S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 16, стр. 1703–1715 (2019) УДК 519.174
DOI 10.33048/semi.2019.16.120 MSC 05C15

ON GARLANDS IN χ-UNIQUELY COLORABLE GRAPHS

P.A. GEIN

Abstract. A graph G is called χ-uniquely colorable, if all its χ-colorings induce the same partion of the vertex set into one-color components. For χ-uniquely colorable graphs new bound of the number of vertex set partions into $\chi + 1$ cocliques is found.

Keywords: graph, complete multipartite graph, uniquely colorable graph, chromatic uniqueness, chromatic invartiant

1. Introduction

All graphs in this paper are considered to be finite and simple, i. e. they do not contain loops and multiple edges. The main terminology is used with accordance to [1].

Let $G = (V, E)$ be a graph with a vertex set V and an edge set E. A *coloring* of a graph G into t colors is a map $\varphi\colon V \to \{1, 2, \ldots, t\}$, such that $\varphi(u) \neq \varphi(v)$ for any two adjacent vertices u and v of a graph G. We will call numbers $1, 2, \ldots, t$ colors. A graph is called t-colorable if there is its coloring into t colors. A minimum integer t, for which G is t-colorable, is called the *chromatic number* of a graph G and denoted as $\chi(G)$. A number of colorings of graph G into t colors is denoted as $P(G, \lambda)$. It is well known (see, for example, [1]), that function $P(G, \lambda)$ is a polynomial of variable λ. Two graph G and H are called *chromatically equivalent*, if $P(G, \lambda) = P(H, \lambda)$. A graph G is called *chromatically unique*, if for any graph H graphs G and H are chromatically equivalent iff they are isomorphic. Much attention of researches was drawn to the problem of chromatic uniqueness of complete multipartite graphs $K(n_1, n_2, \ldots, n_t)$. List some results, a more complete list can be found in the review [2] and the monograph [3].

1) A graph $K(n_1, n_2)$ where $n_1 \geqslant n_2 \geqslant 2$ is chromatically unique (see. [4]).

Gein, P.A., On garlands in χ-uniquely colorable graphs.

Received August, 9, 2019, published November, 21, 2019.

2) A graph $K(n_1, n_2, \ldots, n_t)$ where $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_t \geqslant 2$ is chromatically unique if $n_1 - n_t \leqslant 4$ or some other conditions are satisfied (see. [5, 6, 7, 8, 9]).

One of the main tools in studying chromatic uniqueness is chromatic invariants. Let a number $\alpha(G)$ be defined for any graph G. A number $\alpha(G)$ is called a *chromatic invariant* if for any two chromatic equivalent graphs G and H the equality $\alpha(G) = \alpha(H)$ is held.

By Zykov's theorem (see, for example, [1]) a chromatic polynomial of a graph G can be written as

$$P(G, \lambda) = \sum_{i=\chi}^{n} pt(G, i)\lambda^{(i)},$$

where $\lambda^{(i)} = \lambda \cdot (\lambda - 1) \cdot \ldots \cdot (\lambda - i + 1)$ and $pt(G, i)$ is a number of partitions of its vertex set into i *cocliques*, i. e. into i sets consisting of mutually non-adjacent vertices. It follows from Zykov's theorem, that integers $pt(G, i)$ are chromatic invariants for all $i = \chi, \chi + 1, \ldots, n$.

Consider an arbitrary t-colorable graph G with vertex set V, its t coloring, which uses all t colors, and consider a vertex set partition into disjoint sets of vertices with the same color. Note, that this partition is a vertex set partition into cocliques. A graph is called *t-uniquely colorable* if its any t-coloring of this type induces the same vertex set partition into cocliques. It is clear that complete t-multipartite graph is t-uniquely colorable (note, that t is equal to its chromatic number).

At the same time, a bound of the number of partitions of the vertex set into $t+1$ parts is still an open problem. Such bounds play an important role in investigations of chromatic uniqueness (see, for example, [10, 11, 9]).

Let G be t-uniquely colorable graph. We will denote the number of partitions of the vertex set V into $t+1$ cocliques as $pt(G)$. Every t-colorable graph can be obtained by deleting some set of edges E from complete t-partite graph. Let graph G be obtained from complete multipartite graph $K(v_1, v_2, \ldots, v_t)$ by deleting set of edges E. For convenience we will denote a complete multipartite graph $K(v_1, v_2, \ldots, v_t)$ as $K(v)$, where $v = (v_1, v_2, \ldots, v_t)$. We will denote parts of graph $K(v)$ as $V_i, |V_i| = v_i$ for $i = 1, 2, \ldots, t$. Note, that every partition of the vertex set of the graph $K(v)$ into $t + 1$ cocliques induces a partition in $t + 1$ cocliques of the vertex set of graph G. However, if set E is nonempty, then there are other partitions.

We will denote the number of partitions of the vertex set of graph G into $t + 1$ colciques, which are not the partitions of the vertex set of graph $K(v)$ into $t + 1$ cocliques, as $\Delta pt(G, K(V))$, i. e.

$$\Delta pt(G, K(v)) = pt(G, t + 1) - pt(K(v), t + 1).$$

In the article [10] Zou obtained the following bound

$$|E| \leq \Delta pt(G, K(v)) \leq 2^{|E|} - 1. \tag{1}$$

In the article [9] a combinatorical interpretation of this expression was found. We will discuss this interpretation in section 2. Bound (1) and its improvements helped to promote investigations of chromatic uniqueness of complete multipartite graphs (see [10, 9, 12]).

We will call a part V_i of a graph $K(v)$ *active*, if there is a vertex from V_i, which is incident with at least one edge from E. Let H be a subgraph of a graph induced by the set of edges E. A subgraph H, isomorphic to a graph $K(s, 1)$, where $s > 1$ is called *coordinated*, if all s vertices of degree 1 lay in the same part of a graph $K(v)$.

The main result of this paper is the following theorem, which is an improvement of the bound (1).

Theorem 1. *Let every active part of a graph $K(v_1, v_2, \ldots, v_t)$ contain at least three vertices. Then if E induces a coordinated subgraph $K(|E|, 1)$, then $\Delta pt(G, K(v)) = 2^{|E|} - 1$; otherwise,*

$$\Delta pt(G, K(v)) \leq 2^{|E|-1} + 1.$$

2. Auxiliary statements

A complete multipartite subgraph G_1 of a graph $K(v)$ is called an *E-subgraph*, if every part of graph G_1 is contained in the part of graph $K(v)$ and the edge set of graph G_1 is contained in a set E. In the article [9] an arbitrary nonempty set of disjoint E-subgraphs was called *a garland*.

We will say that a garland G' *destroys* part V_i if every vertex of part V_i lays in some E-subgraph of garland G'. We will call a garland containing exactly p elements, which destroys exactly $p-1$ parts, *interesting*. The set of edges of garland E-subgraphs is called *an edge aggregate*. In the article [9] the authors established the following properties:

1) if the chromatic number of a graph G is equal to t, then every garland which contains exactly p elements destroys at most $p-1$ parts;
2) a garland is uniquely identified by its edge aggregate;
3) a number $\Delta pt(G, K(v))$ is equal to the number of interesting garlands.

Let $G' = \{G'_1, G'_2, \ldots, G'_p\}$ be a garland. We will say, that a garland G' is of type $H_1 \dot\cup H_2 \dot\cup \ldots \dot\cup H_p$, where $\{H_1, H_2, \ldots, H_p\}$ is a set of graphs, if $G'_i \simeq H_i$ for all $i = 1, 2, \ldots, p$. All possible types of garlands, which edge aggregates contain at most four edges, are shown in Fig. 1.

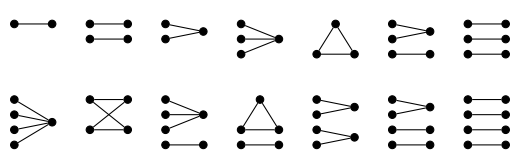

Fig. 1. The types of garlands, which edge aggregates contains at most four edges

Denote as E_{ij} the set of edges, such that one end is in V_i and another one is in V_j.

Lemma 1. *Let an edge $e \in E_{ij}$, G' is an interesting garland containing exactly $p \geq 2$ elements, e is in E-subgraph $H \in G'$ and H is of type $K(s_1, s_2)$. Then*

1) the garland G' destroys at least one part from V_i and V_j;
2) if the garland G' destroys exactly one part from V_i and V_j; then the garland $G' \setminus \{h\}$ is an interesting garland.
3) if there are two E-subgraphs, which edges are in E_{ij} and $p \geq 3$, then G' destroys both parts V_i and V_j.

Proof. Since the garland G' is an interesting garland, it destroys exactly $p-1$ parts. Denote $G_1 = G' \setminus \{H\}$. It is clear that G_1 is nonempty (it has exactly $p - 1 > 0$

distinct E-subgraphs) and consists of E-subgraphs, therefore, it is a garland. Also note that G' contains exactly $p-1$ elements.

Note, that all edges of the graph H are in E_{ij}, since otherwise G' is not a garland. If garland G' does not destroy any of V_i and V_j, then garland G_1 destroy the same parts as G' does, so garland G_1 containing exactly $p-1$ elements destroys exactly $p-1$ parts, which is a contradiction.

Now we assume without loss of generality that garland G' destroys part V_i, but does not destroy part V_j. Then garland G_1 destroy the same parts as garland G does without part V_i, therefore the garland G_1 destroys exactly $p-2$ parts, so it is an interesting garland.

We are to prove the last statement of Lemma. Let H_1 and H_2 be two distinct E-subgraphs from garland G, which edges are in E_{ij} and $G_2 = G' \setminus \{H_1, H_2\}$. Note that G_2 is a garland containing exactly $p-2$ elements. If garland G' destroys at most one part from V_i and V_j, then garland G_2 destroys $p-2$ parts, which is a contradiction. □

Fact. An arbitrary edge of complete multipartite graph $K(s_1, s_2, \ldots, s_t)$, nonisomorpihc to $K(s_1, 1)$, where $t \geq 2$, lays in some cycle.

Lemma 2. *Let e_1 and e_2 be two distinct edges from E. Assume that there exists a set of edges $\hat{E} \subseteq E \setminus \{e_1, e_2\}$, such that sets $\{e_1\} \dot\cup \hat{E}$ and $\{e_2\} \dot\cup \hat{E}$ induce garlands G_1 and G_2 respectively. Then E-subgraph, which is the element of garland G_1 and which contains edge e_1, is of type $K(s, 1)$ for some positive integer s.*

Proof. Assume by contradiction, that E-subgraph, which contains the edge e_1 is not of type $K(s, 1)$. Denote its edge set as F. Since the edge e_1 lays in some cycle, which is induced by edges from F, then both its ends are in the graph induced by a set of edges $F \setminus \{e_1\}$. Consequently, the set of edges $F \setminus \{e_1\}$ is in the edge aggregate of garland G_2, therefore, the edge e_1 is in the edge aggregate of garalnd G_2, and this is a contradiction. □

Definition. Let $E_1, E_2 \subseteq \hat{E} \subset E$. A set E_1 is called *(interesting) continuable outside of set* E, if there is an (ineresting) garland with an edge aggregate $\hat{E}$, such that $\hat{E} \cap E = E_1$. Sets E_1 and E_2 are called *simultaneously interestingly continuable outside of set* $\hat{E}$ if there is a nonempty set of edges $\tilde{E} \subseteq E \setminus \hat{E}$, such that sets $\tilde{E} \cup E_1$ and $\tilde{E} \cup E_2$ are edge aggregates of interesting garlands.

The number of vertices in the largest clique is caled a *clique number.*

Let $\hat{E} \subseteq E$ be an arbitrary set of edges. Let $Cnt(\hat{E})$ be a set of all subsets of $\hat{E}$, which are continuable outside of the set $\hat{E}$. Construct a graph $C(\hat{E})$ with vertex set $Cnt(\hat{E})$, where two vertices are adjacent, if they are simultaneously interestingly continuable outside of a set $\hat{E}$. Let C be the clique number of a graph $C(\hat{E})$. Then, there are at most $C \cdot (2^{|E|-|\hat{E}|} - 1) + N$ interesting garlands, where N is a number of interesting garlands, which edge aggregatte lay in set $\hat{E}$.

3. Case: the set E contains a triangle

Lemma 3. *Let a set of edges $\{e_1, e_2, e_3\} = E_1 \subset E$ induces a triangle. Then there does not exist a nonempty set $\hat{E} \subseteq E \setminus E_1$, such that sets $\{e_i\} \dot\cup E_1$ generate interesting garlands for all $i = 1, 2, 3$.*

Proof. By contradiction assume that there is a set of edges $\hat{E} \subseteq E \setminus E_1$, such that sets $\hat{E}\dot{\cup}\{e_i\}$ are edge aggregates of some interesting garlands for all $i = 1, 2, 3$. Without loss of generality we may assume, that the edges lay in the way, shown in Fig. 2.

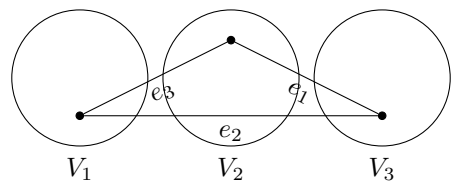

FIG. 2. A triangle, which lays in a set E

Denote a garland with an edge aggregate $\hat{E}\dot{\cup}\{e_i\}$ as G_i. By lemma 2 no edge can be adjacent to all edges e_i for $i = 1, 2, 3$. Therefore, each garland G_i contains at least two elements, and all garlands G_i for $i = 1, 2, 3$ have the same number of elements. Consequently, each garland G_i destroys the same number of parts among parts V_1, V_2 and V_3.

Note that each garland G_i cannot destroy a part V_i, but since it contains at least two elements, it destroys at least one part from V_j and V_k, where $i \neq j, k$. As a result, a garland $\{e_1, e_2, e_3\}\dot{\cup}\hat{E}$ destroys more parts, than G_1 does, by it is impossible, since garland G_1 is an interesting garland and garlands G_1 and $\{e_1, e_2, e_3\}\dot{\cup}\hat{E}$ have the same number of elements. □

The next proposition follows from Lemma 3.

Proposition 1. *If the set E contains a triangle, then there are at most $2^{|E|-1}$ interesting garlands.*

Proof. Denote the set of edges of some triangle in E as E' and let $F = E \setminus E'$. For an arbitrary nonempty set of edges $F_1 \subseteq F$ bound a number of interesting garlands, such that an intersection of its edge aggregates with F is equal to F_1.

Denote an intersection of an edge aggregate and E' as E_1. Note, that E_1 can be one of the five sets: empty, one of the three one-element subsets and E'. By Lemma 3 there are at most four subsets $E_1 \subseteq E'$, such that $E_1\dot{\cup}F_1$ is an edge aggregate of some interesting garland. Therefore, the number of interesting garlands which edge aggregates contain some edges from F does not exceed $4 \cdot (2^{|F|} - 1) = 4 \cdot (2^{|E|-3} - 1)) = 2^{|E|-1} - 4$.

Note that there are exactly 4 garlands, which edge aggregates do not intersect with F, i. e. which edge aggregates are in E'. This garlands are triangle and three garlands which edge aggregates contain exactly one edge.

Thus, there are exactly 4 interesting garlands which edge aggregates do not intersect with F and at most $2^{|E|-1} - 1$ interesting garlands, which edge aggregates intersect with F, therefore, there are at most $2^{|E|-1} - 4 + 4 = 2^{|E|-1}$ interesting garlands. □

4. Case: the set E contains a chain of length 3

Lemma 4. *Let $e_1 = x_1y_1$ and $e_2 = x_2y_2$ be two nonadjacent edges, which are in a set E_{ij} for some $i, j = 1, 2 \ldots, t$. Then there does not exist a nonempty subset*

$\hat{E}$ of a set $E \setminus \{e_1, e_2, x_1y_2, x_2y_1\}$, such that $\{e_1\}\dot{\cup}\hat{E}$ and $\{e_2\}\dot{\cup}\hat{E}$ simultaneously generate interesting garlands.

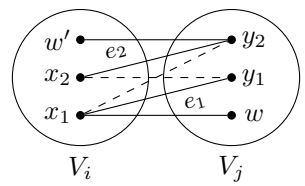

FIG. 3. Two nonintersecting subgraghs, which are in a set E_{ij}

Proof. By contradiction assume that there exist such subset $\hat{E} \subseteq E\setminus\{e_1, e_2, x_1y_2, x_2y_1\}$, that sets $\{e_1\}\dot{\cup}\hat{E}$ and $\{e_2\}\dot{\cup}\hat{E}$ are edge aggregates of interesting garlands G_1 and G_2 respectively.

Let $x_1, x_2 \in V_i$ and $y_1, y_2 \in V_j$.

By Lemma 2 we can deduce that E-subgraph from the garland G_1, containing an edge e_1, is a coordinated subgraph of type $K(s, 1)$.

Assume that no edges from E are adjacent to the edge e_1 or to the edge e_2. Then garlands G_1 and G_2 contain at least two elements because the set $\hat{E}$ is nonempty, therefore by Lemma 1 one can conclude, that each of them destroys some part of V_i and V_j. On the other hand, no one of this garlands cannot destroy V_i or V_j, because vertices x_2 and y_2 are adjacent to no edges from the garland G_1 and vertices x_1 and y_1 are adjacent to no edges from the garland G_2.

Without loss of generality, we can assume that there is an edge $x_1w \in \hat{E}$ and $w \neq y_2$, since xy_2 is not in $\hat{E}$. Note, that edges e_2 and x_1w cannot be in the same E-subgraph of the garland G_2, therefore, it contains at least two elements, consequently, it should destroy at least one part from V_i and V_j. Since it cannot destroy a part V_j because no edges from the garland G_2 are incident with the vertex y_1. Thus, the garland G_1 should destroy the part V_i.

Denote the number of elements in G_2 as p_2.

If $p_2 \geq 3$, then by Lemma 1 the garland G_2 should destroy parts V_i and V_j, but it is impossible. Therefore, $p_2 = 2$.

Since the garland G_2 destroys the part V_1 and $|V_1| \geq 3$, then there is an edge g in $\hat{E}$, which is incident with a vertex w'. Note that edges g and e_1 cannot be in the same E-subgraph of the garland G_1 or the garland G_2, therefore, edges g and e_2 are in the same E-subgraph of the garland G_2. Without loss of generality, we may assume that the edge e connects vertices w' and y_2.

By analogy, one can obtain that the garland G_1 should destroy the part V_j and that the garland G_1 contains exactly two elements. Note that a set $\{e_1, e_2\}\dot{\cup}\hat{E}$ induces a garland G_3, which destroys parts V_1 and V_2, but it contains exactly two elements, and it is a contradiction. □

Lemma 5. *Let $E_1 \subseteq E$ and E_1 induce a graph of type $K(2, 2)$. Then the following pairs of sets (see Fig. 4)*

1) $\{e_1, e_2\}$ and $\{e_2, e_3\}$,

2) $\{e_1\}$ and E_1,

3) $\{e_1, e_3\}$ *and* E_1,
4) empty and $\{e_1, e_3\}$
are not simultaneously interestingly continuable outside of the set E_1.

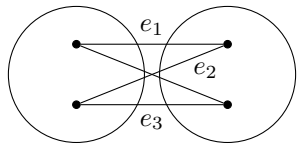

FIG. 4. Subgraph of type $K(2, 2)$

The proof is similar to the proof of Lemma 4.

Lemma 6. *Let* $e_1, e_2, e_3 \in E_{ij}$ *are three edges, which lay as shown in Fig. 5. Let* $X = V_j \setminus \{x_1, x_2\}$. *Then if a nonempty set of edges* $E_1 \subset E$, *such that sets of edges* $E_1 \cup \{e_1\}$ *and* $E_1 \cup \{e_2, e_3\}$ *induce interesting garlands, then* $E_1 = \{zx \colon x \in X\}$.

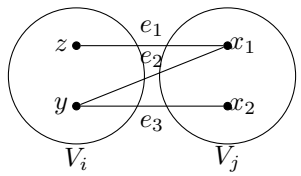

FIG. 5. A chain of length 3

Proof. Let G_1 and G_2 be interesting garlands with edge aggregates $E_1 \cup \{e_1\}$ and $E_1 \cup \{e_2, e_3\}$ respectively. Using Lemma 2 one can obtain that edges from E_1 cannot be incident with x_1 or x_2. Note that the garland G_1 cannot destroy the part V_j.

Assume that there are edges from E_1 which are not incident with the vertex y. Then the garland G_1 contains at least three elements because it should destroy one of parts V_i and V_j, but it cannot destroy the part V_j. Delete from this garland two E-subgraphs, which edges are in E_{ij}. Then the number of elements decreases by two, and the number of destroyed parts decreases by one, and it is a contradiction. Thus, no edges from E_1 are incident with a vertex y, therefore, a garland G_1 cannot destroy any part from V_i and V_j, consequently, it contains exactly one element and all edges from E_1 are incident with a vertex z.

If for some $x \in X$ an edge zx is not in the set E_1, then the garland G_2 destroys no part from V_i and V_j, and it is impossible. □

Lemma 7. *Let* $e_1, e_2 \in E_{ij}$ *be a pair of adjacent edges and* $\langle E \rangle$ *does not contain a triangle. Let a set of edges* $\hat{E} \subseteq E \setminus \{e_1, e_2, x_1y_2, x_2y_1\}$ *and sets* $\hat{E} \cup \{e_1\}$ *and* $\hat{E} \cup \{e_1, e_2\}$ *simultaneously generate interesting garlands. Then* $\hat{E}$ *should be equal to* $\{zx_1 | z \in V_j \setminus \{y_1, y_2\}\}$ *or* $\{wy_1 | w \in V_i \setminus \{x_1, x_2\}\}$.

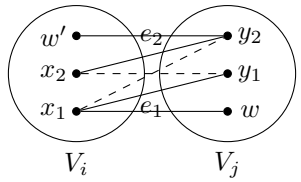

FIG. 6. To Lemma 7

Proof. Denote interesting garlands generated by sets $\hat{E} \cup \{e_1\}$ and $\hat{E} \cup \{e_1, e_2\}$ as G_1 and G_2 respectively.

Assume, that there is an edge e in the set $\hat{E}$, which is incident with the vertex x_2. Then note that edges e and e_1 are in different E-subgraphs of the garland G_1, therefore, the garland G_2 contains at least two elements and there are two distinct E-subgrpahs which edges are in the set E_{ij}. Consequently, this garland should destroy at least one part among V_i and V_j. Since there no edges in the set $\hat{E}$ which are incident with the vertex y_2, the garland G_2 cannot destroy the part V_j. Therefore, the garland G_1 contains exactly two elements (if it contains at least three elements, it contradicts Lemma 1). But in this case, the garland G_1 cannot destroy the part V_i, and this is a contradiction.

By analogy, there no edges in $\hat{E}$ which are incident with the vertex y_2, therefore, the garland G_1 cannot destroy the part V_i or the part V_j, then the garland G_1 contains exactly one element, i. e. all edges from $\hat{E}$ are adjacent to the edge e_1. Since the garland G_2 contains exactly two elements, the garland G_2 should destroy one part from V_i and V_j, and the lemma statement follows. □

Proposition 2. *If a graph $\langle E \rangle$ contains E-subgraph of a type $K(2,2)$, then there are at most $2^{|E|-2} + 5$ interesting garlands.*

Proof. Denote an edge aggregate of the garland as $\hat{E}$. Note that there is not a set E_1, which satisfies conditions of Lemma 6, since otherwise there is a garland with exactly two elements, which destroys two parts, and this is impossible.

There are only 12 sets among all subsets of the set $\hat{E}$, which can be continuable outside of the set E: empty set, 4 one-edge sets, 2 pairs of nonadjacent edges, 4 pairs of adjacent edges and the set $\hat{E}$. Let E' be an arbitrary nonempty subset of $E \setminus \hat{E}$. By Lemma 4 and Lemma 6, there are no more than four subsets F of the set $\hat{E}$, such that a set $E' \sup F$ generates an interesting garland.

So it is sufficient to compute the number of interesting garlands, which edge aggregates are in the set E_1. There are exactly 9 such garlands, namely 4 one-edge garlands, 4 garland of type $K(2,1)$ and 1 garland of type $K(2,2)$.

Therefore, there are no more than $4 \cdot (2^{|E|-4} - 1) + 9 = 2^{|E|-2} + 5$ interesting garlands. □

Proposition 3. *Let there be three edges in the set E, which are located as shown in Fig. 5. Then the number of interesting garlands does not exceed $2^{|E|-1} + 1$.*

Proof. By Proposition 2, it is sufficient to consider a case, when the graph $\langle E \rangle$ does not contain E-subgraph of type $K(2,2)$.

Let $E_1 = \{e_1, e_2, e_3\}$.

Note, that by Lemma 6 at least one pair of sets $\{e_1\}, \{e_2, e_3\}$ and $\{e_3\}, \{e_1, e_2\}$ is not simultaneously interestingly continuable outside of the set $\{e_1, e_2, e_3\}$ (otherwise, there is a two-element garland, which destroys two parts, and it is impossible).

If no one of its pairs is not simultaneously interestingly continuable, then the number of interesting garlands does not exceed $4 \cdot (2^{|E|-3} - 1) + 5 = 2^{|E|-1} + 1$.

Without loss of generality, assume that a pair $\{e_1\}, \{e_2, e_3\}$ is simultaneously interestingly continuable. Let $\hat{E}$ be arbitrary nonempty subset of the set $E \setminus E_1$. If $\hat{E} \neq \{zx \colon x \in X\}$, then there are at most four subsets in E_1, such that their union with $\hat{E}$ is an edge aggregate of some interesting garland. If $\hat{E} = \{zx \colon x \in X\}$, then there is exactly 3 subsets of the set E_1, such that their union with $\hat{E}$ is an edge aggregates of some interesting garland, namely the following subsets: empty, $\{e_1\}$ add $\{e_2, e_3\}$. Therefore, the number of interesting garlands, whose edge aggregates contains edges outside of the set E_1, does not exceed $4 \cdot (2^{|E|-3} - 1)$.

Taking into account, that the number of garlands, which edge aggregates are in the set E_1, equals to 4, the number of interesting garlands does not exceed $4 \cdot (2^{|E|-3} - 1) + 4 = 2^{|E|-1} + 1$, □

5. A general case

Proposition 4. *Let there not be E-subgraphs of type $K(2,2)$, edges e_1 and e_2 are nonadjacent edges from E_{ij}, and there not be a chain of length 3 in set E_{ij}. Then the number of interesting garlands does not exceed $2^{|E|-1} + 1$.*

Proof. Consider two cases.

Case 1. Let there be an edge $e \in E_{ij} \setminus \{e_1, e_2\}$ which is adjacent to the edge e_1 or to the edge e_2. Without loss of generality, we assume that the edge e adjacent to the edge e_1. Let k be the number of edges of the maximum with respect to the number of edges coordinated subgraph of type $K(k, 1)$, which contains the edge e_1. Then $k \geq 2$. Denote its edge set as F.

Let $\hat{E}$ be an arbitrary proper nonempty subset of the set F. Assume that there is a set $E_1 \subset E \setminus \{e_1, e_2\}$, such that sets $E_1 \cup \hat{E}$ and $E_1 \cup \{e_2\}$ simultaneously generate interesting garlands G_1 and G_2 respectively. Note that by maximality of the subgraph $\langle F \rangle$ and the fact that E_1 does not contain subgraphs of type $K(2,2)$ and chains of length 3, the garland G_1 contains at least two elements, therefore, it should destroy at least one part from V_i and V_j.

Denote ends of the edge e_2 as x and y. Note, that edges from E_1 are incident with at most one vertex among x and y, since there is no subgraphs of type $K(2,2)$ in E_{ij}. On the other hand, there should be an edge e, which is incident with the vertex x or the vertex y (otherwise, the garland G_1 cannot destroy the part V_i or the part V_j). Also note, that $e \in E_{ij}$, which implies, that the garland G_1 should destroy parts V_i and V_j, which is impossible.

By analogy, one can prove that sets $\hat{E}$ and $F \cup \{e_2\}$ are not simultaneously interestingly continuable outside of the set $F \cup \{e_2\}$.

The number of interesting garlands, which edge aggregates are not in the set $F \dot{\cup} \{e_2\}$, does not exceed $2^k \cdot (2^{|E|-k-1} - 1)$, the number of interesting garlands, which edge aggregates are in the set F is equal to $2^k + 2 - 1 = 2^k + 1$. So, the number of interesting garlands does not exceed $2^k \cdot (2^{|E|-k-1} - 1) + 2^k + 2 - 1 = 2^{|E|-1} + 1$.

Case 2. Assume that there is not an edge $e \in E_{ij} \setminus \{e_1, e_2\}$ which is adjacent to the edge e_1 or to the edge e_2. By analogy with the previous case, one can prove that

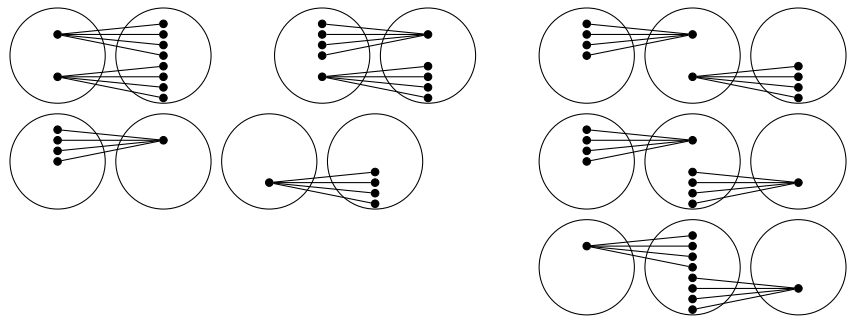

FIG. 7. Mutual locations of two nonintersecting coordinated subgraphs of types $K(|E_1|, 1)$ and $K(|E_2|, 1)$

pairs of sets $\{e_1\}, \{e_1, e_2\}$ and $\{e_2\}, \{e_1, e_2\}$ are not simultaneously interestingly continuable outside of the set $\{e_1, e_2\}$, and the proposition statement follows. □

Proposition 5. *If the subgraph $\langle E \rangle$ does not contain a coordinated subgraph of type $K(2, 1)$ and a triangle, then the number of interesting garlands does not exceed $\lfloor \frac{4}{3}|E| \rfloor$.*

Proof. Let an interesting garland contain p edges, note that it contains exactly p elements. It should destroy $p - 1$ parts, so it should go through at least $3(p - 1)$ vertices. Since p edges have $2p$ edges, $2p \geq 3(p - 1)$. Therefore, $p \leq 3$.

Note that a garland with exactly two elements cannot destroy any part (because its intersection with any part contains no more than two vertices), so there are no interesting garlands with two elements.

Next, we consider a garland with exactly three elements. It should be of type $K(1, 1)\dot{\cup}K(1, 1)\dot{\cup}K(1, 1)$ and should destroy 2 parts. Since for any $i, j = 1, 2, \ldots, t$ any two edges from the set E_{ij} are not adjacent, any two such garlands cannot have a common edge, therefore, there are no more than $\left\lfloor \frac{|E|}{3} \right\rfloor$ such garlands.

Since there is exactly $|E|$ one-element garlands and there are at most $\left\lfloor \frac{|E|}{3} \right\rfloor$ three-element garlands, there are at most $|E| + \left\lfloor \frac{|E|}{3} \right\rfloor = \lfloor \frac{4}{3}|E| \rfloor$ interesting garlands. □

Lemma 8. *Let the set E be disjoint union of two sets E_1 and E_2 and the following conditions held:*

1) sets E_1 and E_2 generate coordinated subgraphs of types $G_1 = K(|E_1|, 1)$ and $G_2 = K(|E_2|, 1)$ respectively;
2) graphs G_1 and G_2 do not have common vertices.

Then the number of interesting garlands does not exceed $2^{|E|-1} + 1$.

Proof. All possible mutual locations of graphs G_1 and G_2 are shown in Fig. 7. The lemma statement can be obtained by direct computing. □

Proposition 6. *Let the graph $\langle E \rangle$ not contain triangle and the set E_{ij} be empty or generate coordinated subgraph of type $K(e_{ij}, 1)$ for all $i, j = 1, 2, \ldots, t, i \neq j$, If the set E does not generate a coordinated subgraph of type $K(|E|, 1)$, than the number of interesting garlands does not exceed $2^{|E|-1} + 1$.*

Proof. Note that if $|E| = 1$ the set E always generates a coordinated subgraph of type $K(1,1)$ and it contains exactly 1 interesting garland.

Prove the proposition by induction on $m = |E| \geq 2$.

Base case. Let $m = 2$. Then, it is obvious, that interesting garlands are generated only by one-element subsets of the set E. So, there are exactly two interesting garlands.

Step case. Assume that the proposition holds for $|E| = m$ and prove it for $m+1$.

Assume that there are two edges e_1 and e_2 from E, such that they have a common vertex, and two other vertices x and y are in different parts and the edge xy is not in the set E. Note that edges e_1 and e_2 can be in the edge aggregates of any interesting garland, since otherwise the edge xy should be in the same aggregate, but it is not in the set E.

Then note that at least one of the sets $\{e_1\} \cup E_1$ and $\{e_2\} \cup E_1$ does not generate a coordinated subgraph of type $K(m,1)$, without loss of generality we may assume, that the set $E' = \{e_2\} \cup E_1$ does not generate a coordinated subgraph of type $K(m,1)$. Note that edge e_1 is in at most 2^{m-1} interesting garlands, because edges e_1 and e_2 cannot be in the same garland. The number of interesting garlands, which edge aggregates are in the set E', by the induction hypothesis, does not exceed $2^{m-1}+1$, which implies that the overall number of interesting garlands does not exceed $2^m + 1$.

Now we assume that such edges do not exist.

Case 1. Assume that there is an active part V_i, such that for some vertex $y \in V_i$ there is no edge in E, incident to the vertex y.

Let $z \in V_i, e \in E$ and the vertex z be incident to the edge e. Let k be the number of edges of the maximum with respect to inclusion order of the coordinated subgraph of type $K(k,1)$, which contains the edge e and denote its edge set as F. Let $E_1 = E \setminus F$

Consider the cases.

Case 1.1. Assume that $k = 1$, i. e. the edge e is not adjacent to any other edge from E.

Case 1.1.1. Assume that the set E_1 generates coordinated subgraph of type $K(m,1)$. Then by Lemma 8 the number of interesting garlands does not exceed $2^m + 1$.

Case 1.1.2. Assume that the set E_1 does not generate a coordinated subgraph of type $K(m,1)$. If $F_1 \subseteq F$ and $\hat{E} \subseteq E_1$ are arbitrary subsets, then a set $F_1 \cup \hat{E}$ is an edge aggregate of some interesting garland only if the set $\hat{E}$ is an edge aggregate of some interesting garland. Also note, since e is not adjacent to any edge from E_1, the edge e are not in any two-edge garland. Therefore, the edge e are in at most $2^{m-1}+1-m$ interesting garlands. The number of interesting garlands, which edge aggregates do not contain the edge e, by the induction hypothesis, does not exceed $2^{m-1}+1$. Thus, there are at most $2^{m-1}+1-m+2^{m-1}+1 = 2^m+2-m \leq 2^m+1$ interesting garlands.

Case 1.2. Assume that $k \geq 2$. Let $F_1 \subseteq F$ be an arbitrary subset and $E' \subseteq E \setminus F$. Then, if a set $E' \cup F_1$ generates an interesting garland, then the set E' also generates an interesting garland.

Case 1.2.1. Let the set $E \setminus F$ not generate a coordinated subgraph of type $K(m+1-k,1)$. Then note that if we add an edge from $E\setminus$ to the set F_1, then

the result set does not generate an interesting garland, so the number of interesting garlands does not exceed

$$\begin{aligned}(2^k-2)(2^{m-k}+1-(m+1-k))+2\cdot(2^{m-k}+1)=\\ =(2^k-2)(2^{m-k}-(m-k))+2^{m-k+1}+2=\\ =2^m-(m-k)2^k-2^{m-k+1}+2(m-k)+2^{m-k+1}+2=\\ =2^m-(m-k)(2^k-2)+2.\end{aligned}$$

Since $m \geq k+1$ and $k \geq 2$, then $-(m-k)(2^k-2)+2<0$, and the bound follows.

Case 1.2.2. Let the set $E \setminus F$ generate a coordinated subgraph of type $K(m+1-k,1)$. The bound follows from Lemma 8.

Case 2. Now we consider a case when for any active part V_i and for any vertex $y \in V_i$ there is an edge e which is incident to the vertex y.

By Proposition 5, it is sufficient to consider a case, when the graph $\langle E \rangle$ contains a coordinated subgraph $K(k,1)$ and $k \geq 2$.

Consider an arbitrary coordinated subgraph of type $K(s_1,1)$, where $s_1 \geq 2$. Denote its edge set as F. Let a vertex x of degree s_1 be in part V_i.

Case 2.2.1. Assume that all other vertices of the part V_i are not destroyed by the single coordinated subgraph of type $K(|V_i|-1,1)$. Denote as T the set of edges, which are incident to any vertex from $V_i \setminus \{x\}$. Note that $|T| \geq |V_i|-1 \geq 2$.

Consider an arbitrary proper subset $F_1 \subset F$. Let a nonempty set $\hat{E} \subseteq E \setminus F$ and the set of edges $\hat{E} \cup F_1$ generate an interesting garland. Note that this garland should destroy the part V_i, therefore, it should contain the set T, so there are at most $2^{m+1-s_1-|T|}$ of such sets.

Also note that the set $T \cup F$ do not generate an interesting garland, since it generates a garland with at least three elements and destroys at most one part. The number of interesting garlands which are formed from elements of the set $E \setminus F$ does not exceed $2^{m+1-s_1-1}+1=2^{m-s_1}+1$ (by the induction hypothesis). The number of such sets $\hat{E}$, that $\hat{E} \cup F$ generates an interesting garland, does not exceed 2^{m+1-s_1}. Thus, the number of interesting garlands does not exceed the number

$$\begin{aligned}(2^{s_1}-2)(2^{m+1-s_1-|T|}-1+1)+2^{m+1-s_1}+2^{m-s_1}+1=\\ =2^{m+1-|T|}-2^{m+2-s_1-|T|}+2^{m+1-s_1}+2^{m-s_1}+1=M.\end{aligned}$$

If $s_1=2$, then $M=2^{m+1-|T|}-2^{m-|T|}+2^{m-1}+2^{m-2}+1=2^{m-|T|}+2^{m-1}+2^{m-2}+1 \leq 2^{m-2}+2^{m-1}+2^{m-2}+1=2^m+1$. If $s_1 \geq 3$, then $M \leq 2^{m+1-|T|}+2^{m+1-3}+2^{m-3}+1 \leq 2^{m-1}+2^{m-2}+2^{m-3}+1<2^m+1$.

Case 2.2.2. Assume that all other vertices of the part V_i are destroyed by the single coordinated subgraph $\hat{G}$ of type $K(|V_i|-1,1)$. Consider a vertex of degree one of the graph $\hat{G}$. Let it be in a part V_j. If all other vertices of the part V_j are not destroyed by single coordinated subgraph of type $K(|V_j-1,1)$, then repeat the proof of Case 2.2.1.

If all other vertices of the part V_j are destroyed by single coordinated subgraph of type $K(|V_j-1,1)$, then consider a vertex of degree one of the graph $\hat{G}_2$ and repeat arguments.

If on every step all vertices except one are destroyed by single coordinated subgraph of type $K(s,1)$, then in some step (denote the number of this step as l) the process returns to the start part, but it means that there is an l-element garland which destroys exactly l parts. This contradiction finishes the proof.

□

Now we are ready to prove the main result.

The proof of the theorem. Assume the the set E induces a coordinated subgraph $K(|E|,1)$. Then every nonempty subset is an edge aggregate of an interesting garland, therefore, there are exactly $2^{|E|-1}$ interesting garlands.

Now we assume that a set E does not induce a coordinated subgraph $K(|E|,1)$. In this case, the proof follows from prepositions 1, 2, 3, 4, 6.

□

The author is grateful to his scientific advisor prof. V.A. Baranskiy for constnat attention and remarks.

References

[1] M. O. Asanov, V. A. Baransky, V. V. Rasin, *Discrete mathematics: graphs, algorithhms, matroids*, Saint-Petersburg: "Lan", 2010. [in Russian].

[2] H. Zhao, *Chromaticity and adjoint polynomials of graphs*, Zutphen: Wöhrmann Print Service, 2005.

[3] F.M. Dong, K.M. Koh, K.L. Teo, *Chromatic polynomials and chromaticity of graphs*, World Scientific, 2005.

[4] K .M. Koh, K. L. Teo, *The search of chromatically unique graphs*, Graphs Combin, **6**:3 (1990), 259–285. MR1081201

[5] V. A. Baranskii, T. A. Koroleva, *Chromatic uniqueness of certain complete tripartite graphs*, Izv. Ural. Gos. Univ. Mat. Mekh., **74**:12 (2010), 5–26. MR2906104

[6] T. A. Koroleva, *Chromatic uniquness of some complete tripartite graphs. I*, Trudy Instituta Matematiki i Mekhaniki UrO RAN, **13**:3 (2007), 65–83.

[7] T. A. Koroleva, *Chromatic uniqueness of some complete tripartite graphs. II*, Izv. Ural. Gos. Univ. Mat. Mekh., **74**:12 (2010), 39–56. MR2906106

[8] T.A. Senchonok, *Chromatic uniqueness of elements of height 2 in latticess of complete multipartite graphs*, Trudy Instituta Matematiki i Mekhaniki UrO RAN, **17**:3 (2011), 271–281.

[9] V. A. Baranskii , T. A. Senchonok, *Chromatic uniqueness of elements of height* ≤ 3 *in lattices of complete multipartite graphs*, Proceedings of the Steklov Institute of Mathematics, **279**, suppl. 1:4 (2012), 1–16. Zbl 1301.05117

[10] Hui Wen Zou, *The chromatic uniqueness of certain complete t-partite graphs*, Discrete Mathematics, **275** (2004), 375–383. MR2026301

[11] F. M. Dong, K. M. Koh, K. L. Teo, C. H. C. Little, M. D. Hendy, *Sharp bounds for the number of 3-independent partitions and the chromaticity of bipartite graphs*, Journal of Graph Theory, **37**:1 (2001), 48–77. MR1822957

[12] Gein, P.A., *About chromatic uniqueness of some complete tripartite graphs*, Siberian Electronic Mathematical Reports, **14** (2017), 1492–1504. MR3744070

Pavel A. Gein
Ural Federal University,
51, Lenina ave.,
Ekaterinburg, 62083, Russia
E-mail address: pavel.gein@gmail.com