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BEHAVIOR OF SOLUTIONS TO AN INVERSE PROBLEM FOR A QUASILINEAR PARABOLIC EQUATION

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Abstract. In this article we consider the inverse problem with an integral condition by redefinition for a parabolic type equation. The existence of a weak solution of the inverse problem is proved by the Galerkin method.In a bounded domain with a homogeneous Dirichlet condition, sufficient conditions for the destruction of its solution in a finite time are obtained, and also the stability of the solution for the inverse problem with the opposite sign on the nonlinearity of the power type.

Keywords: inverse problems, blowing-up solutions, stability, integral overdetermination condition.

1. Introduction

We consider in the cylinder $Q_T = \{(x,t): \ x \in \Omega, \ t \in (0,T)\}$ the inverse problem for the heat equation with a power nonlinearity: determine a pair of functions $(u(x,t), \ f(t))$ satisfying the equation

(1)
$$\frac{\partial}{\partial t}\left(u + a_0|u^{p-2}u\right) - \Delta u + a(x,t,u,\nabla u) = |u|^{p-2}u + f(t)\omega(x), \ \ x \in \Omega, \ \ 0 < t < T,$$

(2)
$$u(x,0) = u_0(x),$$

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$$u|_{\partial\Omega\times(0,T)} = 0, \tag{3}$$

$$\int_\Omega \left(u + a_0|u|^{p-2}u\right)\omega\, dx = \varphi(t),\ \ 0 < t < T. \tag{4}$$

Here $\Omega \subset R^n,\ n \geq 1$ is the bounded area, the border $\partial\Omega$ is sufficiently smooth, p and a_0 positive constants, that $p \geq 2$. $\varphi(t)$ is a differentiable function such that $0 < N_0 < \varphi(t) < N_1$ and $0 < N_0 < \varphi'(t) < N_2$ for some real constants $N_0,\ N_1$ and N_2. $\omega(x)$, $a(x,t,u,\nabla u)$ and $u_0(x)$ satisfy the following conditions:

$$\int_\Omega \omega^2(x)dx = 1,\ \ \omega \in H^2(\Omega) \cap H_0^1(\Omega) \cap L_p(\Omega) \cap L_{\frac{p}{p-1}}(\Omega) \cap L_2(\Omega),\ p \geq 2. \tag{5}$$

$$\int_\Omega u_0 \cdot \omega dx = \varphi(0),\ \ u_0 \in H_0^1(\Omega) \cap L_p(\Omega) \cap L_2(\Omega),\ p \geq 2. \tag{6}$$

and for $a_1 > 0,\ a_2 > 0$ constants are satisfied

$$|a(x,t,u,\nabla u)| \leq a_1|\nabla u| + a_2|u|^{\frac{p}{2}}. \tag{7}$$

The problem of existence of generalized solutions for parabolic equations with double nonlinearity was devoted to the works [1]–[10]. We note the work of P.A.Raviart [2] who first proved the existence of a generalized solution of the initial-boundary value problem for parabolic equations with double nonlinearity.

In the work of [6] F. Kh. Mukminov and E. R. Andriyanova was considered the first mixed problem for a parabolic equation with a double nonlinearity

$$(|u|^{\alpha-2}u)_t + A(u) = f(t,x), \alpha > 1, (t,x) \in D;$$

$$u(t,x)|_s = 0, S = \{t > 0\} \times \partial\Omega;$$

$$u(0,x) = u_0(x), u_0(x) \in L_\alpha(\Omega).$$

Here

$$A(u) = -\sum_{i=1}^{\infty}(|u_{x_i}|^{p-2}u_{x_i})_{x_i}, p > 1.$$

Existence of a global strong solution in time of a parabolic equation with double nonlinearity in an unbounded domain is proved by the method of Galerkin's approximations. A lower bound is obtained for the rate of decrease norm of solutions of this equation.

The problems blow up of solutions of the form

$$\frac{\partial}{\partial t}(u + \sum_{k=1}^{n} a_k(x)|u|^{p_k-2}u) - \Delta(|u|^{q_1}u) = |u|^{q_2}u,$$

$$\frac{\partial}{\partial t}(u + \sum_{k=1}^{n} a_k(x)|u|^{p_k-2}u) - div(h(x,|\nabla u|)\nabla u) + g(x,u) = f(x,u),$$

where the functions $h(x, |\nabla u|)$, $g(x, u)$, $f(x, u)$ have nonlinearity conditions (see [7]), have been investigated in many works [7]–[11]. In the work [7], [8] by the modified method of Levin obtained sufficient conditions for the destruction of its solution in finite time in a bounded domain.

The problem of determining the source for linear and nonlinear equations with a final and integral over determination was considered by various methods by a number of authors [12]–[20].

In the work [19] the inverse problem is investigated for a parabolic equation with power nonlinearity of the form

$$u_t - \Delta u - |u|^p u + b(x,t,u,\nabla u) = F(t)\omega(x), \ \ x \in \Omega, \ \ t > 0,$$

$$u(x,0) = u_0(x),$$

$$u|_{\partial\Omega\times(0,\infty)} = 0,$$

$$\int_\Omega u \cdot \omega \, dx = 1, \ t > 0.$$

In this work we obtain the conditions on known data that guarantee the global absence of solutions of the inverse problem. And also the stability of the solution is established on the bounded domain for the inverse problem with the opposite sign on the nonlinearity of the power type

$$u_t - \Delta u + |u|^p u = F(t)\omega(x), \ x \in \Omega, \ t > 0.$$

In the work [20] the inverse problem is considered for a quasilinear parabolic equation with power nonlinearity of the form

$$u_t - div\left((k_1 + k_2|\nabla u|^{m-2})\nabla u\right) + h(u,\nabla u) - |u|^{p-2}u = f(t)\omega(x), \ x \in \Omega, \ t > 0,$$

with initial-boundary condition (2), (3) and the additional integral condition (4) (when the function $\varphi(t) = 1$ at $t > 0$). The conditions are obtained for the destruction of its solution in finite time.

The inverse problem of determining the right-hand side of a parabolic equation with a non-standard growth condition is investigated in the work [18]. The existence and uniqueness of a solution to this problem are proved. Sufficient conditions for the destruction and disappearance of the solution in a finite time are obtained. The asymptotic behavior of the solutions of the inverse problem for large values of time is investigated.

In this work we use the following notation: $\|u\| = \|u\|_{L_2(\Omega)}$, $\|u\|_{p,\Omega} = \|u\|_{L_p(\Omega)}$, where $L_2(\Omega)$ and $L_p(\Omega)$ of Lebesgue space, scalar product $(u,v) = \int_\Omega u \cdot v dx$.

Jung's Inequality

$$ab \le \beta a^{q_1} + C(\beta, q_1) b^{q_1'}, \quad \frac{1}{q_1} + \frac{1}{q_1'} = 1, \quad \beta > 0, \quad C(\beta, q_1) = \frac{1}{q_1'(\beta q_1)^{\frac{q_1'}{q_1}}}$$

We give a well-known lemma of [21] V. Kalantarov-O.A. Ladyzhenskaya, which is important in obtaining our results.

Theorem 1. *([21]) Suppose that a positive, twice differentiable function $\psi(t)$ satisfies, for $t \geq 0$ the following inequality*

$$\psi(t)\psi''(t) - (1+\alpha)(\psi'(t))^2 \geq -2C_1\psi(t)\psi'(t) - C_2\psi^2(t),$$

where $\alpha > 0$, $C_1, C_2 \geq 0$ and $C_1 + C_2 > 0$. Then, if $\psi(0) > 0$ and $\psi'(0) + \gamma_2\alpha^{-1}\psi(0) > 0$ then the function $\psi(t)$ tends to infinity as

$$t \to t_1 \leq t_2 = \frac{1}{2\sqrt{C_1^2 + \alpha C_2}} \ln \frac{\gamma_1\psi(0) + \alpha\psi'(0)}{\gamma_2\psi(0) + \alpha\psi'(0)}$$

where
$\gamma_1 = -C_1 + \sqrt{C_1^2 + \alpha C_2}$ *and* $\gamma_2 = -C_1 - \sqrt{C_1^2 + \alpha C_2}$.

2. The existence of a solution

Lemma 1. *Problem* (1) - (4) *is equivalent to the following problem for a nonlinear parabolic equation*

$$\frac{\partial}{\partial t}\left(u + a_0|u|^{p-2}u\right) - \Delta u + a(x,t,u,\nabla u) = |u|^{p-2}u + F(t,u)\omega(x), \quad x \in \Omega, \quad 0 < t < T, \tag{8}$$

$$u(x,0) = u_0(x) \ \ in \ \ \Omega, \ u|_{\partial\Omega\times(0,T)} = 0. \tag{9}$$

here

$$F(t,u) = \varphi'(t) + \int_\Omega \nabla u \nabla \omega dx + \int_\Omega a(x,t,u,\nabla u)\cdot\omega dx - \int_\Omega |u|^{p-2}u\cdot\omega dx. \tag{10}$$

Proof. Indeed, from (1) it follows,

$$\begin{aligned}&\textstyle\int_\Omega \frac{\partial}{\partial t}\left(u + a_0|u|^{p-2}u\right)\omega dx - \int_\Omega \Delta u\cdot\omega dx + \int_\Omega a(x,t,u,\nabla u)\omega dx = \\ &= \textstyle\int_\Omega |u|^{p-2}u\cdot\omega dx + f(t)\int_\Omega \omega^2(x)dx,\end{aligned} \tag{11}$$

then if condition (4) and (5) is fulfilled, then

$$F(t,u) = \varphi'(t) - \int_\Omega u\Delta\omega dx + \int_\Omega a(x,t,u,\nabla u)\cdot\omega dx - \int_\Omega |u|^{p-2}u\cdot\omega dx. \tag{12}$$

Therefore, relation (10) issatisfied.
Now, we consider the problem (8)-(9). If relation (10) is satisfied, then equality (12) follows from it in an obvious way.

Then

$$\begin{aligned}&F(t,u) = \varphi'(t) - \textstyle\int_\Omega u\Delta\omega dx + \int_\Omega a(x,t,u,\nabla u)\cdot\omega dx - \int_\Omega |u|^{p-2}u\cdot\omega dx = \\ &= \varphi'(t) - \textstyle\int_\Omega \Delta u\omega dx + \int_\Omega a(x,t,u,\nabla u)\cdot\omega dx - \int_\Omega |u|^{p-2}u\cdot\omega dx.\end{aligned}$$

By virtue of (11)we obtain that

$$\varphi'(t) - \int_\Omega u_t\cdot\omega dx = 0.$$

In this way, $\frac{d}{dt}\left(\varphi(t) - \int_\Omega u\cdot\omega dx\right) = 0$. Denote by $v(t) = \varphi(t) - \int_\Omega u\cdot\omega dx$. Then the function $v(t)$ can be found as a solution to the Cauchy problem: $v'(t) = 0$, $v(0) = 0$. ($v(0) = 0$ follows from the agreement condition (6)). The unique solution to the problem is the function$v(t) = 0$, therefore,$\int_\Omega u \cdot \omega dx = \varphi(t)$.The lemma is proved.

Denote by $W(Q_T)$ space

$$W(Q_T) = \{u: \ u \in L_2(Q_T) \cap L_\infty(0,T;L_2(\Omega)) \cap L_\infty(0,T;L_p(\Omega)), \ \nabla u \in L_2(Q_T)\}$$

and conjugated to him by $W'(Q_T)$.

Definition 1. *The weak generalized solution of problem* (8)-(9) *is the function* $u(x,t) \in W(Q_T)$, $u_t(x,t) \in L_2(0,T;L_2(\Omega))$, $\left(|u|^{\frac{p}{2}}\right)_t' \in L_2(Q_T)$, *satisfying the equality*

$$\begin{aligned}&\int_{Q_T}\left(u_t' + a_0(p-1)\tfrac{2}{p}|u|^{\frac{p}{2}-2}u\left(|u|^{\frac{p}{2}}\right)_t'\right)\eta dxdt+\\ &+\int_{Q_T}\nabla u \nabla \eta dxdt + \int_{Q_T} a(x,t,u,\nabla u)\eta dxdt-\\ &-\int_{Q_T}|u|^{p-2}u\eta dxdt - \int_{Q_T} F(t,u)\omega(x)\eta dxdt = 0,\end{aligned}$$

for all $\eta(x,t) \in W(Q_T)$, $\eta_t(x,t) \in W'(Q_T)$.

Theorem 2. *Let conditions (5) - (7) be fulfilled, then the problem (8), (9), and therefore, the problem (1) - (4) have the weak solution on the interval* $[0, T_{max}) \subset [0,T)$.

We take in space $\overset{0}{W}{}_2^1(\Omega)$ a basis of eigenfunctions $\{\psi^k(x)\}$ of the operatorΔ :

$$\Delta\psi^k = \lambda_k \psi^k, \quad \psi^k\big|_\Gamma = 0, \tag{13}$$

and we will assume it is normalized in $L_2(\Omega)$.

We will build "approximate solutions"$u^N(t) = u^N(x,t)$, $N = 1,2,...$, in the form of a finite sum

$$u^N(x,t) = \sum_{i=1}^{N} C_i^N(t)\psi_i(x) \tag{14}$$

with unknown coefficients $C_i^N(t)$, $k = 1,2,...,N$, $t \in [0,T]$. These coefficients are determined by solving the Cauchy problem

$$\Lambda(C_i^N)\frac{dC_i^N}{dt} = G_i(t, C_1^N, C_2^N, ..., C_N^N), \ \ C_i^N(0) = (u_0(x), \psi_i(x))_{L_2(\Omega)}; \tag{15}$$

here

$$\begin{aligned}&\Lambda(C_i^N) = \left\|a_{kj}(C_i^N)\right\|_{k,j=1}^N,\\ &a_{kj}(C_i^N) = \delta_{kj} + (p-1)\int_\Omega |u^N|^{p-2}\psi_k\psi_j dx,\\ &\delta_{kj} = \begin{cases}1, & k=j,\\ 0, & k \neq j,\end{cases}\\ &G_i(t, C_1^N, C_2^N, ..., C_N^N) = -\lambda_k C_i^N + \int_\Omega |u^N|^{p-2}u^N\psi_i dx + F(t,u^N)\int_\Omega \omega(x)\psi_i dx.\end{aligned}$$

The Cauchy problem (16), by virtue of the Peano theorem, has a solution from the class $C_i^N(t) \in C^1[0,T_N]$ for some $T_N > 0$ and $i = 1,...,N$.

Multiply both sides of equality (15) by$C_i^N(t)$and sum by$i = 1,...,N$, we get equality

$$\frac{d}{dt}\left(\frac{1}{2}\left\|u^N\right\|_{2,\Omega}^2+a_0\frac{p-1}{p}\left\|u^N\right\|_{p,\Omega}^p\right)+\left\|\nabla u^N\right\|_{2,\Omega}^2+$$
$$+\int_\Omega a(x,t,u^N,\nabla u^N)u^N dx=\left\|u^N\right\|_{p,\Omega}^p+F(t,u^N)\int_\Omega u^N\omega dx,$$

From this equality, taking into account the conditions on the functions (5)-(7), we obtain the following inequality

$$\frac{d}{dt}\left(\frac{1}{2}\left\|u^N\right\|_{2,\Omega}^2+a_0\frac{p-1}{p}\left\|u^N\right\|_{p,\Omega}^p\right)+\left\|\nabla u^N\right\|_{2,\Omega}^2\le\left\|u^N\right\|_{p,\Omega}^p+a_1\left\|u^N\right\|_{2,\Omega}\left\|\nabla u^N\right\|_{2,\Omega}+$$
$$+a_2\left\|u^N\right\|_{p,\Omega}^{\frac{p}{2}}\left\|u^N\right\|_{2,\Omega}+|\varphi'|\left\|\omega\right\|_{2,\Omega}\left\|u^N\right\|_{2,\Omega}+$$
$$+\left\|\nabla\omega\right\|_{2,\Omega}\left\|\omega\right\|_{2,\Omega}\left\|u^N\right\|_{2,\Omega}\left\|\nabla u^N\right\|_{2,\Omega}+a_1\left\|\omega\right\|_{2,\Omega}^2\left\|u^N\right\|_{2,\Omega}\left\|\nabla u^N\right\|_{2,\Omega}+$$
$$+a_2\left\|\omega\right\|_{2,\Omega}^2\left\|u^N\right\|_{p,\Omega}^{\frac{p}{2}}\left\|u^N\right\|_{2,\Omega}+\left\|u^N\right\|_{p,\Omega}^p\left\|\omega\right\|_{p,\Omega}\left\|\omega\right\|_{\frac{p}{p-1},\Omega}\le$$
$$\le\left(3+\left\|\omega\right\|_{p,\Omega}\left\|\omega\right\|_{\frac{p}{p-1},\Omega}\right)\left\|u^N\right\|_{p,\Omega}^p+\frac{1}{4}|\varphi'|^2\left\|\omega\right\|_{2,\Omega}^2+\left(2a_1^2+\frac{a_2^2}{4}+1\right)\left\|u^N\right\|_{2,\Omega}^2+$$
$$+\frac{1}{2}\left(\left\|\nabla\omega\right\|_{2,\Omega}\left\|\omega\right\|_{2,\Omega}+a_1\left\|\omega\right\|_{2,\Omega}^2\right)^2\left\|u^N\right\|_{2,\Omega}^2+\frac{1}{2}\left\|\nabla u^N\right\|_{2,\Omega}^2.$$

From this inequality follows

$$\frac{d}{dt}\left(\left\|u^N\right\|_{2,\Omega}^2+2a_0\frac{p-1}{p}\left\|u^N\right\|_{p,\Omega}^p\right)+\left\|\nabla u^N\right\|_{2,\Omega}^2\le$$
$$\le C_1\left(\left\|u^N\right\|_{2,\Omega}^2+2a_0\frac{p-1}{p}\left\|u^N\right\|_{p,\Omega}^p\right)+\frac{1}{4}|\varphi'|^2\left\|\omega\right\|_{2,\Omega}^2.$$

Applying the Gronwall lemma we get

$$\left\|u^N\right\|_{2,\Omega}^2+2a_0\frac{p-1}{p}\left\|u^N\right\|_{p,\Omega}^p+\int_0^t\left\|\nabla u^N\right\|_{2,\Omega}^2d\tau\le C_2, \tag{16}$$

where constant C_2 is independent of m.

Now multiply both sides of (15) by $\frac{dC_i^N(t)}{dt}$ and sum by $i=1,...,N$, we obtain equality

$$\int_\Omega|u_t^N|^2dx+a_0(p-1)\int_\Omega|u^N|^{p-2}|u_t^N|^2dx+\frac{1}{2}\frac{d}{dt}\left\|\nabla u^N\right\|_{2,\Omega}^2+$$
$$+\int_\Omega a(x,t,u^N,\nabla u^N)\cdot u_t^N dx=$$
$$=\frac{1}{p}\cdot\frac{d}{dt}\left\|u^N\right\|_{p,\Omega}^p+F(t,u^N)\int_\Omega\omega(x)u_t^N dx.$$

Integrating this inequality with respect to time, we obtain the equality

$$\int_{Q_t}|u_\tau^N|^2dxd\tau+a_0(p-1)\int_{Q_t}|u^N|^{p-2}|u_\tau^N|^2dxd\tau+\frac{1}{2}\left\|\nabla u^N\right\|_{2,\Omega}^2+$$
$$+\int_{Q_t}a(x,t,u^N,\nabla u^N)\cdot u_\tau^N dxd\tau=$$
$$=\frac{1}{p}\cdot\left\|u^N\right\|_{p,\Omega}^p+\int_0^t F(\tau,u^N)\int_\Omega\omega(x)u_\tau^N dxd\tau.$$

Estimating the right side of the last equality, we get

$\int_{Q_t} |u_\tau^N|^2 dxd\tau + a_0(p-1)\int_{Q_t} |u^N|^{p-2}|u_\tau^N|^2 dxd\tau + \frac{1}{2}\left\|\nabla u^N\right\|_{2,\Omega}^2 \leq$
$\leq \frac{1}{p}\cdot\left\|u^N\right\|_{p,\Omega}^p + a_1\left\|u_\tau^N\right\|_{2,Q_t}\left\|\nabla u^N\right\|_{2,Q_t} +$
$+a_2\left\|u^N\right\|_{p,Q_t}^{\frac{p}{2}}\left\|u_\tau^N\right\|_{2,Q_t} + \int_0^t |\varphi'|\left\|\omega\right\|_{2,\Omega}\left\|u_\tau^N\right\|_{2,\Omega} d\tau +$
$+\int_0^t \left\|\nabla\omega\right\|_{2,\Omega}\left\|\omega\right\|_{2,\Omega}\left\|u_\tau^N\right\|_{2,\Omega}\left\|\nabla u^N\right\|_{2,\Omega} d\tau + a_1\int_0^t \left\|\omega\right\|_{2,\Omega}^2\left\|u_\tau^N\right\|_{2,\Omega}\left\|\nabla u^N\right\|_{2,\Omega} d\tau +$
$+a_2\int_0^t \left\|\omega\right\|_{2,\Omega}^2\left\|u^N\right\|_{p,\Omega}^{\frac{p}{2}}\left\|u_\tau^N\right\|_{2,\Omega} d\tau + \int_0^t \left\|u^N\right\|_{p,\Omega}^{p-1}\left\|u_\tau^N\right\|_{2,\Omega}\left\|\omega\right\|_{p,\Omega}\left\|\omega\right\|_{2,\Omega} d\tau \leq$
$\leq \frac{1}{p}\left\|u^N\right\|_{p,\Omega}^p + \frac{1}{2}\left\|u_\tau^N\right\|_{2,Q_t}^2 + \left(3a_1^2 + 3t\left\|\nabla\omega\right\|_{2,\Omega}^2\left\|\omega\right\|_{2,\Omega}^2 + 3a_1^2 t\left\|\omega\right\|_{2,\Omega}^4\right)\int_0^t \left\|\nabla u^N\right\|_{2,\Omega}^2 d\tau +$
$+3\left\|\omega\right\|_{2,\Omega}^2\int_0^t |\varphi'|^2 d\tau + \left(3a_2^2 + 3a_2^2 t\left\|\omega\right\|_{2,\Omega}^4 + \left\|\omega\right\|_{p,\Omega}^2\left\|\omega\right\|_{2,\Omega}^2\right)\int_0^t \left\|u^N\right\|_{p,\Omega}^{2p-2} d\tau$

This implies

$$\int_{Q_T} |u_t^N|^2 dxdt + \int_{Q_T} |u^N|^{p-2}|u_t^N|^2 dxdt + \left\|\nabla u^N\right\|_{2,\Omega}^2 \leq C_3(T), \tag{17}$$

where constant $C_3(T)$ is independent of m. The obtained estimates (17), (17) allowustocarry out to the limit transition [7] at $m\to\infty$ and prove the existence of a weak solution of problem (8), (9) in the sense of definition 1.

3. Blow-up results

Theorem 3. *Suppose that conditions (5)-(7) are satisfied, and also*

$$-\gamma_2\alpha^{-1}\left(\frac{\sigma}{2p} + C_8\right) < \int_\Omega |u_0|^2 dx + \frac{2a_0(p-1)}{p}\int_\Omega |u_0|^p dx \tag{18}$$

where $C_8 > 0$. Then for a finite time t_1 the solution of the norm tends to infinity $\int_0^t\int_\Omega |u|^2 dxd\tau + \frac{2a_0(p-1)}{p}\int_0^t\int_\Omega |u|^p dxd\tau + \frac{\sigma}{2p} + C_8 \to +\infty$ *at $t\to t_1$.*

Proof. We make a replacement $u = v\cdot e^{\lambda t}$, then the inverse problem (1) - (4) is reduced to the form

(19)
$$\frac{\partial v}{\partial t} + a_0 e^{\lambda(p-2)t}\frac{\partial}{\partial t}(|v|^{p-2}v) + \lambda v + a_0\lambda(p-1)e^{\lambda(p-2)t}|v|^{p-2}v - \Delta v +$$
$$+e^{-\lambda t}a(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v) = e^{\lambda(p-2)t}|v|^{p-2}v + e^{-\lambda t}f(t)\omega(x),\ x\in\Omega,\ 0<t<T,$$

$$v(x,0) = u_0(x), \tag{20}$$

$$v|_{\partial\Omega\times(0,T)} = 0, \tag{21}$$

$$\int_\Omega \left(v + a_0 e^{\lambda(p-2)t}|v|^{p-2}v\right)\omega\, dx = e^{-\lambda t}\varphi(t),\ \ 0<t<T. \tag{22}$$

At first we multiply equation (19) by $\omega(x)$ and integrate over the domain Ω, we obtain

$$\begin{aligned} &e^{-\lambda t}f(t) = e^{-\lambda t}\varphi'(t) - \int_\Omega v\cdot\Delta\omega dx + \\ &+\int_\Omega e^{-\lambda t}a(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v)\cdot\omega dx - e^{\lambda(p-2)t}\int_\Omega |v|^{p-2}v\cdot\omega dx. \end{aligned} \tag{23}$$

Now, multiplying equation (19) by v and $\frac{\partial v}{\partial t}$ and integrating over Ω, then we obtain the following identities

$$\begin{aligned}(24)\quad &\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2dx+\frac{a_0(p-1)}{p}\frac{d}{dt}\left(e^{\lambda(p-2)t}\int_\Omega |v|^pdx\right)-\\ &-\frac{a_0\lambda(p-1)(p-2)}{p}e^{\lambda(p-2)t}\int_\Omega |v|^pdx+a_0\lambda(p-1)e^{\lambda(p-2)t}\int_\Omega |v|^pdx+\\ &+\lambda\int_\Omega |v|^2dx+\int_\Omega |\nabla v|^2dx+e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot vdx=\\ &=e^{\lambda(p-2)t}\int_\Omega |v|^pdx+e^{-\lambda t}f(t)\int_\Omega v\omega dx.\end{aligned}$$

$$\begin{aligned}(25)\quad &\int_\Omega |v_t|^2dx+a_0(p-1)e^{\lambda(p-2)t}\int_\Omega |v|^{p-2}|v_t|^2dx+\frac{\lambda}{2}\frac{d}{dt}\int_\Omega |v|^2dx+\\ &+\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2dx+\frac{a_0\lambda(p-1)}{p}\frac{d}{dt}\left(e^{\lambda(p-2)t}\int_\Omega |v|^pdx\right)-\\ &+\frac{a_0\lambda^2(p-1)(p-2)}{p}e^{\lambda(p-2)t}\int_\Omega |v|^pdx+e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot v_tdx=\\ &=\frac{1}{p}\frac{d}{dt}\left(e^{\lambda(p-2)t}\int_\Omega |v|^pdx\right)-\frac{\lambda(p-2)}{p}e^{\lambda(p-2)t}\int_\Omega |v|^pdx+e^{-\lambda t}f(t)\int_\Omega v_t\omega dx.\end{aligned}$$

Substituting equation (23) in the identities (24) and (25) we get the relation

$$\begin{aligned}(26)\quad &\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2dx+\frac{a_0(p-1)}{p}\frac{d}{dt}\left(e^{\lambda(p-2)t}\int_\Omega |v|^pdx\right)+\\ &+\frac{2a_0\lambda(p-1)}{p}e^{\lambda(p-2)t}\int_\Omega |v|^pdx+\\ &+\lambda\int_\Omega |v|^2dx+\int_\Omega |\nabla v|^2dx+e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot vdx=\\ &=e^{\lambda(p-2)t}\int_\Omega |v|^pdx+\Big(e^{-\lambda t}\varphi'(t)-\int_\Omega v\cdot\Delta\omega dx+\\ &+\int_\Omega e^{-\lambda t}a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot\omega dx-e^{\lambda(p-2)t}\int_\Omega |v|^{p-2}v\cdot\omega dx\Big)\int_\Omega v\omega dx.\end{aligned}$$

$$\begin{aligned}(27)\quad &\int_\Omega |v_t|^2dx+a_0(p-1)e^{\lambda(p-2)t}\int_\Omega |v|^{p-2}|v_t|^2dx+\frac{\lambda}{2}\frac{d}{dt}\int_\Omega |v|^2dx+\\ &+\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2dx+\frac{a_0\lambda(p-1)}{p}\frac{d}{dt}\left(e^{\lambda(p-2)t}\int_\Omega |v|^pdx\right)-\\ &+\frac{a_0\lambda^2(p-1)(p-2)}{p}e^{\lambda(p-2)t}\int_\Omega |v|^pdx+e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot v_tdx=\\ &=\frac{1}{p}\frac{d}{dt}\left(e^{\lambda(p-2)t}\int_\Omega |v|^pdx\right)-\frac{\lambda(p-2)}{p}e^{\lambda(p-2)t}\int_\Omega |v|^pdx+\\ &+\Big(e^{-\lambda t}\varphi'(t)-\int_\Omega v\cdot\Delta\omega dx+\\ &+\int_\Omega e^{-\lambda t}a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot\omega dx-e^{\lambda(p-2)t}\int_\Omega |v|^{p-2}v\cdot\omega dx\Big)\int_\Omega v_t\omega dx.\end{aligned}$$

The following notations are introduced :

$$\begin{aligned}(28)\quad &G(t,v)\equiv e^{\lambda(p-2)t}\int_\Omega |v|^pdx\,,\\ &Lv\equiv\lambda\int_\Omega |v|^2dx+\int_\Omega |\nabla v|^2dx+\frac{2a_0\lambda(p-1)}{p}e^{\lambda(p-2)t}\int_\Omega |v|^pdx\,,\\ &J(t)\equiv\frac{1-a_0\lambda(p-1)}{p}G(t,v)-\frac{\lambda}{2}\int_\Omega |v|^2dx-\frac{1}{2}\int_\Omega |\nabla v|^2dx,\\ &\Psi_1(t)\equiv\int_\Omega |v|^2dx+\frac{2a_0(p-1)}{p}e^{\lambda(p-2)t}\int_\Omega |v|^pdx.\end{aligned}$$

We write equations (26) and (27) by using upper notations

$$\begin{aligned}(29)\quad &\frac{1}{2}\frac{d\Psi_1(t)}{dt}=2pJ(t)+\frac{2a_0\lambda(p-1)^2-p}{p}G(t,v)+\lambda(p-1)\int_\Omega |v|^2dx+\\ &+(p-1)\int_\Omega |\nabla v|^2dx-e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot vdx+\\ &+\Big(e^{-\lambda t}\varphi'(t)-\int_\Omega v\cdot\Delta\omega dx+\\ &+\int_\Omega e^{-\lambda t}a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot\omega dx-e^{\lambda(p-2)t}\int_\Omega |v|^{p-2}v\cdot\omega dx\Big)\int_\Omega v\omega dx.\end{aligned}$$

$$\begin{aligned}(30)\quad &\frac{dJ(t,v)}{dt}=\frac{2\lambda(p-2)(1+a_0\lambda(p-1))}{p}G(t,v)+\\ &+\int_\Omega |v_t|^2dx+a_0(p-1)e^{\lambda(p-2)t}\int_\Omega |v|^{p-2}|v_t|^2dx+\\ &+e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot v_tdx-\\ &-\Big(e^{-\lambda t}\varphi'(t)-\int_\Omega v\cdot\Delta\omega dx+\\ &+\int_\Omega e^{-\lambda t}a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot\omega dx-e^{\lambda(p-2)t}\int_\Omega |v|^{p-2}v\cdot\omega dx\Big)\int_\Omega v_t\omega dx\end{aligned}$$

Estimate the right –hand side of the identity (29):

$$\begin{aligned}(31)\qquad &\left|\int_\Omega v\cdot\Delta\omega dx\int_\Omega v\omega dx\right|\le\\ &\le\left(\int_\Omega|v|^2dx\right)^{\frac12}\left(\int_\Omega|\Delta\omega|^2dx\right)^{\frac12}\left(\int_\Omega|v|^2dx\right)^{\frac12}\left(\int_\Omega|\omega|^2dx\right)^{\frac12}\le\\ &\le C_1\int_\Omega|v|^2dx.\end{aligned}$$

$$\begin{aligned}(32)\qquad &\left|e^{-\lambda t}\varphi'(t)\int_\Omega v\omega dx\right|\le\\ &\le N_2e^{-\lambda t}\left(\int_\Omega|v|^2dx\right)^{\frac12}\left(\int_\Omega|\omega|^2dx\right)^{\frac12}\le\\ &\le\lambda(p-1)\int_\Omega|v|^2dx+\tfrac{N_2^2e^{-2\lambda t}}{4\lambda(p-1)}\|\omega\|_{2,\Omega}^2.\end{aligned}$$

(33)
$$\begin{aligned}&\left|e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot\omega dx\int_\Omega v\omega dx\right|\le a_1\int_\Omega|\nabla v|\cdot|\omega|dx\int_\Omega|v||\omega|dx+\\ &+a_2e^{\frac{\lambda(p-2)t}{2}}\int_\Omega|v|^{\frac p2}\cdot|\omega|dx\int_\Omega|v||\omega|dx\le a_1\|\nabla v\|_{2,\Omega}\|v\|_{2,\Omega}\|\omega\|_{2,\Omega}^2+\\ &+a_2e^{\frac{\lambda(p-2)t}{2}}\|v\|_{p,\Omega}^{\frac p2}\|v\|_{2,\Omega}\|\omega\|_{2,\Omega}^2\le\tfrac{p-1}{2}\|\nabla v\|_{2,\Omega}^2+\tfrac{a_1^2}{2(p-1)}\|\omega\|_{2,\Omega}^4\|v\|_{2,\Omega}^2+\\ &+\tfrac{2a_0\lambda(p-1)^2-p}{2p}e^{\lambda(p-2)t}\|v\|_{p,\Omega}^p+\tfrac{pa_2^2}{4a_0\lambda(p-1)^2-2p}\|\omega\|_{2,\Omega}^4\|v\|_{2,\Omega}^2.\end{aligned}$$

(34)
$$\begin{aligned}&\left|e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot vdx\right|\le a_1\int_\Omega|\nabla v||v|dx+\\ &+a_2e^{\frac{\lambda(p-2)t}{2}}\int_\Omega|v|^{\frac p2}|v|dx\le a_1\|\nabla v\|_{2,\Omega}\|v\|_{2,\Omega}+a_2e^{\frac{\lambda(p-2)t}{2}}\|v\|_{p,\Omega}^{\frac p2}\|v\|_{2,\Omega}\le\\ &\le\tfrac{p-1}{2}\|\nabla v\|_{2,\Omega}^2+\tfrac{a_1^2}{2(p-1)}\|v\|_{2,\Omega}^2+\tfrac{2a_0\lambda(p-1)^2-p}{2p}e^{\lambda(p-2)t}\|v\|_{p,\Omega}^p+\tfrac{pa_2^2}{4a_0\lambda(p-1)^2-2p}\|v\|_{2,\Omega}^2.\end{aligned}$$

$$\begin{aligned}(35)\qquad &\left|e^{\lambda(p-2)t}\int_\Omega|v|^{p-2}v\omega dx\int_\Omega v\omega dx\right|\le\\ &\le e^{\lambda(p-2)t}\int_\Omega|v|^{p-1}|\omega|dx\int_\Omega|v||\omega|dx\le\\ &\le e^{\lambda(p-2)t}\|v\|_{p,\Omega}^p\|\omega\|_{p,\Omega}\|\omega\|_{\frac{p}{p-1},\Omega}=C_2e^{\lambda(p-2)t}\|v\|_{p,\Omega}^p.\end{aligned}$$

We substitute the obtained estimates (31) - (35) in (29), then we get

$$(36)\qquad \frac12\frac{d\Psi_1(t)}{dt}\ge 2pJ(t)-C_3\Psi_1(t)-\frac{N_2^2}{4\lambda(p-1)}\|\omega\|_{2,\Omega}^2e^{-2\lambda t},$$

where

$$(37)\qquad C_3=\max\left\{C_1+\frac{a_1^2(1+\|\omega\|_{2,\Omega}^4)}{2(p-1)}+\frac{pa_2^2(1+\|\omega\|_{2,\Omega}^4)}{4a_0\lambda(p-1)^2-p};\ C_2\right\}.$$

Now, estimating the right-hand side of (30):

$$\begin{aligned}(38)\qquad &\left|\int_\Omega v\cdot\Delta\omega dx\int_\Omega v_t\omega dx\right|\le\\ &\le\left(\int_\Omega|v|^2dx\right)^{\frac12}\left(\int_\Omega|\Delta\omega|^2dx\right)^{\frac12}\left(\int_\Omega|v_t|^2dx\right)^{\frac12}\left(\int_\Omega|\omega|^2dx\right)^{\frac12}\le\\ &\le C_4(\varepsilon_1)\int_\Omega|v|^2dx+\varepsilon_1\int_\Omega|v_t|^2dx.\end{aligned}$$

$$\begin{aligned}
&\left|e^{-\lambda t}\varphi'(t)\int_\Omega v_t\omega dx\right| \leq \\
&\leq N_2 e^{-\lambda t}\left(\int_\Omega |v_t|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\omega|^2 dx\right)^{\frac{1}{2}} \leq \\
&\leq \varepsilon_1 \int_\Omega |v_t|^2 dx + \frac{N_2^2 e^{-2\lambda t}}{4\varepsilon_1}\|\omega\|^2_{2,\Omega}.
\end{aligned} \tag{39}$$

$$\begin{aligned}
&\left|e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v)\cdot\omega dx\int_\Omega v_t\omega dx\right| \leq \\
&\leq a_1\int_\Omega |\nabla v|\cdot|\omega|dx\int_\Omega |v_t||\omega|dx+ \\
&+a_2 e^{\frac{\lambda(p-2)t}{2}}\int_\Omega |v|^{\frac{p}{2}}\cdot|\omega|dx\int_\Omega |v_t||\omega|dx \leq a_1\|\nabla v\|_{2,\Omega}\|v_t\|_{2,\Omega}\|\omega\|^2_{2,\Omega}+ \\
&+a_2 e^{\frac{\lambda(p-2)t}{2}}\|v\|^{\frac{p}{2}}_{p,\Omega}\|v_t\|_{2,\Omega}\|\omega\|^2_{2,\Omega} \leq \frac{a_1^2}{4\varepsilon_1}\|\omega\|^4_{2,\Omega}\|\nabla v\|^2_{2,\Omega}+\varepsilon_1\|v_t\|^2_{2,\Omega}+ \\
&+\frac{a_2^2}{4\varepsilon_1}\|\omega\|^4_{2,\Omega}e^{\lambda(p-2)t}\|v\|^p_{p,\Omega}+\varepsilon_1\|v_t\|^2_{2,\Omega}.
\end{aligned} \tag{40}$$

$$\begin{aligned}
&\left|e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v)\cdot v_t dx\right| \leq \\
&\leq a_1\int_\Omega |\nabla v||v_t|dx + a_2 e^{\frac{\lambda(p-2)t}{2}}\int_\Omega |v|^{\frac{p}{2}}|v_t|dx \leq \\
&\leq a_1\|\nabla v\|_{2,\Omega}\|v_t\|_{2,\Omega}+a_2 e^{\frac{\lambda(p-2)t}{2}}\|v\|^{\frac{p}{2}}_{p,\Omega}\|v_t\|_{2,\Omega} \leq \\
&\leq \frac{a_1^2}{4\varepsilon_1}\|\nabla v\|^2_{2,\Omega}+\varepsilon_1\|v_t\|^2_{2,\Omega}+\frac{a_2^2}{4\varepsilon_1}e^{\lambda(p-2)t}\|v\|^p_{p,\Omega}+\varepsilon_1\|v_t\|^2_{2,\Omega}.
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\left|e^{\lambda(p-2)t}\int_\Omega |v|^{p-2}v\cdot\omega dx\int_\Omega v_t\omega dx\right| \leq \frac{1}{a_0}e^{-\lambda t}|\varphi(t)|\int_\Omega |v_t||\omega|dx+ \\
&+\frac{1}{a_0}\int_\Omega |v||\omega|dx\int_\Omega |v_t||\omega|dx \leq \\
&\leq \frac{N_1}{a_0}e^{-\lambda t}\|v_t\|_{2,\Omega}\|\omega\|_{2,\Omega}+\frac{1}{a_0}\|v_t\|_{2,\Omega}\|v\|_{2,\Omega}\|\omega\|^2_{2,\Omega} \leq \\
&\leq \varepsilon_1\|v_t\|^2_{2,\Omega}+\frac{N_1^2}{a_0^2}\|\omega\|^2_{2,\Omega}e^{-2\lambda t}+\varepsilon_1\|v_t\|^2_{2,\Omega}+\frac{1}{a_0^2}\|\omega\|^4_{2,\Omega}\|v\|^2_{2,\Omega}.
\end{aligned} \tag{42}$$

The obtained estimates (38)-(42) substitute into the identity (30) and we obtain

$$\begin{aligned}
&\frac{dJ(t,v)}{dt} \geq \left(\frac{2\lambda(p-2)(1+a_0\lambda(p-1))}{p}-\frac{a_2^2}{4\varepsilon_1}\|\omega\|^4_{2,\Omega}-\frac{a_2^2}{4\varepsilon_1}\right)G(t,v)+ \\
&+(1-8\varepsilon_1)\int_\Omega |v_t|^2 dx + a_0(p-1)e^{\lambda(p-2)t}\int_\Omega |v|^{p-2}|v_t|^2 dx- \\
&-\left(\frac{a_1^2}{4\varepsilon_1}+\frac{a_1^2}{4\varepsilon_1}\|\omega\|^4_{2,\Omega}\right)\|\nabla v\|^2_{2,\Omega}-\left(C_4(\varepsilon_1)+\frac{1}{a_0^2}\|\omega\|^4_{2,\Omega}\right)\|v\|^2_{2,\Omega}- \\
&-\left(\frac{N_2^2}{4\varepsilon_1}\|\omega\|^2_{2,\Omega}+\frac{N_1^2}{a_0^2}\|\omega\|^2_{2,\Omega}\right)e^{-2\lambda t}.
\end{aligned} \tag{43}$$

Integrating by time the obtained inequality (43) from 0 to t and taking expression $1-e^{-2\lambda t}$ does not exceed 1, we get

(44)

$$\begin{aligned}
&J(t) \geq J(0)+(1-8\varepsilon_1)\int_0^t\int_\Omega |v_\tau|^2 dxd\tau + a_0(p-1)\int_0^t\int_\Omega e^{\lambda(p-2)\tau}|v|^{p-2}|v_\tau|^2 dxd\tau+ \\
&+\left(\frac{2\lambda(p-2)(1+a_0\lambda(p-1))}{p}-\frac{a_2^2}{4\varepsilon_1}\|\omega\|^4_{2,\Omega}-\frac{a_2^2}{4\varepsilon_1}\right)\int_0^t G(\tau,v)d\tau- \\
&-\left(\frac{a_1^2}{4\varepsilon_1}+\frac{a_1^2}{4\varepsilon_1}\|\omega\|^4_{2,\Omega}\right)\int_0^t\int_\Omega |\nabla v|^2 dxd\tau-\left(C_4(\varepsilon_1)+\frac{1}{a_0^2}\|\omega\|^4_{2,\Omega}\right)\int_0^t\|v\|^2_{2,\Omega}d\tau - D_1,
\end{aligned}$$

where

$$D_1 = \frac{N_2^2}{8\lambda\varepsilon_1}\|\omega\|^2_{2,\Omega}+\frac{N_1^2}{2\lambda a_0^2}\|\omega\|^2_{2,\Omega}. \tag{45}$$

Rewriting the identity (24) using the equation (28) as follows:

$$\begin{aligned}(46)\quad &\tfrac{1}{2}\tfrac{d\Psi_1(t)}{dt}+Lv=G(t,v)-e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot vdx+\\ &+\left(e^{-\lambda t}\varphi'(t)-\int_\Omega v\cdot\Delta\omega dx+\right.\\ &\left.+\int_\Omega e^{-\lambda t}a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot\omega dx-e^{\lambda(p-2)t}\int_\Omega|v|^{p-2}v\cdot\omega dx\right)\int_\Omega v\omega dx\end{aligned}$$

We estimate the right-hand side (46):

$$\begin{aligned}(47)\quad &\left|e^{-\lambda(p-1)t}\varphi(t)\int_\Omega v\cdot\Delta\omega dx\right|\le\\ &\le N_1e^{-\lambda(p-1)t}\left(\int_\Omega|v|^2dx\right)^{\frac12}\left(\int_\Omega|\Delta\omega|^2dx\right)^{\frac12}\le\\ &\le\tfrac{\lambda p}{4}e^{-\lambda(p-2)t}\int_\Omega|v|^2dx+\tfrac{N_1^2}{\lambda p}e^{-\lambda pt}\|\Delta\omega\|^2.\end{aligned}$$

$$\begin{aligned}(48)\quad &\left|e^{-\lambda t}\varphi'(t)\int_\Omega v\omega dx\right|\le\\ &\le N_2e^{-\lambda t}\left(\int_\Omega|v|^2dx\right)^{\frac12}\left(\int_\Omega|\omega|^2dx\right)^{\frac12}\le\\ &\le\lambda\int_\Omega|v|^2dx+\tfrac{N_2^2e^{-2\lambda t}}{4\lambda}\|\omega\|_{2,\Omega}^2.\end{aligned}$$

$$\begin{aligned}(49)\quad &\left|e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot\omega dx\int_\Omega v\omega dx\right|\le\\ &\le a_1\int_\Omega|\nabla v|\cdot|\omega|dx\int_\Omega|v||\omega|dx+\\ &+a_2e^{\frac{\lambda(p-2)t}{2}}\int_\Omega|v|^{\frac p2}\cdot|\omega|dx\int_\Omega|v||\omega|dx\le a_1\|\nabla v\|_{2,\Omega}\|v\|_{2,\Omega}\|\omega\|_{2,\Omega}^2+\\ &+a_2e^{\frac{\lambda(p-2)t}{2}}\|v\|_{p,\Omega}^{\frac p2}\|v\|_{2,\Omega}\|\omega\|_{2,\Omega}^2\le\tfrac14\|\nabla v\|_{2,\Omega}^2+a_1^2\|\omega\|_{2,\Omega}^4\|v\|_{2,\Omega}^2+\\ &+e^{\lambda(p-2)t}\|v\|_{p,\Omega}^p+\tfrac{a_2^2}{4}\|\omega\|_{2,\Omega}^4\|v\|_{2,\Omega}^2.\end{aligned}$$

$$\begin{aligned}(50)\quad &\left|e^{-\lambda t}\int_\Omega a(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)\cdot vdx\right|\le a_1\int_\Omega|\nabla v||v|dx+\\ &+a_2e^{\frac{\lambda(p-2)t}{2}}\int_\Omega|v|^{\frac p2}|v|dx\le a_1\|\nabla v\|_{2,\Omega}\|v\|_{2,\Omega}+a_2e^{\frac{\lambda(p-2)t}{2}}\|v\|_{p,\Omega}^{\frac p2}\|v\|_{2,\Omega}\le\\ &\le\tfrac14\|\nabla v\|_{2,\Omega}^2+a_1^2\|v\|_{2,\Omega}^2+\tfrac{4a_0\lambda(p-1)}{p}e^{\lambda(p-2)t}\|v\|_{p,\Omega}^p+\tfrac{a_2^2p}{16a_0\lambda(p-1)}\|v\|_{2,\Omega}^2.\end{aligned}$$

$$\begin{aligned}(51)\quad &\left|e^{\lambda(p-2)t}\int_\Omega|v|^{p-2}v\cdot\omega dx\int_\Omega v\omega dx\right|\le\tfrac{1}{a_0}e^{-\lambda t}|\varphi(t)|\int_\Omega|v||\omega|dx+\\ &+\tfrac{1}{a_0}\int_\Omega|v||\omega|dx\int_\Omega|v||\omega|dx\le\\ &\le\tfrac{N_1}{a_0}e^{-\lambda t}\|v\|_{2,\Omega}\|\omega\|_{2,\Omega}+\tfrac{1}{a_0}\|v\|_{2,\Omega}^2\|\omega\|_{2,\Omega}^2\le\\ &\le\left(1+\tfrac{1}{a_0}\|\omega\|_{2,\Omega}^2\right)\|v\|_{2,\Omega}^2+\tfrac{N_1^2}{4a_0^2}\|\omega\|_{2,\Omega}^2e^{-2\lambda t}.\end{aligned}$$

Applying the estimates (47) - (51) to the identity (24), we obtain:

$$\begin{aligned}&\tfrac12\tfrac{d\Psi_1(t)}{dt}+e^{-\lambda(p-2)t}\int_\Omega|\nabla v|^2dx+\tfrac{\lambda p}{2}e^{-\lambda(p-2)t}\int_\Omega|v|^2dx+\\ &+a_0\lambda(p-1)\int_\Omega|v|^pdx\le G(t,v)+\tfrac14e^{-\lambda(p-2)t}\int_\Omega|\nabla v|^2dx+\\ &+\left(a_1^2+a_2\right)e^{-\lambda(p-2)t}\int_\Omega|v|^2dx+N_1N_2e^{-\lambda pt}+\\ &+\tfrac14e^{-\lambda(p-2)t}\int_\Omega|\nabla v|^2dx+(a_1N_1)^2e^{-\lambda pt}\|\omega\|^2+\tfrac{\lambda p}{4}e^{-\lambda(p-2)t}\int_\Omega|v|^2dx+\\ &+\tfrac{(a_2N_1)^2}{\lambda p}e^{-\lambda pt}\|\omega\|^2+\left(a_0\lambda(p-1)+\tfrac{2a_0(p-1)(a_1^2+a_2)}{p}\right)\int_\Omega|v|^pdx+\\ &+C_6e^{-\lambda pt}\int_\Omega|\omega|^pdx+\tfrac{\lambda p}{4}e^{-\lambda(p-2)t}\int_\Omega|v|^2dx+\tfrac{N_1^2}{\lambda p}e^{-\lambda pt}\|\Delta\omega\|^2+\\ &+G(t,v)+C_7e^{-\lambda pt}\int_\Omega|\omega|^{q+2}dx.\end{aligned}$$

We strengthen the last inequality

$$\begin{aligned}(52)\quad &\tfrac12\tfrac{d\Psi_1(t)}{dt}+\tfrac12\int_\Omega|\nabla v|^2dx\le\\ &\le 2G(t,v)+C_4\Psi_1(t)+\left(\tfrac{N_1^2}{4a_0^2}\|\omega\|_{2,\Omega}^2+\tfrac{N_2^2}{4\lambda}\|\omega\|_{2,\Omega}^2\right)e^{-2\lambda t}.\end{aligned}$$

We integrate the inequality (52) with respect to time from 0 to t and obtain

(53) $$\frac{1}{2}\Psi_1(t)+\frac{1}{2}\int_0^t\int_\Omega|\nabla v|^2dxd\tau\leq\frac{1}{2}\Psi_1(0)+2\int_0^t G(\tau,v)d\tau+ \\ +C_4\int_0^t\Psi_1(\tau)d\tau+D_2,$$

here

(54) $$D_2=\frac{N_1^2}{8\lambda a_0^2}\left\|\omega\right\|_{2,\Omega}^2+\frac{N_2^2}{8\lambda^2}\left\|\omega\right\|_{2,\Omega}^2.$$

Multiplying the inequality (53) by 2, we obtain the following estimate

(55) $$\int_0^t\int_\Omega|\nabla v|^2dxd\tau\leq\Psi_1(0)+4\int_0^t G(\tau,v)d\tau+2C_4\int_0^t\Psi_1(\tau)d\tau+2D_2,$$

Applying the estimate (55) to the inequality (44), we obtain:

(56)
$$J(t)\geq(1-8\varepsilon_1)\int_0^t\int_\Omega\left|v_\tau\right|^2dxd\tau+a_0(p-1)\int_0^t\int_\Omega e^{\lambda(p-2)\tau}|v|^{p-2}\left|v_\tau\right|^2dxd\tau+ \\ +\left(\frac{2\lambda(p-2)(1+a_0\lambda(p-1))}{p}-\frac{a_2^2}{4\varepsilon_1}\left\|\omega\right\|_{2,\Omega}^4-\frac{a_2^2}{4\varepsilon_1}-4\left(\frac{a_1^2}{4\varepsilon_1}+\frac{a_1^2}{4\varepsilon_1}\left\|\omega\right\|_{2,\Omega}^4\right)\right)\int_0^t G(\tau,v)d\tau- \\ -2C_4\left(\frac{a_1^2}{4\varepsilon_1}+\frac{a_1^2}{4\varepsilon_1}\left\|\omega\right\|_{2,\Omega}^4\right)\int_0^t\Psi_1(\tau)d\tau-\left(C_4(\varepsilon_1)+\frac{1}{a_0^2}\left\|\omega\right\|_{2,\Omega}^4\right)\int_0^t\left\|v\right\|_{2,\Omega}^2d\tau+D_3,$$

where

(57)
$$D_3=J(0)-\left(\frac{a_1^2}{4\varepsilon_1}+\frac{a_1^2}{4\varepsilon_1}\left\|\omega\right\|_{2,\Omega}^4\right)(\Psi_1(0)+2D_2)-D_1= \\ =J(0)-D_1-\left(\frac{a_1^2}{4\varepsilon_1}+\frac{a_1^2}{4\varepsilon_1}\left\|\omega\right\|_{2,\Omega}^4\right)\left(\int_\Omega|u_0|^2dx+\frac{2a_0(p-1)}{p}\int_\Omega|u_0|^pdx+2D_2\right).$$

If the inequality is satisfied

(58) $$C_5\equiv\frac{2\lambda(p-2)(1+a_0\lambda(p-1))}{p}-\frac{a_2^2}{4\varepsilon_1}\left\|\omega\right\|_{2,\Omega}^4- \\ -\frac{a_2^2}{4\varepsilon_1}-4\left(\frac{a_1^2}{4\varepsilon_1}+\frac{a_1^2}{4\varepsilon_1}\left\|\omega\right\|_{2,\Omega}^4\right)\geq 0,$$

then

(59)
$$J(t)\geq(1-8\varepsilon_1)\int_0^t\int_\Omega\left|v_\tau\right|^2dxd\tau+a_0(p-1)\int_0^t\int_\Omega e^{\lambda(p-2)\tau}|v|^{p-2}\left|v_\tau\right|^2dxd\tau+ \\ +C_5\int_0^t G(\tau,v)d\tau-C_6\int_0^t\Psi_1(\tau)d\tau+D_3,$$

where

(60) $$C_6=2C_4\left(\frac{a_1^2}{4\varepsilon_1}+\frac{a_1^2}{4\varepsilon_1}\left\|\omega\right\|_{2,\Omega}^4\right)+C_4(\varepsilon_1)+\frac{1}{a_0^2}\left\|\omega\right\|_{2,\Omega}^4.$$

In the inequality (36) we use the obtained estimate (59), then

(61)
$$\frac{1}{2}\frac{d\Psi_1(t)}{dt}\geq 2pJ(t)-C_3\Psi_1(t)-\frac{N_2^2}{4\lambda(p-1)}\left\|\omega\right\|_{2,\Omega}^2e^{-2\lambda t}\geq \\ \geq 2(1-8\varepsilon_1)p\int_0^t\int_\Omega\left|v_\tau\right|^2dxd\tau+2a_0p(p-1)\int_0^t\int_\Omega e^{\lambda(p-2)\tau}|v|^{p-2}\left|v_\tau\right|^2dxd\tau+ \\ +2C_5p\int_0^t G(\tau,v)d\tau-2C_6p\int_0^t\Psi_1(\tau)d\tau+2D_3p-C_3\Psi_1(t)-\frac{N_2^2}{2}\left\|\omega\right\|_{2,\Omega}^2e^{-2\lambda t},$$

We denote by

(62)
$$\Psi_2(t) = \int_0^t \int_\Omega |v|^2 dxd\tau + \frac{2a_0(p-1)}{p} \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^p dxd\tau + \frac{\sigma}{2p} + C_8,$$
$$J_2(t) = \int_0^t \int_\Omega |v_\tau|^2 dxd\tau + a_0(p-1) \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^{p-2} |v_\tau|^2 dxd\tau + C_5 \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^p dxd\tau + \sigma,$$

where $C_8 > 0$, $\sigma = \int_\Omega |u_0|^2 dx + \frac{2a_0(p-1)}{p} \int_\Omega |u_0|^p dx$.

Inequality (61) can be written through $\Psi_2(t)$

(63)
$$\Psi_2''(t) \geq 4p(1-8\varepsilon_1) \int_0^t \int_\Omega |v_\tau|^2 dxd\tau + 4a_0 p(p-1)(1-8\varepsilon_1) \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^{p-2} |v_\tau|^2 dxd\tau +$$
$$+4C_5 p(1-8\varepsilon_1) \int_0^t G(\tau, v) d\tau - 4C_6 p \Psi_2(t) + 4D_3 p - 2C_3 \Psi_2'(t) -$$
$$-\frac{N_2^2}{2\lambda(p-1)} \|\omega\|_{2,\Omega}^2 e^{-2\lambda t} + 2C_6 p \left(\frac{\sigma}{2p} + C_8\right).$$

Now, estimating the expression $[\Psi_2'(t)]^2$, we apply the Cauchy-Bunyakovskii inequality, we get

$$[\Psi_2'(t)]^2 \leq \Big(2 \int_0^t \int_\Omega v v_\tau dxd\tau + 2a_0(p-1) \int_0^t \int_\Omega e^{\frac{\lambda(p-2)\tau}{2}} |v|^{\frac{p-2}{2}} v_\tau e^{\frac{\lambda(p-2)\tau}{2}} |v|^{\frac{p}{2}} dxd\tau +$$
$$+\lambda(p-2) \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^p dxd\tau + \sigma\Big)^2 \leq$$
$$\leq \Big(2 \|v\|_{2,Q_t} \|v_\tau\|_{2,Q_t} + 2a_0(p-1) \left(\int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^{p-2} |v_\tau|^2 dxd\tau\right)^{\frac{1}{2}} \left(\int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^p dxd\tau\right)^{\frac{1}{2}} +$$
$$+\lambda(p-2) \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^p dxd\tau + \sigma\Big)^2 \leq$$
$$\leq \left(4 \int_0^t \int_\Omega |v|^2 dxd\tau + 4a_0(p-1) \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^p dxd\tau + \sigma\right) \times$$
$$\times \Big(\int_0^t \int_\Omega |v_\tau|^2 dxd\tau + a_0(p-1) \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^{p-2} |v_\tau|^2 dxd\tau +$$
$$+\lambda(p-2) \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^p dxd\tau + \sigma\Big) \leq$$
$$\leq 2p \left(\frac{2}{p} \int_0^t \int_\Omega |v|^2 dxd\tau + \frac{2a_0(p-1)}{p} \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^p dxd\tau + \frac{\sigma}{2p}\right) \times$$
$$\times \Big(\int_0^t \int_\Omega |v_\tau|^2 dxd\tau + a_0(p-1) \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^{p-2} |v_\tau|^2 dxd\tau +$$
$$+C_5 \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^p dxd\tau + \sigma\Big) \leq$$
$$\leq 2p \left(\int_0^t \int_\Omega |v|^2 dxd\tau + \frac{2a_0(p-1)}{p} \int_0^t \int_\Omega e^{\lambda(p-2)\tau} |v|^p dxd\tau + \frac{\sigma}{2p} + C_8\right) J_2(t) =$$
$$= 2p \Psi_2(t) J_2(t),$$

If substitute $\varepsilon_1 = \frac{1}{32}$ in the ratio (63) can be written the form

$$\begin{aligned} &\Psi_2''(t)\Psi_2(t) - \tfrac{3}{2}[\Psi_2'(t)]^2 \geq -4C_6 p \Psi_2^2(t) - \\ &-2C_3 \Psi_2'(t)\Psi_2(t) + \left(4C_6 p \left(\tfrac{\sigma}{2p} + C_8\right) + 2C_3 \left(\tfrac{\sigma}{2p} + C_8\right) + \right. \\ &\left. +4pD_3 - \tfrac{N_2^2}{2\lambda(p-1)} \|\omega\|_{2,\Omega}^2 e^{-2\lambda t} + 2C_6 p \left(\tfrac{\sigma}{2p} + C_8\right)\right) \Psi_2(t). \end{aligned} \tag{64}$$

It is necessary to choose $C_8 > 0$ in such a way that the following inequality is satisfied

$$\Big(4C_6 p \left(\frac{\sigma}{2p} + C_8\right) + 2C_3 \left(\frac{\sigma}{2p} + C_8\right) +$$
$$+4pD_3 - \frac{N_2^2}{2\lambda(p-1)} \|\omega\|_{2,\Omega}^2 e^{-2\lambda t} + 2C_6 p \left(\frac{\sigma}{2p} + C_8\right)\Big) > 0 \quad .$$

that is

$$C_8 \geq \frac{\frac{N_2^2}{2\lambda(p-1)} \|\omega\|_{2,\Omega}^2 - (4pC_6 + 2C_3)\frac{\sigma}{2p} - 4pD_3 - C_6\sigma}{6pC_6 + 2C_3}$$

then the inequality (64) reduces to the form

$$\Psi_2''(t)\Psi_2(t) - (1+\alpha)[\Psi_2'(t)]^2 \geq -2\tilde{C}_1\Psi_2'(t)\Psi_2(t) - 2\tilde{C}_2\Psi_2^2(t), \tag{65}$$

where $\alpha = \frac{1}{2}$, $\tilde{C}_1 = C_3$, $\tilde{C}_2 = 2C_6$.

From the condition of the lemma $\Psi_2'(0) + \gamma_2\alpha^{-1}\Psi_2(0) > 0$

$$-\gamma_2\alpha^{-1}\left(\frac{\sigma}{2p} + C_8\right) < \int_\Omega |u_0|^2 dx + \frac{2a_0(p-1)}{p}\int_\Omega |u_0|^p dx$$

then the function $\Psi_2(t)$ tends to infinity for $t \to t_1$. Thus, the solution of inverse problem blows up in finite time. Thereby proved our theorem.

4. Stability of the solution

Consider the following problem

$$\frac{\partial}{\partial t}\left(u + a_0|u|^{p-2}u\right) - \Delta u + |u|^{p-2}u = f(t)\omega(x), \;\; x \in \Omega, \;\; t > 0, \tag{66}$$

$$u(x,0) = u_0(x), \tag{67}$$

$$u|_{\partial\Omega\times(0,\infty)} = 0, \tag{68}$$

$$\int_\Omega u \cdot \omega \, dx = \varphi(t), \;\; t > 0. \tag{69}$$

Here $\Omega \subset R^n$, $n \geq 1$ a bounded domain, the boundary $\partial\Omega$ is smooth enough, p positive constant, which are $p > 2$. Functions $\omega(x)$, u_0 satisfies the conditions (5) and (6). $\varphi(t)$ and $\varphi'(t)$ the non-negative functions satisfy the conditions

$$\varphi(t),\, \varphi'(t) \in L_1(0,\infty),\;\; \varphi(t) \text{ and } \varphi'(t) \text{ wcich tend to 0 as } t \to \infty. \tag{70}$$

Theorem 4. *Let the conditions (5), (6) and (70) be fulfilled, then $t \to \infty$ the norm of the solution of the inverse problem (66)-(69) tends to 0, that is*

$$\lim_{t\to\infty}\left(\|\nabla u\|^2 + \|u\|_{p,\Omega}^p\right) = 0. \tag{71}$$

Proof. The proof of this theorem is also applied by the method [19]. Multiply the equation (66) $\omega(x)$ and integrate over the domain Ω, we obtain

$$f(t) = \varphi'(t) + \int_\Omega \nabla u \cdot \nabla\omega dx + \int_\Omega |u|^{p-2}u \cdot \omega dx. \tag{72}$$

Substituting the ratio (72) in (66), then we get the equation

$$\begin{aligned} &\tfrac{\partial}{\partial t}\left(u + a_0|u|^{p-2}u\right) - \Delta u + |u|^{p-2}u = \\ &= \left(\varphi'(t) + \textstyle\int_\Omega \nabla u \cdot \nabla\omega dx + \int_\Omega |u|^{p-2}u \cdot \omega dx\right)\omega(x), \;\; x \in \Omega, \;\; t > 0, \end{aligned} \tag{73}$$

Now multiply (73) u and and integrate over the domain Ω, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\left\|u\right\|^2 + a_0\frac{p-1}{p}\left\|u\right\|^p_{p,\Omega}\right) + \left\|\nabla u\right\|^2 + \left\|u\right\|^p_{p,\Omega} =$$
$$= \left(\varphi'(t) + \int_\Omega \nabla u\cdot\nabla\omega dx + \int_\Omega |u|^{p-2}u\cdot\omega dx\right)\int_\Omega u\omega dx,$$

Estimating the right part, we find

$$\frac{d}{dt}\left(\frac{1}{2}\left\|u\right\|^2_{2,\Omega} + a_0\frac{p-1}{p}\left\|u\right\|^p_{p,\Omega}\right) + \frac{1}{2}\left\|\nabla u\right\|^2_{2,\Omega} + \frac{1}{2}\left\|u\right\|^p_{p,\Omega} \le$$
$$\le |\varphi'(t)\varphi(t)| + \frac{1}{2}\left\|\nabla\omega\right\|^2_{2,\Omega}|\varphi(t)|^2 + \frac{1}{2}|\varphi(t)|^2\int_\Omega |\omega|^{\frac{p}{p-1}}dx.$$

We apply the Poincare - Friedrichs inequality $\left\|\nabla u\right\|^2 \ge \lambda_1^2\left\|u\right\|^2$, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\left\|u\right\|^2_{2,\Omega} + a_0\frac{p-1}{p}\left\|u\right\|^p_{p,\Omega}\right) + C_3\left(\frac{1}{2}\left\|u\right\|^2_{2,\Omega} + a_0\frac{p-1}{p}\left\|u\right\|^p_{p,\Omega}\right) \le$$
$$\le |\varphi'(t)\varphi(t)| + \frac{1}{2}\left\|\nabla\omega\right\|^2_{2,\Omega}|\varphi(t)|^2 + \frac{1}{2}|\varphi(t)|^2\int_\Omega |\omega|^{\frac{p}{p-1}}dx. \qquad (74)$$

where $C_3 = \min\{\lambda_1,\, 1\}$ and λ_1 is a first eigen value of the Laplace operator under the homogeneous Dirichlet's condition. Обозначим через:

$$E(t) \equiv \frac{1}{2}\left\|u\right\|^2_{2,\Omega} + a_0\frac{p-1}{p}\left\|u\right\|^p_{p,\Omega},$$
$$\Phi(t) \equiv |\varphi'(t)\varphi(t)| + \frac{1}{2}\left\|\nabla\omega\right\|^2_{2,\Omega}|\varphi(t)|^2 + \frac{1}{2}|\varphi(t)|^2\int_\Omega |\omega|^{\frac{p}{p-1}}dx.$$

Then the inequality (74) is reduced to the form

$$\frac{d}{dt}E(t) + C_3E(t) \le \Phi(t). \qquad (75)$$

Consider the case when $\varphi(t) \le C\exp(-\mu t),\ \ \mu > C_3$.

In the inequality (75), applying the well-known Gronwall lemma, and under condition (5), (6) and (70) it follows

$$E(t) \le \exp(-C_3t)\left(E(0) + \frac{C}{\mu-C_3}\right),\ \ \mu > C_3,$$
$$E(t) \le \exp(-C_3t)\left(E(0) + Ct\right),\ \ \mu = C_3.$$

Thus proved that the solution decreases according to the exponential law.

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