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ROTA-BAXTER OPERATORS AND NON-SKEW-SYMMETRIC  
SOLUTIONS OF THE CLASSICAL YANG-BAXTER EQUATION  
ON QUADRATIC LIE ALGEBRAS.

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ABSTRACT. We study possible connections between Rota-Baxter operators of non-zero weight and non-skew-symmetric solutions of the classical Yang-Baxter equation on finite-dimensional quadratic Lie algebras. The particular attention is made to the case when for a solution  $r$  the element  $r + \tau(r)$  is  $L$ -invariant.

**Keywords:** Rota-Baxter operator, quadratic Lie algebra, non-associative bialgebra, classical Yang-Baxter equation.

Let  $A$  be an arbitrary algebra over a field  $F$ ,  $\lambda \in F$ . A map  $R : A \mapsto A$  is called a Rota-Baxter operator of weight  $\lambda$  if for all  $x, y \in A$ :

$$(1) \quad R(x)R(y) = R(R(x)y + xR(y) + \lambda xy).$$

As an example of a Rota-Baxter operator of weight zero one can consider the operation of integration on the algebra of continuous functions on  $\mathbb{R}$ : the equation (1) follows from the integration by parts formula.

Rota-Baxter operators for associative algebras first appears in the paper of G. Baxter as a tool for studying integral operators that appears in the theory of probability and mathematical statistics [1]. The combinatorial properties of Rota-Baxter algebras and operators were studied in papers of F.V. Atkinson, P. Cartier, G.-C. Rota and the others (see [2]-[5]). For basic results and the main properties of Rota-Baxter algebras see [6].

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Independently, in early 80-th Rota-Baxter operators on Lie algebras naturally appears in papers of A.A. Belavin, V.G. Drinfeld [7] and M.A. Semenov-Tyan-Shanskii [8] while studying the solutions of the classical Yang-Baxter equation.

There is a standard method for constructing Rota-Baxter operations of weight zero on a quadratic (that is, possessing a non-degenerate invariant form  $\omega$ ) Lie algebra  $(L, \omega)$  from skew-symmetric solutions of the classical Yang-Baxter equations (CYBE): if  $r = \sum a_i \otimes b_i$  is a skew-symmetric solution of CYBE (that is,  $\tau(r) = -r$  where  $\tau$  is the switch morphism), then one can define an operator  $R$  on  $L$  by

$$R(a) = \sum \omega(b_i, a) a_i.$$

It turns out that  $R$  is a Rota-Baxter operator of weight zero [7, 8]. Moreover, Rota-Baxter operators of weight zero that can be obtained from skew-symmetric solutions of CYBE can be easily described: they satisfy  $R + R^* = 0$  where  $R^*$  is the adjoint operator of  $R$  with respect to the form  $\omega$ .

The case when  $r$  is a non-skew-symmetric solution of the classical Yang-Baxter equation was considered in [9]. It was proved that if  $L$  is a simple Lie algebra and  $r$  is a solution of CYBE such that the element  $r + \tau(r)$  is  $L$ -invariant, then there is a non-degenerate invariant form  $\omega$  on  $L$  such that the corresponding to  $r$  linear map  $R$  is a Rota-Baxter operator of a non-zero weight  $\lambda$ . In this paper, we complement this result by proving that the obtained operator  $R$  satisfies  $R + R^* + \lambda id = 0$ , where  $\lambda$  is the weight of  $R$  and  $id$  is the identity operator.

There is a connection between solutions of CYBE on Lie algebras and Lie bialgebras that were introduced by Drinfeld [10] for studying the solutions of the classical Yang-Baxter equation on Lie algebras. If  $L$  is a Lie algebra and  $r = \sum_i a_i \otimes b_i \in L \otimes L$ , then one can define a comultiplication  $\delta_r : L \rightarrow L \otimes L$  as

$$\delta_r(a) = \sum [a_i, a] \otimes b_i + a_i \otimes [b_i, b].$$

If the element  $r$  is a solution of CYBE and  $r + \tau(r)$  is  $L$ -invariant, then the pair  $(L, \delta_r)$  is a Lie bialgebra. Bialgebras of this type are called quasitriangular (or triangular, if  $r$  is skew-symmetric). Triangular Lie bialgebras play an important role as they lead to solutions of quantum Yang-Baxter equation.

In the current paper, we consider the correspondence between solutions of CYBE and Rota-Baxter operators of non-zero weight on arbitrary quadratic Lie algebra  $(L, \omega)$ . In section 3 we describe Rota-Baxter operators that can be obtained from arbitrary solutions of CYBE on arbitrary (not necessarily simple) quadratic finite-dimensional Lie algebra. Unlike the case of simple Lie algebras, here  $R + R^* + \lambda id$  is not necessarily equal to zero.

In section 4, we study the connection between Rota-Baxter operators of nonzero weights and solutions of CYBE for which  $r + \tau(r)$  is  $L$ -invariant. It turns out that the corresponding Rota-Baxter operators also do not necessarily satisfy the equality  $R + R^* + \lambda id = 0$ . Nevertheless, in this case for such a Rota-Baxter operator  $R$  there is an ideal  $I \subset [L, L]$  such that  $R(I) = 0$  and the restriction of  $R$  on the quotient algebra  $L/I$  satisfy  $R([a, b]) + R^*([a, b]) + \lambda[a, b] = 0$  for all  $a, b \in L/I$ .

In section 4 we also consider the situation when for a solution  $r$  of CYBE the map  $R$  and the adjoint map  $R^*$  are Rota-Baxter operators of the same nonzero weight.

1. DEFINITIONS AND PRELIMINARY RESULTS

Throughout the paper the characteristic of the ground field  $F$  is 0 and all spaces are supposed to be finite-dimensional.

Given vector spaces  $V$  and  $U$  over a field  $F$ , denote by  $V \otimes U$  its tensor product over  $F$ . Define the linear mapping  $\tau$  on  $V \otimes V$  by  $\tau(\sum_i a_i \otimes b_i) = \sum_i b_i \otimes a_i$ . Denote by  $V^*$  the dual space of  $V$ . For  $f \in V^*$  and  $v \in V$  by  $\langle f, v \rangle$  we will denote the action of  $f$  on the vector  $v$ , that is  $\langle f, v \rangle = f(v)$ .

Let  $L$  be a Lie algebra with a multiplication  $[\cdot, \cdot]$ . By  $Z(L)$  we will denote the center of  $L$ . Recall, that  $L$  acts on  $L^{\otimes n}$  by

$$[x_1 \otimes x_2 \otimes \dots \otimes x_n, y] = \sum_i x_1 \otimes \dots \otimes [x_i, y] \otimes \dots \otimes x_n$$

for all  $x_i, y \in L$ .

**Definition 1.** An element  $r \in L^{\otimes n}$  is called  $L$ -invariant (or ad-invariant) if  $[r, y] = 0$  for all  $y \in L$ .

For the special linear Lie algebra  $sl_2(F)$  we will denote by  $e, h, f$  the standard basis of  $sl_2(F)$ :  $e = e_{12}, h = e_{11} - e_{22}, f = e_{21}$ .

The next definition of Lie bialgebra that was given in [10].

**Definition 2.** Let  $L$  be a Lie algebra with a comultiplication  $\delta$ . The pair  $(L, \delta)$  is called a Lie bialgebra if and only if  $(L, \delta)$  is a Lie coalgebra and  $\delta$  is a 1-cocycle, i.e., it satisfies

$$\delta([a, b]) = \sum([a_{(1)}, b] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b]) + \sum([a, b_{(1)}] \otimes b_{(2)} + b_{(1)} \otimes [a, b_{(2)}])$$

for all  $a, b \in L$ .

There is an important type of Lie bialgebras called coboundary Lie bialgebras. Namely, let  $L$  be a Lie algebra and  $r = \sum_i a_i \otimes b_i \in L \otimes L$ . Define a comultiplication  $\delta_r$  on  $L$  by

$$\delta_r(a) = [r, a] = \sum_i [a_i, a] \otimes b_i + a_i \otimes [b_i, a]$$

for all  $a \in L$ . It is easy to see that  $\delta_r$  is a 1-cocycle.

Define an element  $C_L(r)$  as

$$C_L(r) = [r_{12}, r_{13}] - [r_{23}, r_{12}] + [r_{13}, r_{23}].$$

Here  $[r_{12}, r_{13}] = \sum_{ij} [a_i, a_j] \otimes b_i \otimes b_j$ ,  $[r_{23}, r_{12}] = \sum_{ij} a_i \otimes [a_j, b_i] \otimes b_j$ , and  $[r_{13}, r_{23}] = \sum_{ij} a_i \otimes a_j \otimes [b_i, b_j]$ .

The dual algebra  $L^*$  of the coalgebra  $(L, \delta_r)$  is skew-symmetric if and only if  $r + \tau(r)$  is  $L$ -invariant. Also,  $L^*$  satisfies the Jacobi identity if and only if  $C_L(r)$  is  $L$ -invariant. In particular, if  $r = \sum a_i \otimes b_i \in L \otimes L$  satisfy

$$(2) \quad \sum_{ij} [a_i, a_j] \otimes b_i \otimes b_j - a_i \otimes [a_j, b_i] \otimes b_j + a_i \otimes a_j \otimes [b_i, b_j] = 0,$$

and  $r + \tau(r)$  is  $L$ -invariant, then  $(L, \delta_r)$  is a Lie bialgebra called quasitriangular Lie bialgebra (or triangular, if  $\tau(r) = -r$ ). The equation (2) is called *the classical Yang–Baxter equation*.

Note that the equation (2), as well as the corresponding to it bialgebra structures, can be considered for every variety of algebras. For Jordan, associative, alternative and Malcev bialgebras it is known that if  $r$  is a skew-symmetric solution of the classical Yang–Baxter equation on an algebra  $A$ , then  $(A, \delta_r)$  is a bialgebra of corresponding variety [12, 13, 14, 15, 16, 17, 18]. It is also worth mentioning papers of C. Bai, L. Guo and X. Ni [19, 20] where it was studied the connection between coboundary Lie bialgebras with a generalization of CYBE (extended CYBE) and with a generalization of Rota-Baxter operators of nonzero weight (extended  $\mathcal{O}$ -operators).

**Definition 3.** Let  $L$  be a Lie algebra. A bilinear form  $\omega$  on  $L$  is called invariant if  $\omega([a, b], c) = \omega(a, [b, c])$  for all  $a, b, c \in L$ .

**Definition 4.** Let  $L$  be a Lie algebra and  $\omega$  be an invariant non-degenerate form on  $L$ . Then the pair  $(L, \omega)$  is called a quadratic Lie algebra.

Let  $(L, \omega)$  be a quadratic finite-dimensional Lie algebra. Then the dual space  $L^*$  is naturally isomorphic to  $L$  and we may present  $L^*$  as  $L^* = \{a^* \mid a \in L\}$ , where  $\langle a^*, b \rangle = \omega(a, b)$  for all  $a, b \in L$ . Note that the invariance of the form  $\omega$  implies  $\langle [a, b]^*, c \rangle = \langle a^*, [b, c] \rangle = \langle b^*, [c, a] \rangle$ .

Also, if  $(L, \omega)$  is a quadratic finite-dimensional Lie algebra, then there is a natural isomorphism between the space of endomorphisms  $End_F(L)$  and  $L \otimes L$ : for every  $\varphi \in End_F(L)$  there is a unique  $r = \sum_i a_i \otimes b_i \in L \otimes L$  such that  $\varphi(a) = \sum_i \omega(b_i, a) a_i$  for all  $a \in L$ .

There is a connection between skew-symmetric solutions of the classical Yang–Baxter equation and Rota-Baxter operators on quadratic Lie algebras: if  $(L, \omega)$  is a quadratic Lie algebra and  $r = \sum_i a_i \otimes b_i \in L \otimes L$  is a skew-symmetric solution of the classical Yang–Baxter equation on  $L$ , then an operator  $R$  defined as

$$(3) \quad R(a) = \sum_i \omega(b_i, a) a_i$$

for all  $a \in L$  is a Rota-Baxter operator of weight zero [7, 8]. Moreover, if  $R^*$  is the adjoint operator of  $R$  with respect to the form  $\omega$ , then  $R + R^* = 0$ .

In [9] it was proved that if  $L$  is a simple Lie algebra,  $r = \sum a_i \otimes b_i$  is a non-skew-symmetric solution of CYBE such that  $\tau(r) + r$  is  $L$ -invariant, then there is a non-degenerate bilinear invariant form  $\omega$  on  $L$  such that the operator  $R$  defined as in (3) is a Rota-Baxter operator of a non-zero weight.

If  $L$  is not a simple quadratic Lie algebra and  $r$  is a non-skew-symmetric solution of CYBE then everything is possible: as the following examples show, the corresponding map  $R$  may be a Rota-Baxter operator of zero or non-zero weight, or not a Rota-Baxter operator at all.

**Example 1.** Consider  $L = gl_2(\mathbb{C})$  — Lie algebra of  $2 \times 2$  matrices over  $F$  and  $r = \frac{1}{2}(E \otimes e_{11} + e_{22} \otimes E)$ , where  $E$  is the identity matrix. It is easy to see that  $r$  is a not skew-symmetric solution of CYBE and  $r + \tau(r) = E \otimes E$  is  $L$ -invariant. But the operator  $R$  defined by (3) is a Rota-Baxter operator of weight zero.

**Example 2.** Let  $r_i \in L_i \otimes L_i$  ( $i = 1, 2$ ) be a solution of CYBE on simple complex Lie algebra  $L_i$  such that  $r_i + \tau(r_i)$  is  $L_i$ -invariant. Let  $L = L_1 \oplus L_2$  be a semisimple complex Lie algebra and define  $r = r_1 + r_2$ . In this case  $r$  is a solution of CYBE on  $L$ ,  $r + \tau(r)$  is  $L$ -invariant and if at least one  $i \in \{1, 2\}$  satisfies  $r_i + \tau(r_i) \neq 0$ , then  $r + \tau(r) \neq 0$ .

It is known that every non-degenerate invariant form  $\omega$  on  $L$  may be presented as  $\omega = \alpha_1\chi_1 + \alpha_2\chi_2$ , where  $\chi_i$  is the Killing form on  $L_i$  and  $\alpha_i \neq 0$ .

In [9] it was proved that if  $r_i + \tau(r_i) \neq 0$ , then the operator  $R_i$ , defined as in (3) (where  $\omega = \chi_i$ ) is a Rota-Baxter operator of some nonzero weight  $\lambda_i$ . And if  $r_i + \tau(r_i) \neq 0$  for  $i = 1, 2$ , then we can find nonzero scalars  $\mu_i \in \mathbb{C}$  ( $i = 1, 2$ ) such that for the form  $\omega = \mu_1\chi_1 + \mu_2\chi_2$  the operator  $R$  defined by (3) is a Rota-Baxter operator of some nonzero weight  $\lambda$ .

But if  $r_1 + \tau(r_1) = 0$  and  $r_2 + \tau(r_2) \neq 0$ , then for every form  $\omega$  the restriction of the corresponding operator  $R$  to  $L_1$  is a Rota-Baxter operator of weight zero while the restriction of  $R$  on  $L_2$  will be a Rota-Baxter operator of some nonzero weight  $\lambda$ . Therefore, in this case  $R$  is not a Rota-Baxter operator.

2. CYBE AND ROTA-BAXTER OPERATORS ON QUADRATIC LIE ALGEBRAS.

Let  $(L, \omega)$  be a quadratic Lie algebra and  $r = \sum_i a_i \otimes b_i \in L \otimes L$ . Define an operator  $R : L \mapsto L$  as in (3). By  $R^*$  we denote the adjoint operator with respect to the form  $\omega$ .

**Proposition 1.** *Let  $r$  be a solution of the CYBE on  $L$ . Then the following equalities hold for all  $x, y \in L$ :*

1.  $[R(x), R(y)] - R([x, R(y)]) + R([R^*(x), y]) = 0,$
2.  $[R^*(x), R^*(y)] + R^*([x, R(y)]) - R^*([R^*(x), y]) = 0,$
3.  $r + \tau(r)$  is  $L$ -invariant if and only if  $R + R^*$  lies in the centroid of  $L$ , that is

$$[(R + R^*)(x), y] = (R + R^*)([x, y]).$$

*Proof.* 1. For  $x, y \in L$  consider a map  $\psi_{x,y} : L \otimes L \otimes L \mapsto L$  defined as

$$\psi_{x,y}(a \otimes b \otimes c) = \omega(x, b)\omega(y, c)a$$

for all  $a, b, c \in L$ .

Since  $r$  is a solution of CYBE, then for all  $x, y \in L$  we have  $\psi_{x,y}(C_L(r)) = 0$ . Direct computations show

$$\begin{aligned} 0 &= \psi_{x,y}(C_L(r)) \\ &= \sum_{i,j} [a_i, a_j]\omega(b_i, x)\omega(b_j, y) - a_i\omega([a_j, b_i], x)\omega(b_j, y) + a_i\omega(a_j, x)\omega([b_i, b_j], y) \\ &= [R(x), R(y)] - R([x, R(y)]) + R([R^*(x), y]). \end{aligned}$$

2. The proof is similar to 1.

3. Let  $a, b \in L$  and consider a map  $\psi_x = L \otimes L \mapsto L$  defined as

$$\psi_x(a \otimes b) = \omega(x, a)b.$$

Since the form  $\omega$  is non-degenerate, an element  $h \in L \otimes L$  is equal to zero if and only if  $\psi_x(h) = 0$  for all  $x \in L$ .

Now let  $x, y \in L$ . Direct computation shows:

$$\begin{aligned} \psi_x([r + \tau(r), y]) &= \omega([a_i, y], x)b_i + \omega(a_i, x)[b_i, y] + \omega([b_i, y], x)a_i + \omega(b_i, x)[a_i, y] = \\ &= R^*([y, x]) + [R^*(x), y] + R([y, x]) + [R(x), y]. \end{aligned}$$

□

The next theorem gives necessary and sufficient conditions when  $R$  is a Rota-Baxter operator of weight  $\lambda$ .

**Theorem 1.** *Let  $(L, \omega)$  be a quadratic Lie algebra over a field  $F$  and  $r = \sum a_i \otimes b_i \in L \otimes L$ . Let  $R$  be the operator defined as in (3) and  $R^*$  be the adjoint operator of  $R$  with respect to the form  $\omega$ . Then*

1. *If  $r$  is a solution of CYBE on  $L$ , then  $R$  is a Rota-Baxter operator of a weight  $\lambda$  if and only if for all  $a, b \in L$ :*

$$(4) \quad [R(a), b] + [R^*(a), b] + \lambda[a, b] \in \ker(R).$$

*This is equivalent to:*

$$\sum \omega(b_j, a) \omega([b_i, a_j], b) a_i + \omega(a_j, a) \omega([b_i, b_j], b) a_i + \lambda \omega(b_i, [a, b]) a_i = 0.$$

2. *Conversely, if  $R$  is a Rota-Baxter operator of a weight  $\lambda$  on  $L$ , then  $r$  is a solution of CYBE on  $L$  if and only if the equality (4) holds for all  $a, b \in L$ .*

*Proof.* The first statement follows from Proposition 1. In order to prove the second statement consider a map  $\psi_{x,y,z} : L \otimes L \otimes L \mapsto L$  defined as  $\psi_{x,y,z}(a \otimes b \otimes c) = \omega(x, a) \omega(y, b) \omega(z, c)$ . It is easy to see that an element  $h \in L \otimes L \otimes L$  is equal to zero if and only if  $\psi_{x,y,z}(h) = 0$  for all  $x, y, z \in L$ . Using the technique from the proof of Proposition 1, we have

$$\begin{aligned} \psi_{x,y,z}(C_L(r)) &= \omega([R(x), R(y)] - R([x, R(y)]) + R([R^*(x), y]), z) = \\ &= \omega([R(x), R(y)] - R([x, R(y)]) - R([R(x), y]) - R(\lambda[x, y]), z) + \\ &\quad + \omega(R([R^*(x), y]) + R([R(x), y]) + R(\lambda[x, y])), z) = \\ &= \omega(R([R^*(x), y]) + R([R(x), y]) + R(\lambda[x, y])), z). \end{aligned}$$

Thus,  $C_L(r) = 0$  if and only if  $R([R^*(x), y]) + R([R(x), y]) + R(\lambda[x, y]) = 0$  for all  $x, y \in L$ . □

**Remark.** The statement of Theorem 1 can be formulated for  $R^*$ . The condition (4) in this case should be replaced by

$$[R(a), b] + [R^*(a), b] + \lambda[a, b] \in \ker(R^*).$$

**Corollary 1.** *Let  $(L, \omega)$  be a quadratic Lie algebra and  $R$  be a Rota-Baxter operator of a nonzero weight  $\lambda$ . If*

$$R(a) + R^*(a) + \lambda a \in Z(L),$$

*for all  $a \in L$ , then the corresponding to  $R$  tensor  $r$  is a solution of CYBE.*

**Example 3.** Let  $L = sl_2(F)$  and  $\chi$  be the Killing form on  $L$ ,  $F[t]$  be the algebra of polynomials in the variable  $t$  and  $F_2[t] = F[t]/(t^2)$  be the quotient algebra modulo the ideal generated by  $t^2$ . Define  $L_2$  as the tensor product  $L \otimes_F F_2[t]$ . The product on  $L_2$  is defined as

$$[a \otimes \overline{f(t)}, b \otimes \overline{g(t)}] = [a, b] \otimes \overline{f(t)g(t)}$$

where  $[a, b]$  is the product in  $L$  and  $\bar{\cdot} : F[t] \mapsto F[t]/(t^2)$  is the natural homomorphism. Define a form  $\omega$  on  $L_2$  as

$$\omega(a \otimes \overline{f(t)}, b \otimes \overline{g(t)}) = \chi(a, b) \pi(f(t)g(t)),$$

where  $\pi(f(t)) = f_0 + f_1$  whenever  $f(t) = f_0 + f_1 t + \dots + f_n t^n$ .

Then  $(L_2, \omega)$  is quadratic Lie algebra [21]. We can identify elements  $a \otimes \bar{1}$  with  $a$  and  $a \otimes \bar{t}$  with  $a\bar{t}$  and present  $L_2$  as sum of the subalgebras  $L$  and  $L\bar{t}$ :  $L_2 = L \oplus L\bar{t}$ . Define an operator  $R : L_2 \mapsto L_2$  as

$$R(a) = a, \quad R(a\bar{t}) = 0$$

for all  $a \in L$ .

It is known that  $R$  is a Rota-Baxter operator of weight  $-1$  (see [6]). Direct computations show that for all  $a \in L$ :  $R^*(a) = R^*(a\bar{t}) = a\bar{t}$ . We have:

$$R(a) + R^*(a) - a = a\bar{t}, \quad R(a\bar{t}) + R^*(a\bar{t}) - a\bar{t} = 0.$$

It means that  $[R(x) + R^*(x) - x, L_2] = L\bar{t} \subseteq \ker(R)$  for all  $x \in L_2$ . The map  $R$  is defined by the tensor  $r = \frac{1}{4}h \otimes h\bar{t} + e \otimes f\bar{t} + f \otimes e\bar{t}$ . By Theorem 1  $r$  is a solution of CYBE.

Note, that  $R^*$  is not a Rota-Baxter operator. And in order to prove this now it is not necessary to check (1). Indeed, we know, that  $r$  is a solution of CYBE and by Theorem 1  $R^*$  is a Rota-Baxter operator of a weight  $\lambda$  if and only if  $[R(x), y] + [R^*(x), y] + \lambda[x, y] \in \ker(R^*)$  for all  $x, y \in L_2$ . We have already proved that  $[R(x) + R^*(x) - x, L_2] = L\bar{t}$  so  $R^*$  is not a Rota-Baxter operator of weight  $-1$ . Let  $\lambda \neq -1$ . Then for  $a, b \in L$ :

$$R^*([R(a\bar{t}) + R^*(a\bar{t}) + \lambda a\bar{t}, b]) = (1 + \lambda)R^*([a, b]\bar{t}) = (1 + \lambda)[a, b]\bar{t}$$

that proves that  $R^*$  is not a Rota-Baxter operator.

**Example 4.** Let  $(L_2, \omega)$  be the quadratic Lie algebra from Example 3. Define a map  $Q$  as the projection on the subalgebra  $L\bar{t}$  with the kernel  $\ker(Q) = L$ . Then  $Q$  is a Rota-Baxter operator of weight  $-1$ . Note, that  $Q = id - R$ , where  $id$  is the identity operator and  $R$  is the Rota-Baxter operator from Example 3. It means that  $Q^* = id - R^*$ , that is  $Q^*(a) = a - a\bar{t}$ ,  $Q^*(a\bar{t}) = 0$  for all  $a, b \in L$ . But

$$Q([Q(a) + Q^*(a) - a, b]) = Q([a, b] - [a, b]\bar{t}) = -[a, b]\bar{t}$$

for all  $a, b \in L$ . And element  $h = \frac{1}{4}h\bar{t} \otimes (h - h\bar{t}) + x\bar{t} \otimes (y - y\bar{t}) + y\bar{t} \otimes (x - x\bar{t})$  is not a solution of CYBE on  $L_2$  by Theorem 1.

### 3. SOLUTIONS OF CYBE WITH $L$ -INVARIANT SYMMETRIC PART AND ROTA-BAXTER OPERATORS ON QUADRATIC LIE ALGEBRAS.

In this section, we will consider possible connections between solutions of CYBE such that  $r + \tau(r)$  is  $L$ -invariant and Rota-Baxter operators on quadratic Lie algebra  $(L, \omega)$ . Note, that it is known that skew-symmetric solutions of CYBE are in one to one correspondence with Rota-Baxter operators  $R$  of weight zero satisfying  $R + R^* = 0$  (see [7, 8]). We will study the case when  $r + \tau(r) \neq 0$ .

Unless otherwise is specified, throughout this section  $(L, \omega)$  is a finite-dimensional quadratic Lie algebra over a field  $F$ ,  $r = \sum a_i \otimes b_i \in L \otimes L$ ,  $R$  is the operator defined as in (3) and  $R^*$  is the adjoint operator of  $R$  with respect to the form  $\omega$  operator.

**Definition.** Given a linear map  $R : L \mapsto L$  and a scalar  $\alpha \in F$  define an operator

$$\theta_\alpha = R + R^* + \alpha id,$$

where  $id$  is the identity operator.

**Theorem 2.** Let  $r = \sum a_i \otimes b_i \in L \otimes L$  be a solution of CYBE on  $L$  such that  $r + \tau(r)$  is  $L$ -invariant. Then a set

$$(5) \quad I_\lambda = \{\theta_\lambda(x) : x \in [L, L]\}$$

is an ideal in  $L$  for every  $\lambda \in F$ . If  $I_\lambda$  is  $R$ -invariant, then  $I_\lambda$  is also  $R^*$ -invariant and the corresponding induced by  $R$  and  $R^*$  operators on the quotient algebra  $L/I_\lambda$  are Rota-Baxter operators of weight  $\lambda$ . Moreover, in this case we have:

$$(6) \quad \overline{\theta_\lambda(x)} = \overline{R(x) + R^*(x) + \lambda x} = 0 \text{ for all } x \in [L/I_\lambda, L/I_\lambda].$$

*Proof.* Consider  $\lambda \in F$  and define

$$I_\lambda = \{R(x) + R^*(x) + \lambda x \mid x \in [L, L]\}.$$

For all  $z \in L$ , using Proposition 1, we have:

$$\begin{aligned} [R([x, y]) + R^*([x, y]) + \lambda[x, y], z] &= [R([x, y]), z] + [R^*([x, y]), z] + \lambda[x, y], z] = \\ &= R([x, y], z) + R^*([x, y], z) + \lambda[x, y], z] \in I_\lambda. \end{aligned}$$

Thus,  $I_\lambda$  is an ideal in  $L$ . Note, that by Proposition 1:

$$R[x, y] + R^*([x, y]) + \lambda[x, y] = [R(x) + R^*(x) + \lambda x, y] \subset [L, L].$$

Suppose that  $R(I_\lambda) \subset I_\lambda$ . First we prove that  $I_\lambda$  is also an  $R^*$ -invariant ideal. Consider  $\theta_\lambda([x, y]) \in I_\lambda$ . We have

$$\begin{aligned} R(\theta_\lambda([x, y])) + R^*(\theta_\lambda([x, y])) &= R([R(x) + R^*(x) + \lambda x, y]) + R^*([R(x) + R^*(x) + \lambda x, y]) = \\ &= R([R(x) + R^*(x), y]) + R^*([R(x) + R^*(x), y]) + \lambda(R([x, y]) + R^*([x, y])) = \\ &= R([R(x) + R^*(x), y]) + R^*([R(x) + R^*(x), y]) + \lambda[R(x) + R^*(x), y] \in I_\lambda. \end{aligned}$$

Since  $R(\theta_\lambda([x, y])) \in I_\lambda$ , therefore  $R^*(\theta_\lambda([x, y])) \in I_\lambda$ .

Consider the quotient algebra  $L/I_\lambda$ . For simplicity we will denote the induced by  $R$  and  $R^*$  operators on  $L/I_\lambda$  as  $R$  and  $R^*$  respectively. Since  $r$  is a solution of CYBE on  $L$ , then by Proposition 1 :

$$[R(a), R(b)] - R([a, R(b)]) + R([R^*(a), b]) = 0$$

for all  $a, b \in L$ . Consequently, the same equation holds in the  $L/I_\lambda$  for all  $a, b \in L/I_\lambda$ . By the definition of the quotient algebra  $L/I_\lambda$ :  $[R^*(a), b] = -[R(a), b] - \lambda[a, b]$  for all  $a, b \in L/I_\lambda$ . Thus,  $R$  is a Rota-Baxter operator of weight  $\lambda$  on  $L/I_\lambda$ .  $\square$

**Corollary 2.** *Let  $r = \sum a_i \otimes b_i \in L \otimes L$  be a solution of CYBE on  $L$  such that  $r + \tau(r)$  is  $L$ -invariant. If  $R$  is a Rota-Baxter operator of a weight  $\lambda$  then there is an ideal  $I_\lambda$  in  $L$  satisfying  $I_\lambda \subset [L, L]$  and  $R(I_\lambda) = 0$ . The induced by  $R$  operator on the quotient algebra  $L/I_\lambda$  is a Rota-Baxter operator of weight  $\lambda$  satisfying (6).*

*Conversely, let  $R$  be a Rota-Baxter operator of weight  $\lambda$  and for all  $a, b \in L$ :  $R([a, b]) + R^*([a, b]) = [R(a), b] + [R^*(a), b]$ . Suppose there is an ideal  $I_\lambda$  in  $L$  satisfying  $I_\lambda \subset [L, L]$  and  $R(I_\lambda) = 0$ . Suppose in addition that the restriction of the operator  $R$  on the quotient algebra  $L/I_\lambda$  is a Rota-Baxter operator of weight  $\lambda$  satisfying (6). Then  $r$  is a solution of the CYBE and  $r + \tau(r)$  is a nonzero  $L$ -invariant element.*

*Proof.* Define  $I_\lambda$  as in (5). By Theorem 1  $R(I_\lambda) = 0$  so  $I_\lambda$  is  $R$ -invariant. The remaining assertions of the first statement follows from Theorem 2.

Now suppose that  $R$  is a Rota-Baxter operator of weight  $\lambda$  and  $R([a, b]) + R^*([a, b]) = [R(a), b] + [R^*(a), b]$  for all  $a, b \in L$ . By Proposition 1 the element  $r + \tau(r)$  is  $L$ -invariant. Let  $x \in [L, L]$ . In the quotient algebra  $L/I_\lambda$  we have that  $\overline{\theta_\lambda(x)} = 0$ . Thus,  $\theta_\lambda(x) \in I_\lambda$  and since  $R(I_\lambda) = 0$  we have that  $R(\theta_\lambda(x)) = 0$  for all  $x \in [L, L]$ . By Theorem 1,  $r$  is a solution of CYBE on  $L$ .  $\square$

We can use the obtained results and specify the main result of [9]:

**Corollary 3.** *Let  $L$  be a simple Lie algebra and  $r = \sum_i a_i \otimes b_i \neq 0$  be a solution of CYBE (2) such that  $r + \tau(r) \neq 0$  and  $r + \tau(r)$  is  $L$ -invariant. Then there exists a non-degenerate symmetric invariant bilinear form  $\omega$  on  $L$  such that an operator  $R : L \rightarrow L$  defined as*

$$R(a) = \sum_i \omega(b_i, a) a_i$$

is a Rota–Baxter operator of a nonzero weight  $\lambda$  and for all  $a \in L$ :

$$R(a) + R^*(a) + \lambda a = 0.$$

*Proof.* The existence of the form  $\omega$  was proved in [9] (Theorem 4). Consider the ideal  $I_\lambda$  defined as in (5). Since  $L$  is simple,  $I_\lambda = L$  or  $I_\lambda = 0$ . Since  $R$  is a Rota-Baxter operator of weight  $\lambda$ , then by Theorem 1  $R(I_\lambda) = 0$ . If  $I_\lambda = L$  then  $R = 0$  that is impossible since  $r \neq 0$ . Thus,  $I_\lambda = 0$  and by Corollary 1  $R(a) + R^*(a) + \lambda a = 0$  for all  $a \in L$ .  $\square$

**Theorem 3.** *Let  $r = \sum a_i \otimes b_i$  be a non-skew-symmetric solution of CYBE such that  $r + \tau(r)$  is  $L$ -invariant. Then  $R$  and  $R^*$  are Rota-Baxter operators of the same nonzero weight  $\lambda$  if and only if the commutator ideal  $[L, L]$  is the sum of two ideals*

$$[L, L] = I_1 \oplus I_2$$

such that  $R(I_1) = R^*(I_1) = 0$  and

$$R(x) + R^*(x) + \lambda x = 0$$

for all  $x \in I_2$ .

*Conversely, suppose that  $R$  and  $R^*$  are Rota-Baxter operators of the same nonzero weight  $\lambda$  such that  $R([a, b]) + R^*([a, b]) = [R(a), b] + [R^*(a), b]$  for all  $a, b \in L$ . Suppose that the commutator ideal  $[L, L]$  is the sum of two ideals  $[L, L] = I_1 \oplus I_2$  and the ideals  $I_j$  satisfy the same properties as above. Then  $r$  is a solution of the CYBE and  $r + \tau(r)$  is a nonzero  $L$ -invariant element.*

*Proof.* Suppose that  $R$  and  $R^*$  are Rota-Baxter operators of the same nonzero weight  $\lambda$ . By Theorem 1,  $R(\theta_\lambda([a, b])) = R^*(\theta_\lambda([a, b])) = 0$  for all  $a, b \in L$ .

Let  $I_1 = \{\theta_\lambda(a) \mid a \in [L, L]\}$ . From Proposition 1,  $\theta_\lambda([a, b]) = [\theta_\lambda(a), b] = [a, \theta_\lambda(b)]$ . Therefore,  $I_1$  is an ideal in  $L$ . Note, that  $R(I_1) = R^*(I_1) = 0$ .

Let  $I_2 = \{a \in [L, L] \mid \theta_\lambda(a) = 0\}$ . From Proposition 1,  $I_2$  is an ideal in  $L$ .

If  $a \in I_1 \cap I_2$ , then  $R(a) = R^*(a) = 0$ . But  $0 = \theta_\lambda(a) = R(a) + R^*(a) + \lambda a = \lambda a$ . Thus,  $I_1 \cap I_2 = 0$ .

Let  $x \in [L, L]$ . Then  $\theta_\lambda(x - \frac{1}{\lambda} \theta_\lambda(x)) = \theta_\lambda(x) - \theta_\lambda(x) = 0$ . Thus,  $x - \frac{1}{\lambda} \theta_\lambda(x) \in I_2$ . It means that  $[L, L] = I_1 \oplus I_2$ .

Conversely, suppose that  $L = I_1 \oplus I_2$  for ideals  $I_1$  and  $I_2$  satisfying the condition of the Theorem. Then  $\theta_\lambda([a, b]) = 0$  if  $a \in I_2$  or  $b \in I_2$ . Let  $a, b \in I_1$ . We have

$$R(\theta_\lambda([a, b])) = R(\lambda[a, b]) = 0 = R^*(\theta_\lambda([a, b])).$$

Thus,  $R(\theta_\lambda([a, b])) = R^*(\theta_\lambda([a, b])) = 0$  and  $R$  and  $R^*$  are Rota-Baxter operators of the same nonzero weight  $\lambda$  by Theorem 1.

Consider the inverse statement. For  $x \in [L, L]$  we have  $x = i_1 + i_2$ , where  $i_j \in I_j$ . It means that  $\theta_\lambda(x) = \theta_\lambda(i_1) + \theta_\lambda(i_2) = \lambda i_1 \in \ker(R)$ . Therefore,  $r$  is a solution of CYBE by Theorem 1 and by Proposition 1  $r + \tau(r)$  is  $L$ -invariant.  $\square$

**Corollary 4.** *Let  $(L, \omega)$  be a quadratic Lie algebra and  $R$  be a Rota-Baxter operator of a nonzero weight  $\lambda$ . Suppose that for all  $a \in L$*

$$R(a) + R^*(a) + \lambda a = 0.$$

*Then the corresponding tensor  $r$  is a solution of CYBE and  $r + \tau(r)$  is  $L$ -invariant.*

**Example 5.** Consider the quadratic Lie algebra  $(L_2, \omega)$  from Example 3. Direct computations show:

$$R(a) + R^*(a) = a + a\bar{t}, \quad R(a\bar{t}) + R^*(a\bar{t}) = a\bar{t},$$

for all  $a \in L$ . Thus, for all  $a, b \in L$ :

$$[R(a) + R^*(a), b] = [a + a\bar{t}, b] = R([a, b]) + R^*([a, b]),$$

$$[R(a) + R^*(a), b\bar{t}] = [a, b]\bar{t} = R([a, b\bar{t}]) + R^*([a, b\bar{t}]),$$

$$[R(a\bar{t}) + R^*(a\bar{t}), b\bar{t}] = 0 = R([a\bar{t}, b\bar{t}]) + R^*([a\bar{t}, b\bar{t}]).$$

This means that for the corresponding tensor  $r = \frac{1}{4}h \otimes h\bar{t} + x \otimes y\bar{t} + y \otimes x\bar{t}$  the element  $r + \tau(r)$  is  $L$ -invariant.

At the end of the section, we will specify the results in the case of algebraically closed field.

**Theorem 4.** *Let  $(L, \omega)$  be a quadratic Lie algebra over an algebraically closed field  $F$  and let  $r = \sum a_i \otimes b_i \in L \otimes L$  be a solution of CYBE such that  $r + \tau(r)$  is  $L$ -invariant. Then  $R$  and  $R^*$  are Rota-Baxter operators of the same nonzero weight  $\lambda$  if and only if  $L$  is the sum of ideals  $L = I_1 \oplus I_2$  such that for all  $a \in I_1$ :*

$$R(a) + R^*(a) + \lambda a \in Z(L)$$

*and for all  $x \in I_2$  and  $y \in L$ :*

$$R([x, y]) = R^*([x, y]) = 0.$$

*Conversely, suppose that  $R$  is a Rota-Baxter operator of a nonzero weight  $\lambda$ , such that for all  $a, b \in L$ :  $R([a, b]) + R^*([a, b]) = [R(a), b] + [R^*(a), b]$ . If  $L$  is a direct sum of two ideals  $L = I_1 \oplus I_2$  and the ideals  $I_j$  satisfy the same properties as above, then  $r$  is a solution of CYBE and  $r + \tau(r)$  is  $L$  invariant.*

*Proof.* Let  $R$  be a Rota-Baxter operator of weight  $\lambda$ . Consider the linear map  $\theta = \theta_0 = R + R^* : L \mapsto L$ . Since the ground field is algebraically closed,  $L$  is the sum of root subspaces

$$L = L(\alpha_1) \oplus \dots \oplus L(\alpha_k),$$

where  $\{\alpha_1, \dots, \alpha_k\}$  is the spectrum of  $\theta$  and  $L(\alpha_i) = \{v \in L \mid (\theta - \alpha_i \text{id})^{n_i}(v) = 0 \text{ for some } n_i\}$  is the root subspace corresponding to the eigenvalue  $\alpha_i$ . Since  $\theta_\lambda([a, b]) = [\theta_\lambda(a), b]$ ,  $L(\alpha_i)$  is an ideal in  $L$ .

Put  $I_1 = L(-\lambda)$  if  $-\lambda$  is an eigenvalue of  $\theta$  and  $I_1 = 0$  otherwise. Then  $L = I_1 \oplus I_2$ , where  $I_2 = \sum_{\alpha \neq -\lambda} L(\alpha)$ . Since every root subspace is an ideal in  $L$ , then  $I_1$  and  $I_2$  are ideals in  $L$ .

Take  $a \in I_1$ . By the definition of  $I_1$ :  $\theta_\lambda^k(a) = 0$  for some  $k \in \mathbb{N}$ . First consider the case when  $a = [x, y]$  for some  $x, y \in I_1$ . We have

$$\begin{aligned} \theta_\lambda^2([x, y]) &= \theta_\lambda(R([x, y]) + R^*([x, y]) + \lambda[x, y]) = \\ &= (R + R^*)([R(x), y] + [R^*(x), y] + \lambda[x, y]) + \lambda\theta_\lambda([x, y]) = \lambda\theta_\lambda([x, y]). \end{aligned}$$

Therefore,

$$0 = \theta_\lambda^k([x, y]) = \lambda^{k-1}\theta_\lambda([x, y]).$$

Since  $\lambda \neq 0$ , we finally obtain that  $\theta_\lambda([x, y]) = R([x, y]) + R^*([x, y]) + \lambda[x, y] = 0$  for all  $x, y \in I_1$ .

Now let  $a$  be an arbitrary element in  $I_1$ . It is obvious that  $[\theta_\lambda(a), I_2] = 0$ . If  $b \in I_1$ , then  $[\theta_\lambda(a), b] = \theta_\lambda([a, b]) = 0$ . Thus,  $\theta_\lambda(a) \in Z(L)$  for all  $a \in I_1$ .

Consider the ideal  $I_2$ . By the properties of root subspaces,  $I_2$  is  $\theta_\lambda$ -invariant and the restriction of  $\theta_\lambda$  on  $I_2$  is invertible. Thus, for every  $x \in I_2$ :  $x = \theta_\lambda(x')$  for some  $x' \in I_2$ .

Let  $x, y \in I_2$ . Then

$$R([x, y]) = R([\theta_\lambda(x'), y]) = R(\theta_\lambda([x', y])) = 0$$

by Theorem 1. Similarly,  $R^*([x, y]) = 0$ .

The remaining statements follows from Theorem 3. Indeed, if  $L = I_1 \oplus I_2$ , then  $[L, L] = [I_1, I_1] \oplus [I_2, I_2]$ . Since  $\theta_\lambda(a) \in Z(L)$  for all  $a \in I_1$ ,  $\theta_\lambda([a, b]) = [\theta_\lambda(a), b] = 0$  for all  $a, b \in I_1$ . Also, it is obvious that  $R(x) = R^*(x) = 0$  for all  $x \in [I_2, I_2]$  and we may use Theorem 3 to prove the remaining two statements of the theorem.  $\square$

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