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ON HOLOMORPHIC SELF-MAPPINGS OF THE UNIT DISK

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Abstract. Using a symmetrization technique, we prove two distortion theorems for holomorphic mappings of the unit disk into itself taking into account the boundary behavior of these mappings.

Keywords: holomorphic functions, distortion theorems, symmetrization of condensers.

1. Introduction

The study of holomorphic mappings of a disk into itself constitutes a significant part of geometric function theory (see, e.g., the papers [1,2] and the bibliography in them). We are interested in the distortion theorems for such mappings that take into account their boundary behavior [3]. The first result of this kind belongs, apparently, to Unkelbach [4]. Let a function $w = f(z)$ be holomorphic in the disk $U = \{z : |z| < 1\}$ and satisfy the conditions: $f(0) = 0$ and $|f(z)| < 1$ when $z \in U$. If E is an arc of the circle $|z| = 1$ such that the set of the limit values of the function f with respect to E belongs to the circle $|w| = 1$, then this set is also an arc $f(E)$ of the circle $|w| = 1$. According to [4], the lengths of these two arcs satisfy the inequality

$$\text{length}(f(E)) \geq \frac{2}{1 + |f'(0)|}\text{length}(E). \tag{1}$$

If at a point $b \in E$ the derivative $f'(b)$ exists, then by the passage to the limit in (1) we obtain the boundary Schwarz Lemma:

$$|f'(b)| \geq \frac{2}{1 + |f'(0)|}.$$

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The above inequality is often called the inequality of Osserman. Other versions of the Schwarz inequality on the boundary, as well as analogues of inequality (1), can be found in [5-7]. Some of these results can be used in the proofs of the inequalities for the complex polynomials [8]. In this note we complement the theorems from the articles [6, 9] taking into account the boundary distortion. The main tool employed in the proofs is Pólya's circular symmetrization (see [10, Sec. 4.1]). The following section is auxiliary.

2. Preliminaries

We will apply the capacity approach and symmetrization of condensers. The notions from the book [10] will be tacitly used below. In this paper, for the most part, we will consider the condensers with two plates of the form

$$C = (U(R), \{E_0, E_1\}, \{0, 1\}) \equiv (U(R), E_0, E_1),$$

where $U(R) = \{z : \ |z| < R\}, \ 0 < R \leq 1$, and $E_0, \ E_1$ are closed disjoint nonempty sets $E_0 \subset \overline{U}(R), \ E_1 \subset U(R)$. The *capacity* $\operatorname{cap} C$ of C is defined as the infimum of the Dirichlet integrals

$$I(v, U(R)) := \int_{U(R)} |\nabla v|^2$$

over all *admissible* functions v, that is, real functions v which are continuous in $\overline{U}(R)$, Lipschitz on compact subsets of $U(R)$ and equal to k on $E_k, \ k = 0, 1$. If the class of admissible functions is reduced to the subclass of functions $v, \ 0 \leq v \leq 1$, equal to k in a neighbourhood of the plate $E_k, \ k = 0, 1$, then the capacity of a condenser C does not change [10, Lemma 1.2].

Denote by $\gamma(\rho)$ the circle $|z| = \rho, \ 0 \leq \rho \leq 1$. Let B be an open subset of $\overline{U}(R)$. The *circular symmetrization* (with respect to the positive real half-axis) assigns to a set B the "circularly symmetric" set

$$\operatorname{Cr} B = \{\rho e^{i\theta} : \ B \cap \gamma(\rho) \neq \emptyset, \ 2|\theta|\rho < \operatorname{meas}(B \cap \gamma(\rho))\} \cup \{-\rho : \ \gamma(\rho) \subset B\},$$

where $\operatorname{meas}(\cdot)$ is the Lebesgue linear measure. In a similar way, we define the symmetrization of a closed subset E of $\overline{U}(R)$ as follows:

$$\operatorname{Cr} E = \{\rho e^{i\theta} : \ E \cap \gamma(\rho) \neq \emptyset, \ 2|\theta|\rho \leq \operatorname{meas}(E \cap \gamma(\rho))\}.$$

For a condenser $C = (E_0, E_1)$ we set

$$\operatorname{Cr} C = (\overline{U}(R) \setminus \operatorname{Cr}(\overline{U}(R) \setminus E_0), \operatorname{Cr} E_1).$$

The following statement goes back to Pólya's [11] (cf. [10, Theorem 4.2]).

Lemma 1. *With the above notation, we have*

$$\operatorname{cap} C \geq \operatorname{cap} \operatorname{Cr} C.$$

The *inner radius* of an open set $B \subset \mathbb{C}$ with respect to a point $z_0 \in B$ is the quantity

$$r(B, z_0) = \exp\{\lim_{z \to z_0} [g_B(z, z_0) + \log |z - z_0|]\},$$

where $g_B(z, z_0)$ is the Green function of the connected component of B containing the point z_0. Let B be a domain in $\mathbb{C}$ bounded by finitely many piecewise analytic curves, Γ be a nonempty closed subset of ∂B consisting of finitely many nonsingular arcs such that $(\partial B) \setminus \Gamma$ is union of open smooth arcs, and z_0 be a point in B. Then there exists a function $g_B(z, z_0, \Gamma)$ which is continuous on $\overline{B} \setminus \{z_0\}$, harmonic in $B \setminus \{z_0\}$, and satisfying the following conditins:

$$g_B(z, z_0, \Gamma) = 0 \quad for \quad z \in \Gamma;$$

$$\frac{\partial}{\partial n} g_B(z, z_0, \Gamma) = 0 \quad for \quad z \in (\partial B) \setminus \Gamma;$$

$g_B(z, z_0, \Gamma) + \log|z - z_0|$ is a bounded harmonic function in a neighbourhood of z_0 ($\partial/\partial n$ denotes differentiation along the inward normal to ∂B). The function $g_B(z, z_0, \Gamma)$ is called the *Robin function* of the domain B and the set Γ with pole at z_0. By the *Robin radius* of a domain B with respect to a point z_0 and a set Γ, we mean the quantity

$$r(B, \Gamma, z_0) = \exp\{\lim_{z \to z_0} [g_B(z, z_0, \Gamma) + \log|z - z_0|]\}.$$

Note that if a, b and c are real numbers, $a < b < c$, $B_1 = \mathbb{C} \setminus \{z : \ \mathrm{Im} z = 0, \ \mathrm{Re} z \geq b\}$, $B_2 = \mathbb{C} \setminus \{z : \ \mathrm{Im} z = 0, \ \mathrm{Re} z \geq c\}$, then

$$r(B_1, [c, +\infty], a) = r(B_2, a) = 4(c - a). \tag{2}$$

Theorem 2.2 from the book [10] gives

Lemma 2. *Let B be a domain, set $\Gamma \subset \partial B$, $z_0 \in B$ as above, and let $B \subset U$, $(\partial B) \setminus \Gamma \subset \partial U$. Then*

$$\mathrm{cap}\,(U, \Gamma, \{z : \ |z - z_0| \leq \varphi(r)\}) = -\frac{2\pi}{\log r} - 2\pi \left[\log \frac{r(B, \Gamma, z_0)}{\mu}\right] \left(\frac{1}{\log r}\right)^2 + $$
$$+ o\left(\left(\frac{1}{\log r}\right)^2\right), \ r \to 0,$$

where $\varphi(r)$ is any real function of the form $\varphi(r) = \mu r(1 + o(1))$, $r \to 0$ and $\mu > 0$.

The behaviour of the Robin radius under conformal map is described in the following statement.

Lemma 3. *Let the domains B, G and the sets $\Gamma \subset \partial B$, $\Lambda \subset \partial G$ be as in definition of the Robin radius, and let f be a function mapping B conformally and univalently onto G so that $f(\Gamma) = \Lambda$. Then*

$$r(G, \Lambda, f(z_0)) = |f'(z_0)| r(B, \Gamma, z_0)$$

for any point $z_0 \in B$.

This lemma is a special case of the majorization principle [7, Theorem 4.2].

3. Distortion theorems

We will start with an analogue of the assertions [6, Theorem 2] and [9, Theorem 1]. Set

$$E(\alpha) = \{z = e^{i\theta} : \ |\theta| < \alpha\}, \ 0 \leq \alpha < \pi.$$

Theorem 1. *Let f be a holomorphic function in the disk U, $f(U) \subset U$, and let $f(E(\alpha)) \subset E(\beta)$ for some α and β, $0 \leq \alpha < \pi$, $0 \leq \beta < \pi$*[1] *(i.e. for each sequence of points $z_n \in U$, approaching the set $E(\alpha)$, the corresponding sequence $f(z_n) \to$*

[1] If $\alpha = 0$ ($\beta = 0$) then the boundary condition is ignored.

$E(\beta)$.) *Suppose that* $\gamma(\rho) \not\subset f(U)$ *for all* ρ, $\tau \le \rho \le 1$, *and some fixed* τ, $0 \le \tau < 1$. *Then for any real points* z_1, z_2 *such that* $-1 < z_1 < z_2 < 1$ *and* $|f(z_1)| \ne |f(z_2)|$

$$\frac{[k(e^{i\beta}) - k(m)][k(M) - k(-\tau)]}{k(M) - k(m)} \ge \frac{[k(e^{i\alpha}) - k(z_1)][k(e^{i\beta}) - k(-\tau)]}{k(z_2) - k(z_1)}, \tag{3}$$

where $k(z) = z(1+z)^{-2}$ *is the Koebe function and* $m = \min\{|f(z_1)|, |f(z_2)|\}$, $M = \max\{|f(z_1)|, |f(z_2)|\}$. *Equality in* (3) *is attained for any points* $-1 < z_1 < z_2 < 1$ *and any conformal map* $f : U \to U \setminus [-1, -\tau]$ *such that* $f(-1) = -\tau$, $f(1) = 1$ *and* $0 \le f(z_1) < f(z_2) < 1$.

Proof. It sufficies to consider $\alpha \ne 0$, $\beta \ne 0$ and a nonconstant function f. Let $C = (U, (\partial U) \setminus E(\alpha), [z_1, z_2])$, and let v be an admissible function for the condenser C which is equal to 0 in a neighbourhood of $(\partial U)\setminus E(\alpha)$ and to 1 in a neighbourhood of $[z_1, z_2]$ and satisfies $0 \le v(z) \le 1$ for $z \in \overline{U}$. Let γ be a closed circular arc connecting the points $e^{\pm i\alpha}$ and lying in a neighbourhood of $(\partial U) \setminus E(\alpha)$, where $v = 0$. Denote by B the domain in U with the boundary $\partial B = \gamma \cup E(\alpha)$. Note that the set $\overline{f(B)}$ does not contain the boundary points on the arc $(\partial U) \setminus \overline{E(\beta)}$. Finally, let R be close to 1, such that $\tau < R < 1$ and $f([z_1, z_2]) \subset U(R)$. Let us inspect the following function on $\overline{U}(R)$:

$$u(w) = \begin{cases} \max\{v(z) : \ f(z) = w\}, & w \in \overline{U}(R) \cap f(B) \\ 0, & w \in \overline{U}(R) \setminus f(B). \end{cases}$$

The function f takes each value in $f(B)$ on a finite or countable infinite set of points in B, which accumulate at the boundary of B. Hence, the maximum in the definition of u is taken over finitely many values of v. It is easy to see that u is continuous in $\overline{U}(R)$ and Lipschitz in a neighbourhood of each point in $U(R)$, with a possible exception of finitely many points w such that $f(z) = w$ and $f'(z) = 0$. From this we conclude that

$$I(v, U) = I(v, B) \ge I(u, U(R)) \ge \operatorname{cap} C(R),$$

where

$$C(R) = (U(R), \overline{U}(R) \setminus f(B), f([z_1, z_2])).$$

By Lemma 1

$$\operatorname{cap} C(R) \ge \operatorname{cap} \operatorname{Cr} C(R).$$

Set

$$C^*(R) = (U(R), [-R, -\tau] \cup (\partial U(R)) \setminus \{w : \ w/R \in E(\beta(R))\}, [m, M]),$$

where $\beta(R)$ is defined by $2\beta(R)R =$meas$(f(B) \cap \gamma(R))$. In view of the hypotheses of Theorem 1 and monotonicity property of capacity [10, Theorem 1.15] we have

$$\operatorname{cap} \operatorname{Cr} C(R) \ge \operatorname{cap} C^*(R).$$

Thus

$$I(v, U) \ge \ \operatorname{cap} C^*(R).$$

Note that

$$\varlimsup_{R \to 1} \beta(R) \le \beta.$$

By passage to the limit we obtain

$$I(v, U) \ge \ \operatorname{cap} C^*,$$

$$C^* := (U, [-1, -\tau]) \cup (\partial U) \setminus E(\beta), [m, M]).$$

Hence,

$$\operatorname{cap} C \geq \operatorname{cap} C^*. \tag{4}$$

The function

$$\xi(z) = \frac{k(z) - k(z_1)}{k(z_2) - k(z_1)}$$

maps the condenser C onto the condenser $\xi(C)$ with two plates on the Riemann sphere [10, Sec. 1.2]:

$$\xi(C) = ([\xi(e^{i\alpha}), +\infty], [0, 1]).$$

The conformal invariance of capacity gives

$$\operatorname{cap} C = \operatorname{cap} \xi(C).$$

Similarly, the function

$$\eta(w) = \frac{(k(w) - k(m))(k(M) - k(-\tau))}{(k(w) - k(-\tau))(k(M) - k(m))}$$

maps the condenser C^* onto the condenser

$$\eta(C^*) = ([\eta(e^{i\beta}), +\infty], [0, 1]),$$

and

$$\operatorname{cap} C^* = \operatorname{cap} \eta(C^*).$$

From inequality (4) we have

$$\xi(e^{i\alpha}) \leq \eta(e^{i\beta}).$$

This yields the inequality in Theorem 1. If f is a conformal map, $f : U \to U \setminus [-1, -\tau]$ such that $f(-1) = -\tau,\ f(1) = 1$ and $0 \leq f(z_1) < f(z_2) < 1$ for some points $z_1,\ z_2,\ -1 < z_1 < z_2 < 1$, then equality in (4) holds. Hence, we have equality in (3). This completes the proof of Theorem 1. □

Let f be a holomorphic function in the disk $U,\ f(U) \subset U,\ f(0) = 0$. Taking in Theorem 1 $\alpha = \beta = 0,\ z_1 = 0$ and taking the limit as $z_2 \to 0,\ \tau \to 1$ from (3) we deduce the classical Schwarz's inequality

$$|f'(0)| \leq 1.$$

In a similar way, letting $z_1 = 0$, $z_2 = 1$ and $\tau \to 1$ we obtain the Löwner's inequality

$$\alpha \leq \beta.$$

If we set $\alpha = \beta = \tau = 0,\ 0 < z_1 < z_2 < 1,\ |f(z_1)| < |f(z_2)|$ in (3), then we obtain the following estimate:

$$\frac{(1 + z_2)^2 (1 - z_1)^2}{(z_2 - z_1)(1 - z_1 z_2)} \leq \frac{4|f(z_2)|(1 - |f(z_1)|)^2}{(|f(z_2)| - |f(z_1)|)(1 - |f(z_1) f(z_2)|)}$$

(cf. [9, Theorem 1]).

Theorem 2. *Under the hypotheses of Theorem* 1 *suppose that a point* $z \in (-1, 1)$. *Then*

$$\left| \frac{f'(z) k'(|f(z)|)}{k'(z)} \right| \leq \frac{[k(|f(z)|) - k(-\tau)][k(e^{i\beta}) - k(|f(z)|)]}{[k(e^{i\alpha}) - k(z)][k(e^{i\beta}) - k(-\tau)]}, \tag{5}$$

where $k(z) = z(1 + z)^{-2}$ *is the Koebe function. Equality holds in* (5) *for any point* $z \in (-1, 1)$ *and any conformal map* $f : U \to U \setminus [-1, -\tau]$ *such that* $f(-1) = -\tau$, $f(1) = 1$ *and* $0 \leq f(z) < 1$.

Proof. Let z_0 be a fixed point of the interval $(-1, 1)$, $f'(z_0) \neq 0$, and let

$$C = (U, (\partial U) \setminus E(\alpha), \{z : \; |z - z_0| \leq r\})$$

for sufficiently small $r > 0$. Repeating the proof of Theorem 1 for the condenser C we get inequality (4), where this time

$$C^* = (U, [-1, -\tau] \cup (\partial U) \setminus E(\beta), \{w : \; |w - |f(z_0)|| \leq \varphi(r)\}),$$

and $\varphi(r) = |f'(z_0)|r(1 + o(1))$, $r \to 0$. Using Lemma 2 from (4) we obtain

$$|f'(z_0)|r(U, \gamma, z_0) \leq r(U \setminus [-1, -\tau], \Gamma, |f(z_0)|), \tag{6}$$

where $\gamma = (\partial U) \setminus E(\alpha)$, $\Gamma = [-1, -\tau] \cup (\partial U) \setminus E(\beta)$.

The Koebe function $\xi = k(z)$ maps the disk U onto the domain $B_1 = \mathbb{C} \setminus [1/4, +\infty]$ so that $k(\gamma) = [k(e^{i\alpha}), +\infty]$. By Lemma 3 and equality (2)

$$|k'(z_0)|r(U, \gamma, z_0) = r(B_1, k(\gamma), k(z_0)) = r(B_2, k(z_0)) = 4[k(e^{i\alpha}) - k(z_0)],$$

$B_2 = \mathbb{C} \setminus [k(e^{i\alpha}), +\infty]$. Similarly, the function

$$\eta(w) = \frac{1}{k(w) - k(-\tau)}$$

maps the domain $U \setminus [-1, -\tau]$ conformally and univalently onto the domain $B_3 = \mathbb{C} \setminus [-\infty, \eta(1)]$, so that $\eta(\Gamma) = [-\infty, \eta(e^{i\beta})]$. By Lemma 3,

$$|\eta'(|f(z_0)|)|r(U \setminus [-1, -\tau], \Gamma, |f(z_0)|) = r(B_3, \eta(\Gamma), \eta(|f(z_0)|)) =$$

$$= r(B_4, \eta(|f(z_0)|)) = 4[\eta(|f(z_0)|) - \eta(e^{i\beta})],$$

where $B_4 = \mathbb{C} \setminus [-\infty, \eta(e^{i\beta})]$. Thus inequality (6) gives

$$\frac{|f'(z_0)\eta'(|f(z_0)|)|}{|k'(z_0)|} \leq \frac{\eta(|f(z_0)|) - \eta(e^{i\beta})}{k(e^{i\alpha}) - k(z_0)} \; .$$

This yields the inequality in Theorem 2 for $z = z_0$. If f is a conformal map, $f : U \to U \setminus [-1, -\tau]$, such that $f(-1) = -\tau$, $f(1) = 1$ and $0 \leq f(z_0) < 1$ for a point $z_0 \in (-1, 1)$, then by Lemma 3 the equality in (6) holds. Hence, we have equality in (5) for $z = z_0$. This completes the proof of Theorem 2. □

For a holomorphic self-mapping f of the unit disk U, $f(0) = 0$, the inequality (5) gives the classical Schwarz's inequality $|f'(0)| \leq 1$ again. If we set $z = 0$, $f(0) = 0$ and $\tau \to 1$, then we obtain the estimate

$$|f'(0)| \cos^2 \frac{\beta}{2} \leq \cos^2 \frac{\alpha}{2},$$

which is an implication of the Schwarz's inequality and the Lowner's inequality. Now let f be a holomorphic function from Theorem 2, $\alpha \neq 0$, $\beta \neq 0$, and suppose, in addition, that f has an angular limit $f(1) = 1$ and a finite angular derivative $f'(1)$. Then f' has a finite angular limit $f'(1)$ at $z = 1$ [3, Proposition 4.7]. Applying Theorem 2 to the function f and taking $\tau \to 1$, $z \to 1$ we arrive at the new inequality:

$$|f'(1)| \operatorname{tg} \frac{\alpha}{2} \leq \operatorname{tg} \frac{\beta}{2}.$$

Setting in Theorem 2 $\alpha = \beta = \tau = 0$ we obtain the inequality (11) from [9].

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