mathbf{v}) + \lambda(\nabla\tau, \nabla h) + ((\hat{\mathbf{u}}\cdot\nabla)\mathbf{w}, \mathbf{v}) + (\beta(\hat{\varphi})h, \tau) +$$

$$+(\hat{\mathbf{u}}\cdot\nabla\tau, h) - (\operatorname{div}\mathbf{v}, r) - (\mathbf{b}\tau, \mathbf{v}) = 0 \ \ \forall(\mathbf{v}, h) \in H, \qquad (60)$$

$$\operatorname{div}\mathbf{w} = 0 \text{ in } \Omega. \qquad (61)$$

Denoting in (60) $\mathbf{v} = \mathbf{0}$, $h = \tau$, we are concluding the relation

$$\lambda(\nabla\tau, \nabla\tau) + (\beta(\hat{\varphi})\tau, \tau) = 0,$$

from which on the strength of lemma 1 it follows that $\tau = 0$ or $\tau_1 = \tau_2$ in Ω. Taking into account all of this and lemma 1 and letting $h = 0$ in (60), we similarly obtain that $\mathbf{w}_1 = \mathbf{w}_2$ in Ω. Then from (60) and (7) it follows that $r_1 = r_2$ in Ω.

In this case the operator $\Phi : X \to Y$ is surjective and invertible, so according to the Banach theorem Φ is an isomorphism.

As $H^1(\Omega)^3$ is compactly embedded in $L^4(\Omega)^3$, from the estimates

$$|(\mathbf{w}\cdot\nabla\hat{\varphi}, h)| \le \gamma_2\|\mathbf{w}\|_{L^4(\Omega)^3}\|\varphi\|_{1,\Omega}\|h\|_{1,\Omega}, \ |((\mathbf{w}\cdot\nabla)\hat{\mathbf{u}}, \mathbf{v})| \le \gamma_1\|\mathbf{w}\|_{L^4(\Omega)^3}\|\hat{\mathbf{u}}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega},$$

that follow from the lemma 1 we can conclude the operator $\hat{\Phi}$ is continuous and compact. Then the operator $F'_{\mathbf{x}}$ is a Fredholm operator as the sum of the isomorphism and the compact operator.

Let us show further that if the conditions (32) hold for all $f \in K$, then any nontrivial Lagrange multiplier which satisfy (48)–(50) is regular, i.e. has a form of $(1, \mathbf{y}^*)$. This is equivalent to the nonexistence of nontrivial solutions of the homogeneous Euler-Lagrange equation or of the system (48)–(50) at $\lambda_0 = 0$.

Let there be at least one nontrivial solution $\mathbf{y}^* = (\xi, \theta, \sigma) \in Y^*$ of the system (48)–(50) at $\lambda_0 = 0$, where the elements $\hat{\mathbf{x}} = (\hat{\mathbf{u}}, \hat{\varphi}, \hat{p})$ and $\hat{f}$ are connected by the relation $F(\hat{\mathbf{x}}, \hat{f}) = 0$. Setting $\tau = 0$, $\mathbf{w} = \xi$ and $r = \sigma$ in this system, we are obtaining the equality

$$\nu(\nabla\xi, \nabla\xi) + ((\xi\cdot\nabla)\hat{\mathbf{u}}, \xi) = -(\xi\cdot\nabla\hat{\varphi}, \theta),$$

from which while the condition $\mathrm{Re} < 1/2$ is holding, we can conclude the estimate

$$\|\xi\|_{1,\Omega} \le 2(\gamma_2/\delta_0\nu)M_\varphi\|\theta\|_{1,\Omega}. \qquad (62)$$

After letting $\mathbf{w} = \mathbf{0}$, $\tau = \theta$ in (48) , we obtain

$$\lambda(\nabla\theta, \nabla\theta) + (\beta(\hat{\varphi})\theta, \theta) = -(\mathbf{b}\theta, \xi).$$

From this with the help of (62) and of the estimates of lemma 1 we conduct

$$\|\theta\|_{1,\Omega} \le 2(\beta_0/\delta_1\lambda)(\gamma_2/\delta_0\nu)M_\varphi\|\theta\|_{1,\Omega} = 2\mathrm{Ra}\|\theta\|_{1,\Omega}. \qquad (63)$$

From (63), (62) it follows that when the conditions $\mathrm{Re} < 1/2$ and $\mathrm{Ra} < 1/2$ are satisfied, which are less strict than (32), $\theta = 0$ and $\xi = \mathbf{0}$ a.e. in Ω and (48) takes the form $(\mathrm{div}\,\mathbf{w}, \sigma) = 0$ for all $\mathbf{w} \in H_0^1(\Omega)^3$. Then from (7) we obtain that $\sigma = 0$ a.e. in Ω. The latter contradicts the supposed nontriviality of (ξ, θ, σ).

The uniqueness of the regular Lagrange multiplier $(1, \mathbf{y}^*)$ at conditions (32) follows from the operator $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{f})$ being Fredholm. ■

The boundary value problem (10), (11), the conjugated problem (48), (49) and the inequality (50) form *an optimality system* which describes necessary conditions for the minimum for the problem (39). In the next section basing on the analysis of the constructed system we will state sufficient conditions for initial data of the particular extremum problems which will provide the uniqueness of their solutions.

5. Uniqueness of the solution of extremum problem

Let us denote by $(\mathbf{u}_i, \varphi_i, p_i, f_i) \in X \times K$, $i = 1, 2$ two arbitrary solutions of the extremum problem (39). Let us also suppose the set K to bounded and obtain the inequality for the difference of solutions of the problem (39), which be useful for the further analysis. Let us remind that due to the theorem 1 the following estimates are true:

$$\|\varphi_i\|_{1,\Omega} \le M_\varphi = C_* \sup_{f \in K} \|f\|_\Omega, \ \|\mathbf{u}_i\|_{1,\Omega} \le M_{\mathbf{u}} = (\nu\delta_0)^{-1} \sup_{f \in K} (\|\mathbf{f}\|_\Omega + \beta_0 C_* \|f\|_\Omega),$$

$$\|p_i\|_\Omega \le M_p = \beta_2^{-1} \sup_{f \in K} [(\nu + \gamma_2 M_{\mathbf{u}}) M_{\mathbf{u}} + \|\mathbf{f}\|_\Omega + \beta_0 M_\varphi], \ i = 1, 2, \ C_* \equiv \lambda_*^{-1}. \tag{64}$$

It is clear that $M_{\mathbf{u}}$, M_φ and M_p are bounded when K is bounded.

Denote by $(1, \mathbf{y}_i^*) \equiv (1, \xi_i, \theta_i, \sigma_i)$, $i = 1, 2$ nontrivial Lagrange multipliers, which correspond to the solutions $(\mathbf{x}_i, f_i)$ and defined uniquely when (32) holds. They satisfy the following relations:

$$\nu(\nabla \mathbf{w}, \nabla \xi_i) + \lambda(\nabla \tau, \nabla \theta_i) + ((\mathbf{u}_i \cdot \nabla)\mathbf{w}, \xi_i) + ((\mathbf{w} \cdot \nabla)\mathbf{u}_i, \xi_i) + 3æ(\varphi_i^2 \tau, \theta_i) - (\mathrm{div}\,\mathbf{w}, \sigma_i) +$$

$$+ (\mathbf{w} \cdot \nabla \varphi_i, \theta_i) + (\mathbf{u}_i \cdot \nabla \tau, \theta_i) - (\mathbf{b}\tau, \xi_i) = -(\mu_0/2)(\langle I'_{\mathbf{u}}(\mathbf{x}_i), \mathbf{w}\rangle + \langle I'_\varphi(\mathbf{x}_i), \tau\rangle) \ \forall (\mathbf{w}, \tau) \in H, \tag{65}$$

$$(\mathrm{div}\,\xi_i, r) = \lambda_0(\mu_0/2)(I'_p(\mathbf{x}_i), r) \ \ \forall r \in L_0^2(\Omega), \tag{66}$$

Let

$$\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2, \ \varphi = \varphi_1 - \varphi_2, \ p = p_1 - p_2,$$

$$f = f_1 - f_2, \ \xi = \xi_1 - \xi_2, \ \theta = \theta_1 - \theta_2, \ \sigma = \sigma_1 - \sigma_2. \tag{67}$$

Subtract (11), (12) at $(\mathbf{x}_2, f_2)$ from (11), (12) at $(\mathbf{x}_1, f_1)$. Taking into account that

$$(\mathbf{u}_1 \cdot \nabla \varphi_1, h) - (\mathbf{u}_2 \cdot \nabla \varphi_2, h) = (\mathbf{u}_1 \cdot \nabla \varphi, h) + (\mathbf{u} \cdot \nabla \varphi_2, h),$$

$$\varphi_1^3 - \varphi_2^3 = (\varphi_1 - \varphi_2)(\varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2) \equiv k_0 \varphi, \ k_0 = \varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2 \ge 0,$$

in terms of (67) we obtain

$$\lambda(\nabla \varphi, \nabla h) + æ(k_0 \varphi, h) + (\mathbf{u}_1 \cdot \nabla \varphi, h) = (f, h) - (\mathbf{u} \cdot \nabla \varphi_2, h) \ \ \forall h \in H_0^1(\Omega), \tag{68}$$

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u}_1 \cdot \nabla)\mathbf{u}, \mathbf{v}) = (\mathbf{b}\varphi, \mathbf{v}) - ((\mathbf{u} \cdot \nabla)\mathbf{u}_2, \mathbf{v}) \ \forall \mathbf{v} \in V. \tag{69}$$

As $k_0 \in L_+^p(\Omega)$, $p \ge 3/2$ then after using lemma 1 and accounting to (64) we get the following estimate from (68):

$$\|\varphi\|_{1,\Omega} \le C_*(\|f\|_\Omega + \gamma_2 M_\varphi \|\mathbf{u}\|_{1,\Omega}). \tag{70}$$

Also after applying lemma 1 to (69) and taking into account (64) we derive an inequality

$$\nu_*\|\mathbf{u}\|_{1,\Omega} \le \beta_0\|\varphi\|_{1,\Omega} + \gamma_1 M_{\mathbf{u}}\|\mathbf{u}\|_{1,\Omega}. \tag{71}$$

Similarly to the section 1 at the condition Re $< 1/2$ we obtain the estimate from (71) $\|\mathbf{u}\|_{1,\Omega} \le 2(\beta_0/\delta_0\nu)\|\varphi\|_{1,\Omega}$, which coincides with (30). Putting this estimate in (70) one can get an inequality

$$\|\varphi\|_{1,\Omega} \le C_*(\|f\|_\Omega + 2\gamma_2 M_\varphi(\beta_0/\delta_0\nu)\|\varphi\|_{1,\Omega}) = C_*\|f\|_\Omega + 2\mathrm{Ra}\|\varphi\|_{1,\Omega},$$

from which at the condition Ra $< 1/4$ the estimate follows

$$\|\varphi\|_{1,\Omega} \le 2C_*\|f\|_\Omega. \tag{72}$$

Substituting (72) in (30), derive the estimate of the difference $\mathbf{u}$ via the difference of the controls f:

$$\|\mathbf{u}\|_{1,\Omega} \le 4C_*(\beta_0/\delta_0\nu)\|f\|_\Omega. \tag{73}$$

Let $f = f_2$ in the inequality (50) written at $\hat f = f_1,\ \theta = \theta_1$ and let $f = f_1$ in the inequality (50) at $\hat f = f_2,\ \theta = \theta_2$. We obtain

$$\mu_1(f_1, f_2 - f_1) - (f_2 - f_1, \theta_1) \ge 0,\ \mu_1(f_2, f_1 - f_2) - (f_1 - f_2, \theta_2) \ge 0.$$

Add these inequalities and derive the estimate

$$-(f,\theta) \le -\mu_1\|f\|^2_\Omega. \tag{74}$$

Subtract (65), (66) at $i = 2$ from (65), (66) at $i = 1$. Taking into account that

$$(\varphi_1^2\tau,\theta_1) - (\varphi_2^2\tau,\theta_2) = (\varphi_1^2\tau,\theta) + ((\varphi_1^2 - \varphi_2^2)\tau,\theta_2) = (\varphi_1^2\tau,\theta) + ((\varphi_1+\varphi_2)\varphi\tau,\theta_2),$$

we will have

$$\nu(\nabla\mathbf{w},\nabla\xi) + \lambda(\nabla\tau,\nabla\theta) + ((\mathbf{u}_1\cdot\nabla)\mathbf{w},\xi) + ((\mathbf{w}\cdot\nabla)\mathbf{u}_1,\xi) + 3æ(\varphi_1^2\tau,\theta) + 3æ((\varphi_1^2-\varphi_2^2)\tau,\theta_2) +$$
$$+ (\mathbf{u}_1\cdot\nabla\varphi,\theta) - (\operatorname{div}\mathbf{w},\sigma) - (\mathbf{b}\tau,\xi) - ((\mathbf{u}\cdot\nabla)\mathbf{w},\xi_2) - (\mathbf{u}\cdot\nabla\tau,\theta_2) - ((\mathbf{w}\cdot\nabla)\mathbf{u},\xi_2) -$$
$$= -(\mu_0/2)(\langle I'_{\mathbf{u}}(\mathbf{x}_1) - I'_{\mathbf{u}}(\mathbf{x}_2),\mathbf{w}\rangle + \langle I'_\varphi(\mathbf{x}_1) - I'_\varphi(\mathbf{x}_2),\tau\rangle)\ \forall(\mathbf{w},\tau)\in H, \tag{75}$$

$$(\operatorname{div}\xi, r) = \lambda_0(\mu_0/2)(I'_p(\mathbf{x}_1) - I'_p(\mathbf{x}_2), r)\ \ \forall r \in L^2_0(\Omega), \tag{76}$$

After letting $\mathbf{w} = \mathbf{u}$, $\tau = \varphi$ and $r = \sigma$ in (75), (76) we derive

$$\nu(\nabla\mathbf{u},\nabla\xi) + \lambda(\nabla\varphi,\nabla\theta) + ((\mathbf{u}_1\cdot\nabla)\mathbf{u},\xi) + ((\mathbf{u}\cdot\nabla)\mathbf{u}_1,\xi) + 3æ(\varphi_1^2\varphi,\theta) + 3æ((\varphi_1^2-\varphi_2^2)\varphi,\theta_2) +$$
$$+ (\mathbf{u}_1\cdot\nabla\varphi,\theta) - (\mathbf{b}\varphi,\xi) = -((\mathbf{u}\cdot\nabla)\mathbf{u},\xi_2) - (\mathbf{u}\cdot\nabla\varphi,\theta_2) - ((\mathbf{u}\cdot\nabla)\mathbf{u},\xi_2) -$$
$$-(\mu_0/2)(\langle I'_{\mathbf{u}}(\mathbf{x}_1) - I'_{\mathbf{u}}(\mathbf{x}_2),\mathbf{u}\rangle + \langle I'_\varphi(\mathbf{x}_1) - I'_\varphi(\mathbf{x}_2),\varphi\rangle), \tag{77}$$

$$(\operatorname{div}\xi,\sigma) = \lambda_0(\mu_0/2)(I'_p(\mathbf{x}_1) - I'_p(\mathbf{x}_2),\sigma). \tag{78}$$

Let us suppose further $\mathbf{v} = \xi$, $h = \theta$ in (68), (69) and add obtained relations

$$\nu(\nabla\mathbf{u},\nabla\xi) + \lambda(\nabla\varphi,\nabla\theta) + ((\mathbf{u}_1\cdot\nabla)\mathbf{u},\xi) + æ(k_0\varphi,\theta) + (\mathbf{u}_1\cdot\nabla\varphi,\theta) =$$
$$= (\mathbf{b}\varphi,\xi) + (f,\theta) - ((\mathbf{u}\cdot\nabla)\mathbf{u}_2,\xi) - (\mathbf{u}\cdot\nabla\varphi_2,\theta) \tag{79}$$

Subtract (79) from (77) and add with (74). Accounting that

$$2((\mathbf{u}\cdot\nabla)\mathbf{u},\xi_2) + ((\mathbf{u}_1\cdot\nabla)\mathbf{u},\xi) - ((\mathbf{u}_2\cdot\nabla)\mathbf{u},\xi) = ((\mathbf{u}\cdot\nabla)\mathbf{u},\xi_1+\xi_2),$$
$$2(\mathbf{u}\cdot\nabla\varphi,\theta_2) + (\mathbf{u}_1\cdot\nabla\varphi,\theta) - (\mathbf{u}_2\cdot\nabla\varphi,\theta) = (\mathbf{u}\cdot\nabla\varphi,\theta_1+\theta_2),$$
$$3(\varphi_1^2\varphi,\theta) + 3((\varphi_1^2-\varphi_2^2)\varphi,\theta_2) - ((\varphi_1^2+\varphi_1\varphi_2+\varphi_2^2)\varphi,\theta) =$$

$$= ((2\varphi_1^2 - \varphi_2^2 - \varphi_1\varphi_2)\varphi, \theta) + 3((\varphi_1 + \varphi_2)\varphi^2, \theta_1),$$

we are deriving the relation

$$((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1+\xi_2)+(\mathbf{u}\cdot\nabla\varphi, \theta_1+\theta_2)+æ[((2\varphi_1^2-\varphi_2^2-\varphi_1\varphi_2)\varphi, \theta)+3((\varphi_1+\varphi_2)\varphi^2, \theta_1)] \leq$$

(80)

$$-(\mu_0/2)(\langle I'_{\mathbf{u}}(\mathbf{x}_1)-I'_{\mathbf{u}}(\mathbf{x}_2), \mathbf{u}\rangle+\langle I'_\varphi(\mathbf{x}_1)-I'_\varphi(\mathbf{x}_2), \varphi\rangle+\langle I'_p(\mathbf{x}_1)-I'_p(\mathbf{x}_2), p\rangle)-\mu_1\|f\|^2_\Omega.$$

Inequality (80) plays a key role when deriving sufficient conditions for the uniqueness of optimal problems' solutions. Let us begin with the analysis of the extremum problem that corresponds to the cost functional $I_1(\varphi) = \|\varphi - \varphi^d\|^2_Q$:

$$J(\varphi, f) = (\mu_0/2)I_1(\varphi) + (\mu_1/2)\|f\|^2_\Omega \to \inf,\ F(\mathbf{x}, f) = 0,\ (\mathbf{x}, f) \in X \times K. \tag{81}$$

Denote by $(\mathbf{x}_1, f_1)$ and $(\mathbf{x}_2, f_2)$ two solutions of problem (81). For the functional $I_1(\varphi)$ these relations $\langle I'_1(\varphi_i), \tau\rangle = 2(\varphi_i - \varphi^d, \tau)_Q$, $\langle I'_1(\varphi_1) - I'_1(\varphi_2), \tau\rangle = 2(\varphi, \tau)_Q$, $i = 1, 2$ are true. Due to them the inequality (80) takes the form

$$((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1+\xi_2)+(\mathbf{u}\cdot\nabla\varphi, \theta_1+\theta_2)+æ[((2\varphi_1^2-\varphi_2^2-\varphi_1\varphi_2)\varphi, \theta)+3((\varphi_1+\varphi_2)\varphi^2, \theta_1)] \leq$$

$$\leq -\mu_0\|\varphi\|^2_Q - \mu_1\|f\|^2_\Omega. \tag{82}$$

Let us estimate the Lagrange multiplier $(1, \mathbf{y}_i^*) \equiv (1, \xi_i, \theta_i, \sigma_i)$, $i = 1, 2$ from the relations (65), (66), which at $I = I_1(\varphi)$ look like

$$\nu(\nabla\mathbf{w}, \nabla\xi_i) + \lambda(\nabla\tau, \nabla\theta_i) + ((\mathbf{u}_i\cdot\nabla)\mathbf{w}, \xi_i) + ((\mathbf{w}\cdot\nabla)\mathbf{u}_i, \xi_i) + 3æ(\varphi_i^2\tau, \theta_i)+$$

(83)

$$+(\mathbf{w}\cdot\nabla\varphi_i, \theta_i)+(\mathbf{u}_i\cdot\nabla\tau, \theta_i)-(\operatorname{div}\mathbf{w}, \sigma_i)-(\mathbf{b}\tau, \xi_i) = -\mu_0(\varphi_i-\varphi^d, \tau)_Q\ \forall(\mathbf{w}, \tau) \in H,$$

$$(\operatorname{div}\xi_i, r) = 0\ \ \forall r \in L^2_0(\Omega), \tag{84}$$

Let $\mathbf{w} = \xi_i$, $\tau = 0$ in (83) and due to (84) we obtain

$$\nu(\nabla\xi_i, \nabla\xi_i) + ((\xi_i\cdot\nabla)\mathbf{u}_i, \xi_i) = -(\xi_i\cdot\nabla\varphi_i, \theta_i). \tag{85}$$

On the strength of lemma 1 the following estimates are satisfied

$$|((\xi_i\cdot\nabla)\mathbf{u}_i, \xi_i)| \leq \gamma_1 M_{\mathbf{u}}\|\xi_i\|^2_{1,\Omega},\ |(\xi_i\cdot\nabla\varphi_i, \theta_i)| \leq \gamma_2 M_\varphi\|\xi_i\|_{1,\Omega}\|\theta_i\|_{1,\Omega}.$$

At $\mathrm{Re} < 1/2$ with respect to the estimates from (85) we deduct

$$\|\xi_i\|_{1,\Omega} \leq 2(\gamma_2/\delta_0\nu)M_\varphi\|\theta_i\|_{1,\Omega}. \tag{86}$$

Let $\mathbf{w} = \mathbf{0}$, $\tau = \theta_i$ in (83) then

$$\lambda(\nabla\theta_i, \nabla\theta_i) + 3æ(\varphi_i^2\theta_i, \theta_i) = (\mathbf{b}\theta_i, \xi_i) - \mu_0(\varphi_i - \varphi^d, \theta_i)_Q. \tag{87}$$

From (87) with respect to (86) we obtain the inequality

$$(\delta_1\lambda)\|\theta_i\|^2_{1,\Omega} \leq 2\beta_0(\gamma_2/\delta_0\nu)M_\varphi\|\theta_i\|^2_{1,\Omega} + \mu_0 M^0_\varphi\|\theta_i\|_{1,\Omega},\ M^0_\varphi = M_\varphi + \|\varphi^d\|_Q,$$

from which due to $\mathrm{Ra} < 1/4$ we get an estimate for the multiplier θ_i

$$\|\theta_i\|_{1,\Omega} \leq \mu_0(2/\delta_1\lambda)M^0_\varphi,\ i = 1, 2. \tag{88}$$

Finally, with the help of (88), from (86) we conclude the estimate for ξ_i:

$$\|\xi_i\|_{1,\Omega} \leq 4\mu_0(\gamma_2/\delta_0\nu\delta_1\lambda)M_\varphi M^0_\varphi = 4(\mu_0/\beta_0)\mathrm{Ra}\, M^0_\varphi,\ i = 1, 2. \tag{89}$$

Obtained estimates for Lagrange multipliers ξ_i and η_i together with the inequalities (72), (73) give an opportunity to estimate the first and the second summands in left part of (82) via the L^2-norm of the difference $f = f_1 - f_2$:

$$|(\mathbf{u}\cdot\nabla\varphi, \theta_1+\theta_2)| \le 32\mu_0 C_*^2(\gamma_2\beta_0/(\delta_1\lambda\delta_0\nu))M_\varphi^0\|f\|_\Omega^2 = 32\mu_0 C_*^2\mathrm{Ra}^0\|f\|_\Omega^2,$$

$$\mathrm{Ra}^0 = (M_\varphi^0/M_\varphi)\mathrm{Ra},$$

$$|((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1+\xi_2)| \le 128\mu_0\gamma_1 C_*^2(1/\nu\delta_0)^2\mathrm{Ra}\|f\|_\Omega^2. \tag{90}$$

The ratio (75) for the functional $I_1(\varphi)$ at $\mathbf{w}=\mathbf{0}$ takes the form

$$\lambda(\nabla\tau,\nabla\theta)+3æ(\varphi_1^2\tau,\theta)+(\mathbf{u}_1\cdot\nabla\tau,\theta) = -3æ((\varphi_1+\varphi_2)\varphi\tau,\theta_2)-(\mathbf{u}\cdot\nabla\tau,\theta_2)-\mu_0(\varphi,\tau)_Q. \tag{91}$$

From (91) we will get an estimate for the difference $\theta = \theta_1 - \theta_2$ via the L^2-norm of the controls' difference. That will allow to estimate the third and the fourth summands in (82) via $\|f\|_\Omega^2$.

With the help of the estimates (72), (73), (88) we obtain the inequalities for the summands from the right part of (91):

$$|(\mathbf{u}\cdot\nabla\tau,\theta_2)| \le \gamma_2\|\mathbf{u}\|_{1,\Omega}\|\theta_2\|_{1,\Omega}\|\tau\|_{1,\Omega} \le 2\mu_0 C_*\mathrm{Ra}^0\|f\|_\Omega\|\tau\|_{1,\Omega},$$

$$|(\varphi,\tau)_Q| \le \|\varphi\|_{1,\Omega}\|\tau\|_{1,\Omega} \le 2C_*\|f\|_\Omega\|\tau\|_{1,\Omega},$$

$$|((\varphi_1+\varphi_2)\varphi\tau,\theta_2)| \le \|\varphi_1+\varphi_2\|_{L^4(\Omega)}\|\varphi\|_{L^4(\Omega)}\|\theta_2\|_{L^4(\Omega)}\|\tau\|_{L^4(\Omega)} \le$$

$$\le C_4^4\|\varphi_1+\varphi_2\|_{1,\Omega}\|\theta_2\|_{1,\Omega}\|\varphi\|_{1,\Omega}\|\tau\|_{1,\Omega} \le 8\mu_0 C_4^4 C_*(1/\delta_1\lambda)M_\varphi M_\varphi^0\|f\|_\Omega\|\tau\|_{1,\Omega}.$$

Setting $\tau=\theta$ in (91) leads to the following inequality for the difference θ:

$$\|\theta\|_{1,\Omega} \le \mu_0\alpha\|f\|_\Omega,\ \ \alpha = 2C_*(4\mathrm{Ra}^0 + 12C_4^4(1/\delta_1\lambda)M_\varphi M_\varphi^0 + 1). \tag{92}$$

It allows to get the following estimates for the mentioned summands from (82):

$$|((2\varphi_1^2-\varphi_2^2-\varphi_1\varphi_2)\varphi,\theta)| \le 8\mu_0 M_\varphi^2 C_4^4 C_*\alpha\|f\|_\Omega^2,$$

$$3|((\varphi_1+\varphi_2)\varphi^2,\theta_1)| \le 48\mu_0 C_4^4 C_*^2(1/\delta_1\lambda)M_\varphi M_\varphi^0\|f\|_\Omega^2. \tag{93}$$

Denote by A the sum of the first four summands in (82), and then obtain from (90), (93)

$$|A| \le \mu_0\omega^2\|f\|_\Omega^2, \tag{94}$$

$$\omega^2 = 32C_*^2(\mathrm{Ra}^0 + 4(\gamma_1/\nu\delta_0)^2\mathrm{Ra}) + 8M_\varphi C_4^4 C_*(M_\varphi\alpha + 6C_*(1/\delta_1\lambda)M_\varphi^0), \tag{95}$$

where the constant α is defined in (92).

Let the input data of the problem (81) be such that the condition

$$\mu_0\omega^2 < \mu_1(1-\varepsilon). \tag{96}$$

is satisfied, where $\varepsilon \in (0,1)$ is an arbitrary number. If the condition (96) is met, the estimate (82) takes the form

$$\varepsilon\|f\|_\Omega^2 + \mu_0\|\varphi\|_Q^2 \le 0.$$

From the last inequality it follows that $f=0$ or $f_1=f_2$ a.e. in Ω. Then from the estimates (73) and (72) we obtain that $\mathbf{u}=\mathbf{0}$ and $\varphi=0$ or $\mathbf{u}_1=\mathbf{u}_2$ and $\varphi_1=\varphi_2$ a.e. in Ω. Subtracting (10) at $(\mathbf{x}_2, f_2)$ from (10) at $(\mathbf{x}_1, f_1)$ and taking into account that $\mathbf{u}=\mathbf{0}$ and $\varphi=0$ we obtain that the difference $p=p_1-p_2$ satisfies the equation (33). From (33) we arrive to $p=0$ due to (7).

The following theorem holds.

Theorem 4. *In addition to the conditions (i) and (j), let K be a bounded set,* $\mathrm{Re} < 1/2$ *and* $\mathrm{Ra} < 1/4$ *and the condition (96) are satisfied, where ω is defined in (95). Then there exists a unique solution $(\mathbf{x}, f) \in X \times K$ of the problem (81).*

With minor changes in the technique of the proving sufficient conditions for the uniqueness of the optimal solutions for the "hydrodynamic" cost functionals from (40) can be stated.

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Roman Victorovich Brizitskii
Institute of Applied Mathematics,
7, Radio str.,
Vladivostok, 690041, Russia
Far Eastern Federal University,
8, Sukhanova str.,
Vladivostok, 690091, Russia
E-mail address: mlnwizard@mail.ru

Zhanna Yurievna Saritskaya
Far Eastern Federal University,
8, Sukhanova str.,
Vladivostok, 690091, Russia
E-mail address: zhsar@icloud.com

Roman Romanovich Kravchuk
Far Eastern Federal University,
8, Sukhanova str.,
Vladivostok, 690091, Russia
E-mail address: sheild54@gmail.com