ELEMENTARY FORMULAS FOR KIRCHHOFF INDEX OF
MÖBIUS LADDER AND PRISM GRAPHS

G.A. BAIGONAKOVA, A.D. MEDNYKH

Abstract. Let \( G \) be a finite connected graph on \( n \) vertices with Laplacian spectrum \( 0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \). The Kirchhoff index of \( G \) is defined by the formula

\[
Kf(G) = n \sum_{j=2}^{n} \frac{1}{\lambda_j}
\]

The aim of this paper is to find an explicit analytical formula for the Kirchhoff index of Möbius ladder graph \( M_n = C_2 \times C_n \) and Prism graph \( Pr_n = C_n \times P_2 \). The obtained formulas provide a simple asymptotical behavior of both invariants as \( n \) is going to the infinity.

Keywords: Laplacian matrix, circulant graph, Kirchhoff index, Wiener index, Chebyshev polynomial.

1. Introduction

Let \( G \) be a finite connected graph on \( n \) vertices. Denote by \( D(G) \) the diagonal matrix formed by degrees of vertices and by \( A(G) \) the adjacency matrix of the graph \( G \). The matrix \( L(G) = D(G) - A(G) \) is called the Laplacian matrix of \( G \). It is well known [17] that \( L(G) \geq 0 \) and \( \det(L(G)) = 0 \). For a connected graph \( G \), all eigenvalues of \( L(G) \) except one are strictly positive. Hence, the Laplacian spectrum of \( G \) can be represented in the form \( 0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \). The Kirchhoff index of \( G \) was originally defined by Klein and Randić [12] as a new distance function named resistance distance framed in terms of electrical network theory. More precisely, let vertices of the graph \( G \) are labeled by \( 1, 2, \ldots, n \). Then...
the resistance distance between vertices \( i \) and \( j \), denoted by \( r_{ij} = r_{ij}(G) \) is defined to be the effective resistance between them when unit resistors are placed on every edge of \( G \). Following [12] define

\[
Kf(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij}
\]

to be the Kirchhoff index of \( G \). The motivation for such a definition was a famous Wiener index \( W(G) \), which counts the sum of distances between pairs of vertices in \( G \), that is

\[
W(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij},
\]

where \( d_{ij} \) is the distance between vertices \( i \) and \( j \). The Wiener index is named after Harry Wiener, who introduced it in [24]. See also [13] for more advance properties. Klein and Randić [12] proved that \( Kf(G) \leq W(G) \) with equality, if and only if \( G \) is a tree. Closed-form formulae for the Kirchhoff index have been given for some classes of graphs, such as cycles [12, 15], complete graphs [15], distance-transitive graphs [15], circulant graphs [25], ladder-like chains [4] and others [1, 14, 10, 18, 19]. Numerical values of the Kirchhoff index have been computed for platonic solids [15, 16] and some fullerene graphs [1, 2, 3, 8, 10]. In [26] the Kirchhoff index of join, corona and cluster of graphs have been investigated.

There is a nice relationship [9] between the Laplacian spectrum and the Kirchhoff index given by the formula

\[
(1) \quad Kf(G) = n \sum_{j=2}^{n} \frac{1}{\lambda_j}.
\]

2. **AN EXPLICIT FORMULA FOR THE KIRCHHOFF INDEX**

Consider the Möbius ladder \( M_n = C_{2n}(1, n) \) and the Prism graph \( Pr_n = C_n \times P_2 \). Here \( C_{2n}(1, n) \) is a 3-valent circulant graphs on \( 2n \) vertices \( v_1, v_2, \ldots, v_{2n} \) with edges \( v_i v_{i+1} \) and \( v_i v_{i+n}, i \mod 2n \), \( C_n \) is the cycle graph on \( n \) vertices and \( P_2 \) is the path graph on two vertices. For \( n = 6 \) the graphs are shown in Fig. 1 and Fig. 2 respectively. The main results of this paper are the following two theorems.

**Theorem 1.** The Kirchhoff index of Möbius ladder \( M_n = C_{2n}(1, n) \) is given by the formula

\[
Kf(M_n) = \frac{n^3 - n}{6} + \frac{n^2 \tanh \left( \frac{n}{2} \arccosh 2 \right)}{\sqrt{3}}.
\]

**Theorem 2.** The Kirchhoff index of Prism graph \( Pr_n = C_n \times P_2 \) is given by the formula

\[
Kf(Pr_n) = \frac{n^3 - n}{6} + \frac{n^2 \coth \left( \frac{n}{2} \arccosh 2 \right)}{\sqrt{3}}.
\]
Remark. By Besot’s theorem, the Kirchhoff indices in Theorem 1 and Theorem 2 are rational numbers. Their explicit values up to \( n = 30 \) are given in Table 1.

Proof of Theorems 1 and 2. Consider an order \( n \) polynomial \( P(w) = \prod_{k=1}^{n} (w - w_k) \). Then we have

\[
\sum_{k=1}^{n} \frac{1}{w - w_k} = \frac{P'(w)}{P(w)},
\]

where the both sides of the equation are considered as meromorphic functions of \( w \).

Let \( T_n(w) \) and \( U_{n-1}(w) \) be the Chebyshev polynomials of the first and the second kind respectively. We take \( P(w) = T_n(w + a) - 1 \). Then \( P'(w) = n U_{n-1}(w + a) \). Since \( T_n(w) = \cos(n \arccos w) \), all the roots of \( P(w) \) are given by the formulas \( w_k = -a + \cos \frac{2k\pi}{n}, \ n = 1, 2, \ldots, n \). Hence

\[
\sum_{k=1}^{n} \frac{1}{w + a - \cos \frac{2k\pi}{n}} = \frac{n U_{n-1}(w + a)}{T_n(w + a) - 1}.
\]

Taking \( w = 0 \) in (3) and dividing by 2 we obtain
Indeed, by (17), Theorem 3.5) the Laplacian eigenvalues of $Pr$ are given by the formulas $\mu_j = 3 - 2 \cos \frac{2k\pi}{n} = (-1)^j$, $j = 1, 2, \ldots, 2n - 1$. Hence, we have the following two elementary identities

\begin{equation}
\sum_{k=1}^{n} \frac{1}{2a - 2 \cos \frac{2k\pi}{n}} = \frac{n U_{n-1}(a)}{2(T_n(a) - 1)}.
\end{equation}

As a result, we have

\begin{equation}
\sum_{k=1}^{n} \frac{1}{4 - 2 \cos \frac{2k\pi}{n}} = \frac{n U_{n-1}(2)}{2(T_n(2) - 1)}
\end{equation}

and

\begin{equation}
\sum_{k=1}^{n-1} \frac{1}{2 - 2 \cos \frac{2k\pi}{n}} = \lim_{a \to 1} \left( \frac{n U_{n-1}(a)}{2(T_n(a) - 1)} - \frac{1}{2a - 2} \right) = \frac{n^2 - 1}{12}.
\end{equation}

Now, we are going to find the Kirchhoff index for Möbius ladder $M_n$. Recall that by ([7], p. 71) the non-zero Laplacian eigenvalues of $M_n$ are given by the formulas $\mu_j = 3 - 2 \cos \frac{2k\pi}{n} = (-1)^j$, $j = 1, 2, \ldots, 2n - 1$. Hence, we have

\begin{equation}
Kf(M_n) = 2n \sum_{j=1}^{2n-1} \mu_j = 2n \left( \sum_{k=1}^{n} \frac{1}{4 - 2 \cos \frac{2k\pi}{n}} + \sum_{k=1}^{n-1} \frac{1}{2 - 2 \cos \frac{2k\pi}{n}} \right)
\end{equation}

By making use of identities (5) and (6) we get

\begin{equation}
Kf(M_n) = 2n \left( \frac{2n U_{2n-1}(2)}{2(T_{2n}(2) - 1)} - \frac{n U_{n-1}(2)}{2(T_n(2) - 1)} + \frac{n^2 - 1}{12} \right).
\end{equation}

We set $\theta = \text{arccosh} 2$ and note that $T_n(2) = \cosh(n\theta)$, $U_{n-1}(2) = \frac{\sinh(n\theta)}{\sinh(\theta)}$. Observing that $\coth(n\theta) - \frac{1}{2} \coth(\frac{\theta}{2}) = \frac{1}{2} \tanh(\frac{\theta}{2})$, we have

\begin{equation}
\frac{2n U_{2n-1}(2)}{2(T_{2n}(2) - 1)} - \frac{n U_{n-1}(2)}{2(T_n(2) - 1)} = \frac{n \tanh(\frac{\theta}{2})}{2 \sinh(\theta)} = \frac{n \tanh(\frac{n}{2} \text{arccosh} 2)}{2\sqrt{3}}.
\end{equation}

Putting the last equation (8) into (7), we obtain the statement of Theorem 1:

\begin{equation}
Kf(M_n) = \frac{n^3 - n}{6} + \frac{n^2 \tanh(\frac{n}{2} \text{arccosh} 2)}{\sqrt{3}}.
\end{equation}

The Kirchhoff index for prism graph $Pr_n$ can be found by similar arguments. Indeed, by ([17], Theorem 3.5) the Laplacian eigenvalues of $Pr_n$ are given by the following list $4 - 2 \cos \frac{2k\pi}{n}$, $2 - 2 \cos \frac{2k\pi}{n}$, $k = 1, 2, \ldots, n$. Hence,

\begin{equation}
Kf(Pr_n) = 2n \left( \sum_{k=1}^{n} \frac{1}{4 - 2 \cos \frac{2k\pi}{n}} + \sum_{k=1}^{n-1} \frac{1}{2 - 2 \cos \frac{2k\pi}{n}} \right).
\end{equation}

Again, by (5) and (6) we have

\begin{equation}
Kf(Pr_n) = 2n \left( \frac{n U_{n-1}(2)}{2(T_n(2) - 1)} + \frac{n^2 - 1}{12} \right).
\end{equation}
Since
\[ \frac{U_{n-1}(2)}{2(T_n(2) - 1)} = \frac{\coth(\frac{\theta}{2})}{2\sinh(\theta)} = \frac{\coth(\frac{\theta}{2} \arccosh 2)}{2\sqrt{3}}, \]
the statement of Theorem 2 follows.

3. Asymptotical behavior and estimates

The asymptotic formulas for Kirchhoff index of the above graphs can be derived from the following theorem.

**Theorem 3.** Let \( K(n) = \frac{1}{6}(n^3 + 2\sqrt{3}n^2 - n) \). Then

(i) \( Kf(M_n) < K(n) < Kf(Pr_n) \);

(ii) \( 0 < Kf(Pr_n) - Kf(M_n) \leq \frac{2n^2}{\sqrt{3}} \).

**Proof.** The first statement of the theorem is a consequence of Theorems 1 and 2 and elementary inequalities \( \tanh(\frac{\theta}{2}) < 1 < \coth(\frac{\theta}{2}) \), where \( \theta = \arccosh 2 \). To prove the second statement we note that by virtue of Theorems 1 and 2,

\[ Kf(Pr_n) - Kf(M_n) = \frac{n^2(\coth(\frac{\theta}{2}) - \tanh(\frac{\theta}{2}))}{\sqrt{3}} = \frac{2n^2}{\sqrt{3}\sinh(n\theta)} \leq \frac{2n^2}{3^n}. \]

The latter inequality follows from the observation
\[ \sqrt{3}\sinh(n\theta) = \frac{\sqrt{3}}{2}((2 + \sqrt{3})^n - (2 - \sqrt{3})^n) \geq 3^n \text{ for } n \geq 1. \]

One can easily check from Theorem 3 that for \( n \geq 16 \) the values \( Kf(M_n), Kf(Pr_n) \) and \( K(n) \) are accurate to within \( 10^{-5} \). The respective values are given in Table 2.

As an immediate consequence of the above theorem we have the following result.

**Theorem 4.** Let \( K(n) = \frac{1}{6}(n^3 + 2\sqrt{3}n^2 - n) \). Then

\[ Kf(M_n) = K(n) + O\left(\frac{n^2}{3^n}\right), \quad n \to \infty, \]

and

\[ Kf(Pr_n) = K(n) + O\left(\frac{n^2}{3^n}\right), \quad n \to \infty. \]

**Remark.** When the paper was already prepared for publication, we discovered that the results equivalent to Theorems 1 and 2 were obtained by completely different methods in [5] and [6] respectively. A weak form of the asymptotics for Kirchhoff index in the Möbius ladder was done in [25].
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4. The table of Kirchhoff indices for graphs $M_n$ and $Pr_n$

Table 1. Kirchhoff indices for graphs $M_n$ and $Pr_n$

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### Table 2. Numerical values of $Kf(M_n)$, $K(n)$ and $Kf(Pr_n)$

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### References


ELEMENTARY FORMULAS FOR KIRCHHOFF INDEX


Galya Amanboldynova Baigonakova
Gorno-Altaysk State University,
34, Socialisticheskaya str.,
Gorno-Altaysk, 639000, Russia
E-mail address: galyaab@mail.ru

Aleksandr Dmitrievich Mednykh
 Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Novosibirsk State University,
1, Pirogova str.,
Novosibirsk, 630090, Russia
E-mail address: ilyamednykh@mail.ru