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ON BOROVKOV'S ESTIMATE IN THE INVARIANCE PRINCIPLE

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Abstract. The first general estimate in the functional CLT (i.e. in the Invariance Principle) was obtained by A. Borovkov in 1973 in terms of the Lyapunov fractions of order not greater than three. Here we present explicit numerical bounds for the constants in Borovkov's estimate. In addition, we have found similar but more precise estimates in terms of truncated Lyapunov fractions.

Keywords: estimates in the Invariance Principle, Prokhorov distance, Lyapunov fraction, truncated Lyapunov fraction.

1. Introduction

1.1. Suppose that for all $n = 1, 2, \dots$ we are given n independent random variables $\xi_{1,n}, \xi_{2,n}, \dots, \xi_{n,n}$ such that

$$\forall n\ \forall j \quad \mathbf{E}\xi_{j,n} = 0, \qquad \sum_{j\le n} \mathbf{E}\xi_{j,n}^2 = 1. \tag{1}$$

Introduce the random function $S_n = S_n(t)$, by setting

$$S(t_{k,n}) = \sum_{j\le k} \xi_{j,n} \qquad \text{with} \qquad t_{k,n} = \sum_{j\le k} \mathbf{E}\xi_{j,n}^2, \quad k = 1, 2, \dots, \tag{2}$$

and extending $S_n(t)$ so that $t_{0,n} = S_n(0) = 0$ and

$$\forall k \ge 1 \quad |S_n(t) - S_n(t_{k-1,n})| \le |\xi_{k,n}| \qquad \text{for all} \quad t \in [t_{k-1,n}, t_{k,n}]. \tag{3}$$

We treat $S_n = S_n(\cdot)$ as a random process with values in the space $\mathbb{D}_u[0, 1]$ of all real-valued functions on $[0, 1]$ which are right continuous with left limits; and

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we consider the space $\mathbb{D}_u[0,1]$ having the uniform norm $\|\cdot\|$. It is well known that under the classical Lindeberg condition the sequence $S_n = S_n(\cdot)$ satisfies functional CLT, known also as the Invariance Principle, introduced in [2] and [3]. It means that for any real bounded and continuous functional $f(\cdot)$, defined on $\mathbb{D}_u[0,1]$, we have

$$\mathbf{E}[f(S_n)] \to \mathbf{E}[f(W)] \qquad \text{as} \quad n \to \infty,$$

where $W = W(\cdot)$ is a standard Wiener process. This fact is also equivalent to the following convergence

$$\Pi(S_n, W) \to 0 \qquad \text{as} \quad n \to \infty,$$

where $\Pi(S_n, W)$ is the Prokhorov distance between the distributions in $\mathbb{D}_u[0,1]$ of the processes S_n and W (we will revisit the corresponding definitions at the beginning of the next Section 2).

1.2. Prokhorov distance was introduced in the seminal paper [3] of Prokhorov as a useful tool for investigation of convergence of probability distributions in separable metric spaces. It is also a useful tool for investigation of rates of convergence in functional CLT. Indeed, if the functional $f(\cdot)$ is Lipschitz continuous, i.e.,

$$\forall X, Y \in \mathbb{D}_u[0,1] \quad \big|f(X) - f(Y)\big| \le L\big\|X - Y\big\|, \quad L < \infty,$$

and if $f(W)$ has a bounded density, i.e.,

$$\forall a \in \mathbb{R} \;\; \forall h \ge 0 \quad \mathbf{P}\big(a \le f(W) \le a + h\big) \le Kh, \quad K < \infty,$$

then (see [1])

$$\sup_a \big|\mathbf{P}(f(S_n) \le a) - \mathbf{P}(f(W) \le a)\big| \le (KL+1)\Pi(S_n, W). \tag{4}$$

Inspired by inequality (4), a difficult task of obtaining estimates for the Prokhorov distance $\Pi(S_n, B)$ attracted a lot of attention. The first such estimate was obtained by Prokhorov in [3] when $L_{3,n} < \infty$, where

$$L_{\alpha,n} := \sum_{j \le n} \mathbf{E}|\xi_{j,n}|^\alpha \qquad \text{for} \quad \alpha \ge 2$$

is the Lyapunov fraction of order α. The first general estimate is due to Borovkov [1], who proved that

$$\Pi(S_n, W) \le C_0(\alpha)\big(L_{\alpha,n}\big)^{1/(\alpha+1)} \qquad \text{for} \quad \alpha \in [2,3], \tag{5}$$

where $C_0(\alpha)$ depends only on α, and

$$C_1 := \sup_{\alpha \in [2,3]} C_0(\alpha) < \infty$$

is an absolute constant. Later, Sakhanenko in [5] extended (5) for all $\alpha \ge 2$.

In [5] and [7] you may also find the simple proofs of the fact that inequality (5) is unimprovable for all $\alpha \ge 2$, up to a constant $C_0(\alpha)$.

1.3. In the present paper we are going to obtain explicit numerical estimates for constants $C_0(\alpha)$ and C_1. Our main results for Prokhorov distance and its generalizations are presented in Section 2, with all explicit numerical estimates for the constants. In particular, it will be shown in Theorem 2 that $C_1 < 6.3$. As far as we know it is the first case when estimates in the classical functional CLT (i.e. in the Invariance Principle) are obtained with the explicit numerical constants (see Remark 5 at the beginning of Section 3 about the previous attempt).

Let us stress that our estimates from Theorems 1 – 3 are, in a strong sense, better than unimprovable Borovkov's estimate (5). Indeed, it follows from Theorems 2 and 3 that for the classical Prokhorov distance we have the following inequality

$$\Pi(S_n, W) \le C(\alpha)\big(L_{\alpha,n}(1/2)\big)^{1/(\alpha+1)} \le C(\alpha)\big(L_{\alpha,n}\big)^{1/(\alpha+1)} \quad \forall\ \alpha \ge 2,$$

where $C(\alpha)$ depends only on α, $C(\alpha) \le 6.3$ for $\alpha \in [2,3]$, and

$$L_{\alpha,n}(\varepsilon) := \sum_{j\le n} \mathbf{E}\min\{|\xi_{j,n}|^\alpha, \varepsilon^{\alpha-2}\xi_{j,n}^2\} \le L_{\alpha,n}, \quad \varepsilon > 0,\ \alpha \ge 2. \tag{6}$$

So, the value $L_{\alpha,n}(\varepsilon)$ may be treated as a truncation of the Lyapunov fraction $L_{\alpha,n}$.

Thus, Theorems 1 – 3 give estimates in the Invariance Principle in terms of the values $L_{\alpha,n}(\varepsilon)$ whose convergence to zero when $\alpha > 2$ is necessary for the validity of the Principle (see details in Remark 4). The Lyapunov fractions $L_{\alpha,n}$, used in Borovkov's estimate (5), do not possess this remarkable property: they may be equal to infinity for each $n \ge 1$ and all $\alpha > 2$.

2. Main results

2.1. On Prokhorov distance. We first remind a definition of the Prokhorov distance, which we use in the present paper. Denote by $\mathbb{D}_u[0,1]$ the set of all real-valued functions on $[0,1]$ which are right continuous with left limits, and we suppose that

$$\|X - Y\| := \max\{|X(t) - Y(t)| : t \in [0,1]\}$$

is the distance between functions $X, Y \in \mathbb{D}_u[0,1]$. Later on for each measurable set $A \subset \mathbb{D}_u[0,1]$ symbol A^ε denotes the ε-neighborhood of A, i.e.,

$$A^\varepsilon := \big\{X \in \mathbb{D}_u[0,1] : \exists Y \in A \quad \text{with} \quad \|X - Y\| \le \varepsilon\big\}.$$

Now for any two processes S and Z with values in $\mathbb{D}_u[0,1]$ introduce the following useful notation:

$$\Pi(S, Z, \varepsilon) := \sup_A \big[\mathbf{P}(S \in A) - \mathbf{P}(Z \in A^\varepsilon)\big] \le 1, \qquad \varepsilon > 0,$$

where the supremum is taken over all measurable sets $A \subset \mathbb{D}_u[0,1]$. After that we may define the Prokhorov metric between the distributions of such S and Z:

$$\Pi(S, Z) := \inf\big\{\varepsilon > 0 : \Pi(S, Z, \varepsilon) \le \varepsilon\big\} \le 1. \tag{7}$$

2.2. Main results. We first introduce two constants:

$$C_2(\alpha) := 3(1 + 4/\sqrt{2\pi}) \cdot 2^{3+\alpha} < 31,15 \cdot 2^{\alpha+1} < 498,4 < 500, \quad \alpha \in [2,3];$$
$$C_3(\alpha) := \big(C_2(\alpha)\big)^{1/(\alpha+1)} < 2(31,15)^{1/(\alpha+1)} < 6.3, \qquad \alpha \in [2,3].$$

Theorem 1. *Under assumptions from the first paragraph of the paper, for any $\varepsilon > 0$ and $n \ge 1$ on the same probability space with the process S_n it is possible to construct a Wiener process W_n such that for all $\alpha \in [2,3]$*

$$\mathbf{P}(||S_n - W_n|| > \varepsilon) \le \frac{C_2(3)L_{3,n}(\varepsilon/2)}{\varepsilon^3} \le \frac{C_2(\alpha)L_{\alpha,n}(\varepsilon/2)}{\varepsilon^\alpha} \le \frac{C_2(\alpha)L_{\alpha,n}}{\varepsilon^\alpha}. \tag{8}$$

We can always assume by [8] that the process W_n is only a function of S_n, ε and some random variable with the uniform distribution. Recall that if processes W_n and S_n are constructed on the same probability space, then it is easy to prove (see e.g. [1]) that

$$\Pi(S_n, W, \varepsilon) \le \mathbf{P}(||S_n - W_n|| > \varepsilon) \qquad \forall \varepsilon > 0. \tag{9}$$

Using (9) we prove below the following estimate for the Prokhorov distance

Theorem 2. *Under conditions of Theorem 1*

$$\Pi(S_n, W) \le C_3(\alpha)\big(L_{\alpha,n}(1/2)\big)^{1/(\alpha+1)} \le C_3(\alpha)\big(L_{\alpha,n}\big)^{1/(\alpha+1)} \qquad \forall \alpha \in [2,3]. \tag{10}$$

Thus, we obtained a generalization of unimprovable Borovkov's estimate (5) with an explicit absolute constant.

2.3. Additional Remarks.

Remark 1. *It follows from the last assertion of Theorem 1 in* [7] *that under assumptions from the first paragraph of the present paper, for any $\varepsilon > 0$, $n \ge 1$ and $\alpha \ge 2$ there exists a Wiener process W_n such that*

$$\mathbf{P}\big(||S_n - W_n|| > C_{abs}\alpha\varepsilon\big) \le \frac{L_{\alpha,n}(\varepsilon)}{\varepsilon^\alpha} \le \frac{L_{\alpha,n}}{\varepsilon^\alpha}, \tag{11}$$

where $C_{abs} \ge 1$ is an absolute constant, which was not estimated in [7].

Since (11) takes place for all $\alpha \ge 2$, it may be considered as a generalization of (8). But for $\alpha \le 3$ we found in (8) explicit upper bounds for the absolute constants.

Using the idea from the proof of Theorem 2 we obtain

Theorem 3. *Under conditions of Theorem 1*

$$\Pi(S_n, W) \le C_{abs}\alpha\big(L_{\alpha,n}(1/2)\big)^{1/(\alpha+1)} \le C_{abs}\alpha\big(L_{\alpha,n}\big)^{1/(\alpha+1)} \qquad \forall \alpha \ge 2, \tag{12}$$

where $C_{abs} \ge 1$ is the absolute constant from (11).

Remark 2. *It follows from the second assertion of Corollary 5 in* [7] *that under assumptions of Theorem 1*

$$\Pi(S_n, W) \le C_{abs}\alpha\pi_{\alpha,n} \qquad \forall \alpha \ge 2,$$

where $\pi_{\alpha,n}$ was defined in [7] *in the following implicit way:*

$$\pi_{\alpha,n} := \inf\big\{\varepsilon > 0 : L_{\alpha,n}(\varepsilon) \le \varepsilon\big\}.$$

It is clear that our estimates (10) and (12) with explicit values $L_{\alpha,n}(1/2)$ are more convenient for further applications. It is one of the achievements of the present paper.

Remark 3. *If in the assertions of Theorems 2 and 3 we change the value $L_{\alpha,n}(1/2)$ by $L_{\alpha,n}(\varepsilon_0)$ for some fixed $\varepsilon_0 > 0$ then we obtain similar assertions, but with different constants $C_3(\alpha)$ and C_{abs} which will depend on the chosen value ε_0. The only reason why we use $\varepsilon_0 = 1/2$ is that in this case Lemma 1 (see Section 3) allows us to obtain estimate (10) with the minimal value $C_3(\alpha)$.*

Remark 4. *The following assertions are equivalent:*

1) the Invariance Principle (i.e. functional CLT) is valid;

2) the Prokhorov distance $\Pi(S_n, W) \to 0$ *as* $n \to \infty$*;*

3) the Lindeberg condition is satisfied;

4) $\exists \alpha > 2\ \exists \varepsilon > 0$ *such that* $L_{\alpha,n}(\varepsilon) \to 0$*;*

5) $\forall \alpha > 2\ \forall \varepsilon > 0$ *we have* $L_{\alpha,n}(\varepsilon) \to 0$.

Thus, Theorems 1 – 3 give estimates in the Invariance Principle in terms of the values $L_{\alpha,n}(\varepsilon)$ *whose convergence to zero is necessary for validity of the Principle. The Lyapunov fractions* $L_{\alpha,n}$*, used earlier, do not possess this property.*

The above-mentioned fact is one of the reasons that motivated the author to write this article.

The rest of the paper is devoted to proofs of the main results presented above.

3. Proofs

3.1. Main Lemma.

Lemma 1. *For any* $x > 0$ *and* $n \geq 1$ *on the same probability space with the process* S_n *it is possible to construct a Wiener process* W_n *such that*

$$\mathbf{P}\big(\Delta_n > x\big) \leq 16 L_{3,n}(x)/x^3 + 16 L_{3,n}^{(W)}(x)/x^3, \tag{13}$$

where

$$\Delta_n := \max_{k \leq n} \big|S_n(t_{k,n}) - W_n(t_{k,n})\big|, \quad \eta_{j,n} := W_n(t_{j,n}) - W_n(t_{j-1,n}), \tag{14}$$

$$L_{\alpha,n}^{(W)}(x) := \sum_{j \leq n} \mathbf{E} \min\big\{|\eta_{j,n}|^{\alpha}, x^{\alpha-2}\eta_{j,n}^2\big\}, \quad x > 0,\ \alpha > 0. \tag{15}$$

This estimate was obtained in Corollary 3.1 in Sakhanenko [6], using arguments similar to Sakhanenko [4].

Consider the random step functions

$$\forall k \geq 1 \quad \tilde{S}_n(t) = S_n(t_{k-1,n}) \qquad \text{for all} \qquad t \in [t_{k-1,n}, t_{k,n}], \tag{16}$$

$$\forall k \geq 1 \quad \tilde{W}_n(t) = W_n(t_{k-1,n}) \qquad \text{for all} \qquad t \in [t_{k-1,n}, t_{k,n}]. \tag{17}$$

So, $\tilde{S}_n$ is the simplest partial case of S_n, since it satisfies (2) and (3). It is evident from (16), (17) and (14) that

$$||\tilde{S}_n - \tilde{W}_n|| = \Delta_n. \tag{18}$$

Remark 5. *Estimate (13) may be treated as the first estimate in the Invariance Principle (Functional Central Limit Theorem) with explicit constants. But it follows from (18) that (13) is not a "real"estimate in the Invariance Principle because it is not an estimate for the value* $||S_n - W_n||$ *which we have to estimate in the Invariance Principle and which is estimated in Theorem 1.*

3.2. Auxiliary lemmas. Having in mind (1), (2) and (14) define

$$\sigma_{j,n}^2 := \mathbf{E}\xi_{j,n}^2 = \mathbf{E}\eta_{j,n}^2; \quad L_{\alpha,n}^{(W)} := \sum_{j \leq n} \mathbf{E}|\eta_{j,n}|^{\alpha}, \quad \alpha > 0. \tag{19}$$

Lemma 2. *For any* $\alpha \geq 1$ *and* $x > 0$

$$\mathbf{P}\big(\|\tilde{W}_n - W_n\| \geq x\big) \leq L_{\alpha,n}^{(W)}/x^{\alpha}. \tag{20}$$

Proof. It is well known that function $|x|^\alpha$ is convex for $\alpha \geq 1$ and $W_n(t)$ is a martingale. So, by maximal inequality we have

$$p_j := \mathbf{P}\Big(\sup_{t\in[t_{j-1,n},t_{j,n}]} |W_n(t) - W_n(t_{j-1,n})| \geq x\Big) = \mathbf{P}\Big(\sup_{t\in[0,\sigma_{j,n}^2]} |W_n(t)| \geq x\Big)$$
$$\leq \mathbf{E}\big|W_n(\sigma_{j,n}^2))\big|^\alpha / x^\alpha = \mathbf{E}|\eta_{j,n}|^\alpha / x^\alpha =: \tilde{p}_j.$$

Note that $\tilde{W}_n(t_{j-1,n}) = W_n(t_{j-1,n})$ for $j = 1, \dots, n$. Hence

$$\mathbf{P}\big(\|\tilde{W}_n - W_n\| \geq x\big) = \mathbf{P}\big(\max_{t\in[0,1]} |\tilde{W}_n(t) - W(t)| \geq x\big) \leq \sum_{j\leq n} \tilde{p}_j \leq L_{\alpha,n}^{(W)} / x^\alpha.$$

Here we also used (19). □

For any $c > 0$ introduce functions

$$f_c(x) := \min\big\{|x|^3, cx^2\big\}, \qquad g_c(x) = \int_0^{|x|} \min\big\{(3/2)y^{1/2}, c\big\} dy. \tag{21}$$

It follows from definitions (21) and (6) that

$$\sum_{j\leq n} \mathbf{E} f_c(\xi_{j,n}) = L_{3,n}(c) \leq L_{3,n}(C) \quad \forall C \geq c. \tag{22}$$

Lemma 3. *For each $j = 1, \dots, n$ we have for any $c > 0$ that*

$$\mathbf{E} f_c(\xi_{j,n}) \geq \mathbf{E} g_c(\xi_{j,n}^2) \geq g_c(\sigma_{j,n}^2) \geq \min\big\{\sigma_{j,n}^3, (2c/3)^3\big\}. \tag{23}$$

Proof. Function $g_c(x)$ is even and convex because its derivative $g_c^{'}(x)$ is non-decreasing for $x \geq 0$. By Jensen's inequality we have with $\xi = \xi_{j,n}$ and $\sigma = \sigma_{j,n}$ the following

$$\mathbf{E} g_c(\xi^2) \geq g_c(\mathbf{E}\xi^2) = g_c(\sigma^2) \geq \min\big\{g_c(\sigma^2), g_c(2c/3)\big\} = \min\big\{\sigma^3, (2c/3)^3\big\}, \tag{24}$$

because $g_c(\sigma^2) = \sigma^3$ when $\sigma \leq 2c/3$ as it follows from definition (21).

Next, it is easy to see from (21) that

$$\forall\ x \geq 0 \quad f_c^{'}(x) = \begin{cases} 3x^2, & \text{if } \ x \leq c \\ 2cx, & \text{if } \ x \geq c \end{cases} \geq 2x \min\big\{(3/2)x, c\big\} = \frac{dg_c(x^2)}{dx}.$$

Hence, $f_c(x) = f_c(|x|) \geq g_c(x^2)$ for all x. From this fact and (24) we obtain (23). □

Throughout the rest of the paper the random variable η has the standard normal distribution and

$$C_4 := \mathbf{E}|\eta|^3 = 4/\sqrt{2\pi} < 1,5958. \tag{25}$$

Lemma 4. *Suppose that for some constant $c > 0$*

$$\max_{j\leq n} \sigma_{j,n} \leq 2c/3. \tag{26}$$

Then

$$L_{3,n}^{(W)} \leq C_4 \sum_{j\leq n} \mathbf{E} f_c(\xi_{j,n}) = C_4 L_{3,n}(c). \tag{27}$$

Proof. First, note that

$$\mathbf{E}|\eta_{j,n}|^3 = \mathbf{E}|\sigma_{j,n}\eta|^3 = \sigma_{j,n}^3\mathbf{E}|\eta|^3 = C_4\sigma_{j,n}^3 = C_4 \min\left\{\sigma_{j,n}^3, (2c/3)^3\right\} \le C_4\mathbf{E}f_c(\xi_{j,n}). \tag{28}$$

Here we also used (23) and (26). From (28) and (22) we immediately obtain (27). □

3.3. Proof of Theorem 1. Suppose that a Wiener process W_n is constructed on the same probability space with the process S_n, so that all assertions of Lemma 1 take place. In this case

$$\|S_n - W_n\| \le \|S_n - \tilde{S}_n\| + \|\tilde{S}_n - \tilde{W}_n\| + \|\tilde{W}_n - W_n\|$$

with processes $\tilde{S}_n$ and $\tilde{W}_n$ introduced in (16) and (17). Hence, for all $\varepsilon > 0$

$$\mathbf{P}\big(||S_n - W_n|| > \varepsilon\big) \le \delta_1(\varepsilon/4) + \delta_2(\varepsilon/2) + \delta_3(\varepsilon/4), \tag{29}$$

where

$$\delta_1(x) := \mathbf{P}\big(\|S_n - \tilde{S}_n\| > x\big), \quad \delta_3(x) := \mathbf{P}\big(\|\tilde{W}_n - W_n\| > x\big), \tag{30}$$

$$\delta_2(x) := \mathbf{P}\big(\|\tilde{S}_n - \tilde{W}_n\| > x\big) = \mathbf{P}\big(\Delta_n > x\big) \le 16L_{3,n}(x)/x^3 + 16L_{3,n}^{(W)}/x^3. \tag{31}$$

In the last formula we also used (18) and the assertion of Lemma 1.

Suppose now that

$$\max_{j\le n} \sigma_{j,n} \le 2c/3 = \varepsilon/3 \quad \text{with} \quad c = \varepsilon/2. \tag{32}$$

In this case, substituting (27) into (31) with $x = \varepsilon/2$ we obtain that

$$\delta_2(\varepsilon/2) \le 16L_{3,n}(\varepsilon/2)/(\varepsilon/2)^3 + 16C_4L_{3,n}(\varepsilon/2)/(\varepsilon/2)^3 = 128(1 + C_4)L_{3,n}(\varepsilon/2)/\varepsilon^3. \tag{33}$$

Using again (32) and (27), we have from (20) and (25) with $\alpha = 3$ and $x = \varepsilon/4$ that

$$\delta_3(\varepsilon/4) \le L_{3,n}^{(W)}/(\varepsilon/4)^4 \le 4^3C_4L_{3,n}(\varepsilon/2)/\varepsilon^3 = 64C_4L_{3,n}(\varepsilon/2)/\varepsilon^3. \tag{34}$$

At last, using (3) we obtain from Chebyshev inequality that

$$\delta_1(\varepsilon/4) \le \mathbf{P}\big(\sup_{j\le n} |\xi_{j,n}| > \varepsilon/4\big) = \mathbf{P}\big(\sup_{j\le n} f_c(|\xi_{j,n}|) > f_c(\varepsilon/4)\big) \tag{35}$$

$$\le \sum_{j\le n} \mathbf{P}\big(f_c(|\xi_{j,n}|) > f_c(\varepsilon/4)\big) \le \sum_{j\le n} \frac{\mathbf{E}f_c(\xi_{j,n})}{f_c(\varepsilon/4)} = \frac{L_{3,n}(\varepsilon/2)}{(\varepsilon/4)^3} = \frac{64L_{3,n}(\varepsilon/2)}{\varepsilon^3}.$$

Substituting now (33), (34) and (35) into (29) we obtain that

$$\mathbf{P}\big(||S_n - W_n|| > \varepsilon\big) \le \frac{C_5L_{3,n}(\varepsilon/2)}{\varepsilon^3} \quad \text{with} \quad C_5 := (128 + 64)(1 + C_4) = C_2(3). \tag{36}$$

Thus, inequality (36) is proved if (32) holds. Consider now the case when (32) is not true. Then we have from (22) and (23) that

$$L_{3,n}(c) = \sum_{j\le n} \mathbf{E}f_c(\xi_{j,n}) \ge \max_{j\le n} \mathbf{E}f_c(\xi_{j,n}) \ge \max_{j\le n} \min\left\{\sigma_{j,n}^3, (2c/3)^3\right\} = (2c/3)^3$$

with $2c/3 = \varepsilon/3$. Hence, in this case the right-hand side in (36) has the following property

$$\frac{C_5 L_{3,n}(\varepsilon/2)}{\varepsilon^3} \geq \frac{C_5(2c/3)^3}{\varepsilon^3} = \frac{C_5(\varepsilon/3)^3}{\varepsilon^3} = \frac{C_5}{27} > 1. \tag{37}$$

But the probability on the left-hand side of (36) does not exceed 1. Thus, (36) is proved in all cases.

If now $2 \leq \alpha \leq 3$ then we have from definition (6) that

$$\begin{aligned} L_{3,n}(\varepsilon/2) &= \sum_{j\leq n} \mathbf{E}\min\left\{|\xi_{j,n}|^3, (\varepsilon/2)\xi_{j,n}^2\right\} \\ &\leq \sum_{j\leq n} \mathbf{E}\min\left\{(\varepsilon/2)^{3-\alpha}|\xi_{j,n}|^{\alpha-2}, (\varepsilon/2)\xi_{j,n}^2\right\} = (\varepsilon/2)^{3-\alpha} L_{\alpha,n}(\varepsilon/2). \end{aligned} \tag{38}$$

Substituting (38) into (36) we obtain that

$$\mathbf{P}\big(||S_n - W_n|| > \varepsilon\big) \leq \frac{C_5 L_{3,n}(\varepsilon/2)}{\varepsilon^3} \leq \frac{C_5(\varepsilon/2)^{3-\alpha} L_{\alpha,n}(\varepsilon/2)}{\varepsilon^3} = \frac{C_5 L_{\alpha,n}(\varepsilon/2)}{2^{3-\alpha}\varepsilon^3}.$$

So, (8) is true with any $C_2(\alpha) = C_5/2^{3-\alpha}$. Thus, Theorem 1 is proved in all cases.

3.4. Proof of Theorem 2. We are going to use the assertion of Theorem 1 with

$$\varepsilon := C_3(\alpha)\big(L_{\alpha,n}(1/2)\big)^{1/(\alpha+1)}. \tag{39}$$

If $\varepsilon \leq 1$ then $L_{\alpha,n}(\varepsilon/2) \leq L_{\alpha,n}(1/2)$ and we have from (9) and (8) that

$$\Pi(S_n, W, \varepsilon) \leq \mathbf{P}(||S_n - W_n|| > \varepsilon) \leq \frac{C_2(\alpha) L_{\alpha,n}(\varepsilon/2)}{\varepsilon^\alpha} \leq \frac{C_2(\alpha) L_{\alpha,n}(1/2)}{\varepsilon^\alpha} = \varepsilon.$$

Thus, it follows from definition (7) that the value ε from (39) is an estimate for $\Pi(S_n, W)$, and (10) follows when $\varepsilon \leq 1$.

If $\varepsilon > 1$ then (10) again takes place because the probability on the left-hand side of (10) does not exceed 1.

3.5. Proof of Theorem 3. Consider the following numbers

$$\varepsilon := \big(L_{\alpha,n}(1)\big)^{1/(\alpha+1)} \qquad \text{and} \qquad \tilde\varepsilon := C_{abs}\alpha\varepsilon. \tag{40}$$

If $\varepsilon \leq 1/2$ then $L_{\alpha,n}(\varepsilon) \leq L_{\alpha,n}(1/2)$ and we have from (9) and (11) that

$$\begin{aligned} \Pi(S_n, W, \tilde\varepsilon) &\leq \mathbf{P}(||S_n - W_n|| > \tilde\varepsilon) = \mathbf{P}\big(||S_n - W_n|| > C_{abs}\alpha\varepsilon\big) \\ &\leq \frac{L_{\alpha,n}(\varepsilon)}{\varepsilon^\alpha} \leq \frac{L_{\alpha,n}(1/2)}{\varepsilon^\alpha} = \varepsilon < C_{abs}\alpha\varepsilon = \tilde\varepsilon, \end{aligned}$$

because $C_{abs} \geq 1$ and $\alpha \geq 2$ by assumptions of Theorem 3. Thus, it follows from definition (7) that the value $\tilde\varepsilon$ from (40) is an estimate for $\Pi(S_n, W)$, and (12) follows when $\varepsilon \leq 1/2$.

If $\varepsilon > 1/2$ then the right-hand side in (12) has the following property

$$\tilde\varepsilon = C_{abs}\alpha\varepsilon \geq \alpha\varepsilon \geq 2\varepsilon > 1.$$

But the probability on the left-hand side of (12) does not exceed 1. So, (12) is proved in all cases. □

Thus, all assertions of the paper are proved.

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