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A REGULARITY CRITERION TO THE 3D BOUSSINESQ EQUATIONS

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ABSTRACT. The paper deals with the regularity criterion for the weak solutions to the 3D Boussinesq equations in terms of the partial derivatives in Besov spaces. It is proved that the weak solution (u, θ) becomes regular provided that $(\nabla_h u, \nabla_h \theta) \in L^{\frac{8}{3}}(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3))$. Our results improve and extend the well-known results of Fang-Qian [13] for the Navier-Stokes equations.

Keywords: Boussinesq equations, regularity criterion, weak solutions, Besov space.

1. INTRODUCTION AND MAIN RESULT

This paper is devoted to the study of the Cauchy problem for the Boussinesq equations in $\mathbb{R}^3 \times (0, T)$:

$$(1.1) \quad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = \theta e_3, \\ \partial_t \theta - \Delta \theta + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases}$$

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where $u = u(x, t)$ is the velocity of the fluid, $\theta = \theta(x, t)$ is the scalar quantity such as the concentration of a chemical substance or the temperature variation in a gravity field, $\pi = \pi(x, t)$ is the scalar pressure, while u_0 and θ_0 are given initial velocity and initial temperature with $\nabla \cdot u_0 = 0$ in the sense of distributions. $e_3 = (0, 0, 1)^T$ denotes the vertical unit vector.

The Cauchy problem (1.1) for the Boussinesq equation, has been studied extensively by many authors (see, for example, [1, 2, 3, 5, 6, 7, 10, 11, 12, 14, 19, 20, 21, 22, 23, 24] and references cited therein).

When $\theta = 0$, (1.1) is the well-known incompressible Navier-Stokes equations, which the global regularity is an outstanding open problem, as well as the famous millennium prize problem. Since the global existence of weak solutions is well-known and strong solutions are unique and smooth in $(0, T)$, it is an interesting problem on the regularity criterion of the weak solutions if some partial derivatives of the velocity satisfy certain growth conditions (see, e.g. [4, 13, 15, 16, 18, 25, 30, 32, 33]). One of the most significant achievements in this direction is the celebrated Fang and Qian criterion [13]. More precisely, they showed that a weak solution with H^1 -data is a strong solution provided that

$$(1.2) \quad \nabla_h u \in L^{\frac{8}{3}}(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)).$$

where $\nabla_h = (\partial_1, \partial_2)$ denotes the horizontal gradient operator and $\dot{B}_{\infty, \infty}^{-1}$ denotes the homogeneous Besov space. For details see [31].

Recall that the weak solutions satisfy the following energy inequality

$$(1.3) \quad \|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2 \int_0^t (\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla \theta(\tau)\|_{L^2}^2) d\tau \leq \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2,$$

for all $0 \leq t \leq T$.

Motivated by the reference mentioned above, our aim of the present paper is to improve and extend the above regularity criterion (1.2) to the Boussinesq equations (1.1).

Our main result reads as follows.

Theorem 1.1. *Suppose $T > 0$, $(u_0, \theta_0) \in H^1(\mathbb{R}^3)$ with $\text{div } u_0 = 0$ in \mathbb{R}^3 , in the sense of distributions. Let (u, θ) be a weak solution of (1.1) in $(0, T)$. Assume that*

$$(1.4) \quad (\nabla_h u, \nabla_h \theta) \in L^{\frac{8}{3}}(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)),$$

then the weak solution (u, θ) is regular on $\mathbb{R}^3 \times (0, T]$.

Remark 1.1. *In the case $\theta = 0$, the above theorem reduces to the well-known Fang and Qian result [13] for the Navier-Stokes equations.*

1.1. Proof of Theorem 1.1. In this section, we shall give the proof of Theorem 1.1, we first need to prove the following lemma.

Lemma 1.2. *Let (u, θ) be a smooth solution to (1.1). Then, there exists a positive universal constant C such that the following a priori estimates hold :*

$$(1.5) \leq \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u dx + \int_{\mathbb{R}^3} (u \cdot \nabla) \theta \cdot \Delta \theta dx + C \int_{\mathbb{R}^3} |\nabla_h u| |\nabla u|^2 dx + C \int_{\mathbb{R}^3} |\nabla_h u| |\nabla \theta|^2 dx + C \int_{\mathbb{R}^3} |\nabla_h \theta| |\nabla u| |\nabla \theta| dx.$$

Proof: Due to the divergence-free condition $\nabla \cdot u = 0$, one shows that

$$\begin{aligned} \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i \partial_k u_j \partial_k u_j dx &= \frac{1}{2} \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i (\partial_k u_j)^2 dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^3} \left(\sum_{i=1}^3 \partial_i u_i \right) \left(\sum_{j,k=1}^3 (\partial_k u_j)^2 \right) dx = 0, \\ \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i \partial_k \theta \partial_k \theta dx &= 0, \end{aligned}$$

Hence,

$$\begin{aligned} I &= \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u dx + \int_{\mathbb{R}^3} (u \cdot \nabla) \theta \cdot \Delta \theta dx \\ &= - \int_{\mathbb{R}^3} \nabla (u \cdot \nabla) u \cdot \nabla u dx - \int_{\mathbb{R}^3} \nabla (u \cdot \nabla) \theta \cdot \nabla \theta dx \\ &= - \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx - \sum_{i,k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_i \theta \partial_k \theta dx. \end{aligned}$$

In order to estimate the right hand side of I , we split each of the above integrals according to the following rules :

Case 1 : when $1 \leq k \leq 2$ or $1 \leq i \leq 2$, then the integral I has at least $\nabla_h u$ or $\nabla_h \theta$ in the integrand and can be dominated by

$$(1.6) \quad I \leq C \int_{\mathbb{R}^3} |\nabla_h u| |\nabla u|^2 dx + C \int_{\mathbb{R}^3} |\nabla_h u| |\nabla \theta|^2 dx + C \int_{\mathbb{R}^3} |\nabla_h \theta| |\nabla u| |\nabla \theta| dx$$

Case 2 : when $k = i = 3$, then we use the divergence-free condition to rewrite

$$\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2,$$

then the integral can be controlled by (1.6). Hence the proof of Lemma is complete. \square

1.2. Proof of Theorem 1.1. Before going to the proof, we recall the following inequality established in [8] :

$$(1.7) \quad \begin{aligned} \|f\|_{L^r} &\leq C \|f\|_{L^2}^{\frac{6-r}{2r}} \|\partial_1 f\|_{L^2}^{\frac{r-2}{2r}} \|\partial_2 f\|_{L^2}^{\frac{r-2}{2r}} \|\partial_3 f\|_{L^2}^{\frac{r-2}{2r}} \\ &\leq C \|f\|_{L^2}^{\frac{6-r}{2r}} \|\nabla_h f\|_{L^2}^{\frac{r-2}{r}} \|\nabla f\|_{L^2}^{\frac{r-2}{2r}}, \end{aligned}$$

for every $f \in H^1(\mathbb{R}^3)$ and $r \in [2, 6]$.

Now we are ready to present the proof of Theorem 1.1.

Proof: Since the initial data $(u_0, \theta_0) \in H^1(\mathbb{R}^3)$ with $\text{div } u_0 = 0$ in \mathbb{R}^3 , there exists a unique local strong solution (u, θ) of the 3D Boussinesq equations on $(0, T)$ (see [2, 6, 7, 23]). By using a standard method, we only need to show the following a priori estimate

$$(1.8) \quad \begin{aligned} &\sup_{0 \leq t \leq T} \left(\|\nabla u(\cdot, t)\|_{L^2}^2 + \|\nabla \theta(\cdot, t)\|_{L^2}^2 \right) \\ &\leq \left(\|\nabla u_0\|_{L^2}^2 + \|\nabla \theta_0\|_{L^2}^2 + C \|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} + C \|\nabla_h \theta_0\|_{L^2}^{\frac{8}{3}} \right) e^{CK(T)}, \end{aligned}$$

where

$$\mathcal{K}(T) = \int_0^T \left(\|\nabla_h u(s)\|_{\dot{B}_{\infty,\infty}^{\frac{8}{3}-1}}^{\frac{8}{3}} + \|\nabla_h \theta(s)\|_{\dot{B}_{\infty,\infty}^{\frac{8}{3}-1}}^{\frac{8}{3}} \right) ds.$$

Let

$$\begin{aligned} \mathcal{J}(t) &= \|\nabla_h u(t)\|_{L^2}^2 + \|\nabla_h \theta(t)\|_{L^2}^2 + \int_0^t \left(\|\nabla \nabla_h u(\tau)\|_{L^2}^2 + \|\nabla \nabla_h \theta(\tau)\|_{L^2}^2 \right) d\tau, \\ \mathcal{X}(t) &= \int_0^t \left(\|\nabla_h u(\tau)\|_{\dot{B}_{\infty,\infty}^{\frac{8}{3}-1}}^2 + \|\nabla_h \theta(\tau)\|_{\dot{B}_{\infty,\infty}^{\frac{8}{3}-1}}^2 \right) \left(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla \theta(\tau)\|_{L^2}^2 \right) d\tau, \\ \mathcal{Z}(t) &= \|\nabla u(\cdot, t)\|_{L^2}^2 + \|\nabla \theta(\cdot, t)\|_{L^2}^2 + \int_0^t \left(\|\Delta u(\cdot, \tau)\|_{L^2}^2 + \|\Delta \theta(\cdot, \tau)\|_{L^2}^2 \right) d\tau, \\ \mathcal{W}(t) &= \int_0^t \left(\|\nabla_h u(\tau)\|_{\dot{B}_{\infty,\infty}^{\frac{8}{3}-1}}^{\frac{8}{3}} + \|\nabla_h \theta(\tau)\|_{\dot{B}_{\infty,\infty}^{\frac{8}{3}-1}}^{\frac{8}{3}} \right) \left(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla \theta(\tau)\|_{L^2}^2 \right) d\tau. \end{aligned}$$

We start with the estimates of $\|\nabla_h u\|_{L^2}$ and $\|\nabla_h \theta\|_{L^2}$. Multiplying the first equation of (1.1) by $(-\Delta_h u)$, where $\Delta_h = \partial_1 \partial_1 + \partial_2 \partial_2$ is the horizontal Laplacian and integrating by parts and using the divergence free condition $\nabla \cdot u = 0$ into account, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla_h u(t)\|_{L^2}^2 + \|\nabla \nabla_h u\|_{L^2}^2 &= \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta_h u dx - \int_{\mathbb{R}^3} \theta e_3 \cdot \Delta_h u dx \\ &= \int_{\mathbb{R}^3} \sum_{j=1}^3 \sum_{l=1}^2 u_j \partial_j u \cdot \partial_l^2 u dx - \int_{\mathbb{R}^3} \sum_{l=1}^2 \theta e_3 \cdot \partial_l^2 u dx \\ &= - \int_{\mathbb{R}^3} \sum_{j=1}^3 \sum_{l=1}^2 \partial_l u_j \partial_j u \partial_l u dx + \int_{\mathbb{R}^3} \sum_{l=1}^2 \partial_l (\theta e_3) \partial_l u dx \\ (1.9) \qquad \qquad \qquad &= - \int_{\mathbb{R}^3} \nabla_h u \cdot \nabla u \cdot \nabla_h u dx + \int_{\mathbb{R}^3} \nabla_h (\theta e_3) \cdot \nabla_h u dx, \end{aligned}$$

where we have used

$$\int_{\mathbb{R}^3} \sum_{j=1}^3 \sum_{l=1}^2 u_j \partial_j \partial_l u \cdot \partial_l u dx = 0.$$

Similarly, multiplying the second equation of (1.1) by $(-\Delta_h \theta)$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla_h \theta(t)\|_{L^2}^2 + \|\nabla \nabla_h \theta\|_{L^2}^2 &= \int_{\mathbb{R}^3} (u \cdot \nabla) \theta \cdot \Delta_h \theta dx = \int_{\mathbb{R}^3} \sum_{j=1}^3 \sum_{l=1}^2 u_j \partial_j \theta \cdot \partial_l^2 \theta dx \\ (1.10) \qquad \qquad \qquad &= - \int_{\mathbb{R}^3} \sum_{j=1}^3 \sum_{l=1}^2 \partial_l u_j \partial_j \theta \partial_l \theta dx = - \int_{\mathbb{R}^3} \nabla_h u \cdot \nabla \theta \cdot \nabla_h \theta dx, \end{aligned}$$

where we have used

$$\int_{\mathbb{R}^3} \sum_{j=1}^3 \sum_{l=1}^2 u_j \partial_j \partial_l \theta \cdot \partial_l \theta dx = 0.$$

Combining (1.9) and (1.10) yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla_h u(t)\|_{L^2}^2 + \|\nabla_h \theta(t)\|_{L^2}^2) + \|\nabla \nabla_h u\|_{L^2}^2 + \|\nabla \nabla_h \theta\|_{L^2}^2 \\ &= - \int_{\mathbb{R}^3} \nabla_h u \cdot \nabla u \cdot \nabla_h u dx - \int_{\mathbb{R}^3} \nabla_h u \cdot \nabla \theta \cdot \nabla_h \theta dx + \int_{\mathbb{R}^3} \nabla_h(\theta e_3) \cdot \nabla_h u dx \end{aligned}$$

(1.11) $\neq R_1 + R_2 + R_3,$

Attention is now focused on bounding these terms; we start with R_1 . Using Hölder and Young’s inequalities, one has, for R_1 ,

$$\begin{aligned} |R_1| &\leq C \|\nabla_h u\|_{L^4}^2 \|\nabla u\|_{L^2} \\ &\leq C \|\nabla \nabla_h u\|_{L^2} \|\nabla_h u\|_{B_{\infty,\infty}^{-1}} \|\nabla u\|_{L^2} \\ &\leq \frac{1}{4} \|\nabla \nabla_h u\|_{L^2}^2 + C \|\nabla_h u\|_{B_{\infty,\infty}^{-1}}^2 \|\nabla u\|_{L^2}^2. \end{aligned}$$

Here we have used the following inequality due to Meyer-Gerard-Oru [28] (see also [17]) :

$$(1.12) \quad \|f\|_{L^4}^2 \leq C \|\nabla f\|_{L^2} \|f\|_{B_{\infty,\infty}^{-1}}.$$

For R_2 , analogously, using Hölder and Young’s inequalities, we deduce from (1.12) that

$$\begin{aligned} |R_2| &\leq C \|\nabla_h u\|_{L^4} \|\nabla \theta\|_{L^2} \|\nabla_h \theta\|_{L^4} \\ &\leq C \|\nabla \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h u\|_{B_{\infty,\infty}^{-1}}^{\frac{1}{2}} \|\nabla \theta\|_{L^2} \|\nabla \nabla_h \theta\|_{L^2}^{\frac{1}{2}} \|\nabla_h \theta\|_{B_{\infty,\infty}^{-1}}^{\frac{1}{2}} \\ &= C \left(\|\nabla \nabla_h u\|_{L^2}^2 \right)^{\frac{1}{4}} \left(\|\nabla_h u\|_{B_{\infty,\infty}^{-1}}^2 \|\nabla \theta\|_{L^2}^2 \right)^{\frac{1}{4}} \left(\|\nabla \nabla_h \theta\|_{L^2}^2 \right)^{\frac{1}{4}} \\ &\quad \times \left(\|\nabla_h \theta\|_{B_{\infty,\infty}^{-1}}^2 \|\nabla \theta\|_{L^2}^2 \right)^{\frac{1}{4}} \\ &\leq \frac{1}{4} \|\nabla \nabla_h u\|_{L^2}^2 + \frac{1}{4} \|\nabla \nabla_h \theta\|_{L^2}^2 + C \|\nabla_h u\|_{B_{\infty,\infty}^{-1}}^2 \|\nabla \theta\|_{L^2}^2 \\ &\quad + C \|\nabla_h \theta\|_{B_{\infty,\infty}^{-1}}^2 \|\nabla \theta\|_{L^2}^2. \end{aligned}$$

For R_3 , by means of the Hölder and Cauchy inequalities, it follows that

$$\begin{aligned} |R_3| &\leq C \|\nabla_h \theta\|_{L^2} \|\nabla_h u\|_{L^2} \\ &\leq C (\|\nabla_h \theta\|_{L^2}^2 + \|\nabla_h u\|_{L^2}^2). \end{aligned}$$

Inserting the above estimate into (1.11), we derive that

$$\begin{aligned} & \frac{d}{dt} \left(\|\nabla_h u(\cdot, t)\|_{L^2}^2 + \|\nabla_h \theta(\cdot, t)\|_{L^2}^2 \right) + \|\nabla \nabla_h u(\cdot, t)\|_{L^2}^2 + \|\nabla \nabla_h \theta(\cdot, t)\|_{L^2}^2 \\ & \leq C \left(\|\nabla_h u(\cdot, t)\|_{B_{\infty,\infty}^{-1}}^2 + \|\nabla_h \theta(\cdot, t)\|_{B_{\infty,\infty}^{-1}}^2 \right) \left(\|\nabla u(\cdot, t)\|_{L^2}^2 + \|\nabla \theta(\cdot, t)\|_{L^2}^2 \right). \end{aligned}$$

Integrating the above inequality in time variable over $0 \leq \tau \leq t$, we get

$$\mathcal{J}(t) \leq \|\nabla_h u_0\|_{L^2}^2 + \|\nabla_h \theta_0\|_{L^2}^2 + C\mathcal{X}(t).$$

Next, we derive the bounds of $\|\nabla u\|_{L^2}$ and $\|\nabla\theta\|_{L^2}$. Multiplying the two equations of (1.1) by $(-\Delta u)$ and $(-\Delta\theta)$, respectively, integrating and applying the incompressibility condition, we have by (1.5)

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) + \|\Delta u(t)\|_{L^2}^2 + \|\Delta\theta(t)\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u dx + \int_{\mathbb{R}^3} (u \cdot \nabla) \theta \cdot \Delta\theta dx + \int_{\mathbb{R}^3} (\theta e_3) \cdot \Delta u dx \\ (1.13) \quad & \leq C \int_{\mathbb{R}^3} |\nabla_h u| |\nabla u|^2 dx + C \int_{\mathbb{R}^3} |\nabla_h u| |\nabla\theta|^2 dx \\ &+ C \int_{\mathbb{R}^3} |\nabla_h \theta| |\nabla u| |\nabla\theta| dx + C \|\theta\|_{L^2} \|\Delta u\|_{L^2} \end{aligned}$$

$$(1.14) \quad \leq K_1 + K_2 + K_3 + \frac{1}{4} \|\Delta u\|_{L^2}^2 + C \|\theta\|_{L^2}^2.$$

Now we deal with K_1 . It follows that, from the Hölder inequality and (1.7)

$$\begin{aligned} K_1 &\leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^4}^2 \\ &\leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h u\|_{L^2} \|\Delta u\|_{L^2}^{\frac{1}{2}}. \end{aligned}$$

For K_2 , Hölder inequality and (1.7), together give,

$$\begin{aligned} K_2 &\leq C \|\nabla_h u\|_{L^2} \|\nabla\theta\|_{L^4}^2 \\ &\leq C \|\nabla_h u\|_{L^2} \|\nabla\theta\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h \theta\|_{L^2} \|\Delta\theta\|_{L^2}^{\frac{1}{2}}. \end{aligned}$$

Arguing similarly as the estimate of K_1 , thanks to the Hölder inequality and (1.7), one has

$$\begin{aligned} K_3 &\leq C \|\nabla_h \theta\|_{L^2} \|\nabla u\|_{L^4} \|\nabla\theta\|_{L^4} \\ &\leq C \|\nabla_h \theta\|_{L^2} \left(\|\nabla u\|_{L^2}^{\frac{1}{4}} \|\nabla \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{4}} \right) \left(\|\nabla\theta\|_{L^2}^{\frac{1}{4}} \|\nabla \nabla_h \theta\|_{L^2}^{\frac{1}{2}} \|\Delta\theta\|_{L^2}^{\frac{1}{4}} \right). \end{aligned}$$

Combining the above estimates of K_1, K_2 and K_3 and inserting into (1.14), we get

$$\begin{aligned} & \frac{d}{dt} (\|\nabla u(\cdot, t)\|_{L^2}^2 + \|\nabla\theta(\cdot, t)\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 \\ & \leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h u\|_{L^2} \|\Delta u\|_{L^2}^{\frac{1}{2}} \\ & \quad + C \|\nabla_h u\|_{L^2} \|\nabla\theta\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h \theta\|_{L^2} \|\Delta\theta\|_{L^2}^{\frac{1}{2}} \\ & + C \|\nabla_h \theta\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{4}} \|\nabla \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{4}} \|\nabla\theta\|_{L^2}^{\frac{1}{4}} \|\nabla \nabla_h \theta\|_{L^2}^{\frac{1}{2}} \|\Delta\theta\|_{L^2}^{\frac{1}{4}} + C \|\theta\|_{L^2}^2. \end{aligned}$$

Integrating the above inequality in time variable over $0 \leq \tau \leq t$, one shows that

$$\begin{aligned} \mathcal{Z}(t) &\leq C \left(\sup_{0 \leq \tau \leq t} \|\nabla_h u\|_{L^2} \right) A_1(t) + C \left(\sup_{0 \leq \tau \leq t} \|\nabla_h u\|_{L^2} \right) A_2(t) \\ & \quad + C \left(\sup_{0 \leq \tau \leq t} \|\nabla_h \theta\|_{L^2} \right) (A_3(t) \times A_4(t)) + \|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2 \\ & \leq C \left(\|\nabla_h u_0\|_{L^2}^2 + \|\nabla_h \theta_0\|_{L^2}^2 + \mathcal{X}(t) \right) \times A_5(t) + \|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2, \end{aligned}$$

where we denote

$$\begin{aligned}
 A_1(t) &= \left(\int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \left(\int_0^t \|\nabla \nabla_h u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\Delta u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}}, \\
 A_2(t) &= \left(\int_0^t \|\nabla \theta(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \left(\int_0^t \|\nabla \nabla_h \theta(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\Delta \theta(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}}, \\
 A_3(t) &= \left(\int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{8}} \left(\int_0^t \|\nabla \nabla_h u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \left(\int_0^t \|\Delta u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{8}}, \\
 A_4(t) &= \left(\int_0^t \|\nabla \theta(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{8}} \left(\int_0^t \|\nabla \nabla_h \theta(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \left(\int_0^t \|\Delta \theta(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{8}}, \\
 A_5(t) &= \left(\int_0^t \left(\|\Delta u(\cdot, \tau)\|_{L^2}^2 + \|\Delta \theta(\cdot, \tau)\|_{L^2}^2 \right) d\tau \right)^{\frac{1}{4}}.
 \end{aligned}$$

By virtue of the Hölder and Young inequalities and energy inequality (1.3), we get

$$\begin{aligned}
\mathcal{Z}(t) &\leq \|\nabla u_0\|_{L^2}^2 + \|\nabla \theta_0\|_{L^2}^2 + C \left(\|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} + \|\nabla_h \theta_0\|_{L^2}^{\frac{8}{3}} \right) \\
&\quad + C \int_0^t \left(\|\nabla_h u(\tau)\|_{B_{\infty,\infty}}^{\frac{8}{3}-1} \|\nabla u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{4}{3}} \\
&\quad + C \int_0^t \left(\|\nabla_h u(\tau)\|_{B_{\infty,\infty}}^{\frac{8}{3}-1} \|\nabla \theta(\tau)\|_{L^2}^2 d\tau \right)^{\frac{4}{3}} \\
&\quad + C \int_0^t \left(\|\nabla_h \theta(\tau)\|_{B_{\infty,\infty}}^{\frac{8}{3}-1} \|\nabla \theta(\tau)\|_{L^2}^2 d\tau \right)^{\frac{4}{3}} \\
&\quad + C \int_0^t \left(\|\nabla_h \theta(\tau)\|_{B_{\infty,\infty}}^{\frac{8}{3}-1} \|\nabla u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{4}{3}} \\
&\leq \|\nabla u_0\|_{L^2}^2 + \|\nabla \theta_0\|_{L^2}^2 + C \|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} + C \|\nabla_h \theta_0\|_{L^2}^{\frac{8}{3}} \\
&\quad + C \int_0^t \left(\|\nabla_h u(\tau)\|_{B_{\infty,\infty}}^{\frac{8}{3}-1} \|\nabla u(\tau)\|_{L^2}^2 d\tau \right) \left(\int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{3}} \\
&\quad + C \int_0^t \left(\|\nabla_h u(\tau)\|_{B_{\infty,\infty}}^{\frac{8}{3}-1} \|\nabla \theta(\tau)\|_{L^2}^2 d\tau \right) \left(\int_0^t \|\nabla \theta(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{3}} \\
&\quad + C \int_0^t \left(\|\nabla_h \theta(\tau)\|_{B_{\infty,\infty}}^{\frac{8}{3}-1} \|\nabla \theta(\tau)\|_{L^2}^2 d\tau \right) \left(\int_0^t \|\nabla \theta(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{3}} \\
&\quad + C \int_0^t \left(\|\nabla_h \theta(\tau)\|_{B_{\infty,\infty}}^{\frac{8}{3}-1} \|\nabla u(\tau)\|_{L^2}^2 d\tau \right) \left(\int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{3}}.
\end{aligned}$$

Thanks to the energy inequality (1.3), we get

$$\mathcal{Z}(t) \leq \|\nabla u_0\|_{L^2}^2 + \|\nabla \theta_0\|_{L^2}^2 + C \|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} + C \|\nabla_h \theta_0\|_{L^2}^{\frac{8}{3}} + C\mathcal{W}(t).$$

Taking the Gronwall inequality into consideration, we arrive at

$$\mathcal{Z}(t) \leq \left(\|\nabla u_0\|_{L^2}^2 + \|\nabla \theta_0\|_{L^2}^2 + C \|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} + C \|\nabla_h \theta_0\|_{L^2}^{\frac{8}{3}} \right) e^{C\mathcal{K}(t)},$$

for any $t \in [0, T)$, which implies that

$$(1.15) \quad \sup_{0 \leq t \leq T} \left(\|\nabla u(\cdot, t)\|_{L^2}^2 + \|\nabla \theta(\cdot, t)\|_{L^2}^2 \right) < \infty.$$

In the end, by the standard arguments of continuation of local solutions, we complete the proof of Theorem 1.1. \square

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