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# THE KOSTLAN–SHUB–SMALE RANDOM POLYNOMIALS IN THE CASE OF GROWING NUMBER OF VARIABLES

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ABSTRACT. Let $\mathcal{P}_n = \sum_j \mathcal{H}_j$ be the decomposition in $L^2(S^m)$ of the space of homogeneous polynomials of degree $n$ on $\mathbb{R}^{m+1}$ into the sum of irreducible components of the group $\mathrm{SO}(m+1)$. We consider the asymptotic behavior of the sequence $\nu_n(t) = \frac{\mathsf{E}(|\pi_j u|^2)}{\mathsf{E}(|u|^2)}$, where $t = \frac{j}{n}$, $\pi_j$ is the projection onto $\mathcal{H}_j$, and $\mathsf{E}$ stands for the expectation in the Kostlan-Shub–Smale model for random polynomials. Assuming $\frac{m}{n} \to a > 0$ as $n \to \infty$, we prove that $\nu_n(t)$ is asymptotic to $\sqrt{\frac{4+a}{\pi n}}\, e^{-n(1+\frac{a}{4})(t-\sigma_a)^2}$, where $\sigma_a = \frac{1}{2}(\sqrt{a^2+4a} - a)$.

**Keywords:** random polynomials.

## 1. Introduction

Let $\mathcal{P}_n$ be the space of homogeneous polynomials of degree $n$ of $m+1$ real variables and $\mathcal{H}_j$ be the space of all harmonic polynomials in $\mathcal{P}_j$. There is the well known $\mathrm{SO}(m+1)$-invariant decomposition

$$\mathcal{P}_n = \sum_{j \in J_n} |x|^{n-j} \mathcal{H}_j, \tag{1}$$

where

$$J_n = \{j \in \mathbb{Z} : 0 \le j \le n,\ n-j \ \text{even}\}. \tag{2}$$

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The spaces $\mathcal{H}_j$ are irreducible. Both $\mathcal{P}_n$ and $\mathcal{H}_j$ can be treated as function spaces on the unit sphere $S^m$ in $\mathbb{R}^{m+1}$. Note that the traces of $|x|^{n-j}\mathcal{H}_j$ and $\mathcal{H}_j$ on $S^m$ coincide. Thus we may assume that

$$\mathcal{H}_j \subseteq \mathcal{P}_j \subseteq \mathcal{P}_n \quad \text{if} \;\; j \in J_n.$$

The Kostlan–Shub–Smale model for random polynomials is the Gaussian distribution in $\mathcal{P}_n$ whose density is proportional to $e^{-\widetilde{|x|}^2}$ for some special Euclidean norm $\widetilde{|\;|}$ (see Section 2 for the definition). Expectations of some metric quantities of random functions from finite dimensional Euclidean shift-invariant function spaces on compact isotropy irreducible homogeneous Riemannian manifolds were considered in [2]. Let $M$ and $\mathcal{E}$ be such a homogeneous space and a function space, respectively. After normalization, the evaluation mapping $\mathrm{ev}_{\mathcal{E}} : M \to \mathcal{E}$ defines an equivariant immersion of $M$ into the unit sphere in $\mathcal{E}$ which is a local metric homothety. The coefficient $s$ of the homothety is an important ingredient of formulas for the expectations: for example, the mean number of solutions to the system $u_j(x) = 0$, where $j = 1, \dots, m$, $m = \dim M$, and $u_j \in \mathcal{E}_j$, is equal to the product $s_1 \dots s_m$, where $s_j$ is the coefficient of metric homothety for $\mathcal{E}_j$. If $\mathcal{E}$ is irreducible, then it is an eigenspace of the Laplace operator $\Delta$ and $s = \sqrt{\frac{\lambda}{m}}$, where $\lambda$ is the eigenvalue of $-\Delta$ in $\mathcal{E}$. Thus, $s$ is independent of the Euclidean structure in $\mathcal{E}$, which is unique up to a scaling factor in this case. If $\mathcal{E} = \mathcal{E}_1 + \cdots + \mathcal{E}_k$, where the sum is orthogonal and $\mathcal{E}_j$ are irreducible for all $j = 1, \dots, k$, then

$$s^2 = \nu_1 s_1^2 + \cdots + \nu_k s_k^2,$$

where $\nu_j = \frac{c_j^2}{c^2}$ and $c_j = |\,\mathrm{ev}_{\mathcal{E}_j}(p)|$, $c = |\,\mathrm{ev}_{\mathcal{E}}(p)|$ ($c_j$ and $c$ are independent of the choice of $p \in M$). For the Kostlan–Shub–Smale model and the decomposition (1), the coefficients $\nu_{j,n}$ were computed in [3]. As functions of $t = \frac{j}{\sqrt{(m-1)n}}$, $j = 1, \dots, n$, they are asymptotic to $\frac{A_m}{\sqrt{n}}(t^2 e^{1-t^2})^{\frac{m-1}{2}}$.

In the above assertions $m$ is supposed fixed. In this paper, we prove similar results assuming $\frac{m}{n} \to a > 0$ as $n \to \infty$. We give another definition of the coefficients $\nu_{j,n}$:

$$\nu_{j,n} = \frac{\mathsf{E}(|\pi_j u|^2)}{\mathsf{E}(|u|^2)},$$

where $\pi_j$ is the orthogonal projection onto $\mathcal{H}_j$, $|\;|$ stands for the norm in $L^2(S^m)$, and $\mathsf{E}$ is the expectation in the Kostlan–Shub–Smale model for random polynomials. A computation shows that the definitions agree.

Substituting $m = an$ and $j = tn$ into the explicit formulas for $\nu_{j,n}$, we get the functions $\nu_{a,n}(t)$ defined on the interval $(0,1)$. They are log-concave, $\nu_{a,n}(t)$ attains its maximum near $\sigma_a = \frac{1}{2}(\sqrt{a^2+4a} - a)$. The maximum equals to $\sqrt{\frac{a+4}{\pi n}}$. Moreover, $\nu_{a,n}\left(\frac{x}{\sqrt{n}}\right)$ is asymptotic to $\frac{g_a(x - \sqrt{n}\,\sigma_a)}{\sqrt{n}}$, where $g_a$ is the density of the Gaussian distribution with the zero expectation and variance $\frac{4}{4+a}$. The additive approximation error is $O(n^{-1})$ but the relative error is greater than $O(n^{-\frac{1}{2}})$. There is more precise asymptotic formula (see (24) and (29)).

The results above concern the asymptotic behavior of the function $\nu_{a,n}$. They are proved in Theorem 1. The asymptotic of $\nu_{j,n}$ also depends on the rate of approximation of $a$ by $a_n = \frac{m_n}{n}$ as $n \to \infty$. For example, $\nu_{a_n,n}$ is asymptotic to $\nu_{a,n}$

in $L^1$-norm only if $a_n - a = o\left(n^{-\frac{1}{2}}\right)$. The terms $O(n^{-1}), O(n^{-\frac{1}{2}})$ in the asymptotic formulas of Theorem 1 are locally uniform on $a$ and $t$. Since the quotient $a_n = \frac{m_n}{n}$ can be treated as a small perturbation of $a$, this makes it possible to estimate the total contribution of the spaces $\mathcal{H}_j$ such that $\frac{j}{n}$ lies outside the $\varepsilon$-neighborhood of $\sigma_a$: the ratio $\frac{\mathsf{E}(|u-\pi_\varepsilon u|^2)}{\mathsf{E}(|u|^2)}$ decays exponentially as $n \to \infty$ for any $\varepsilon > 0$. Here $\pi_\varepsilon$ is the projection in $\mathcal{P}_n$ onto the sum of the spaces $\mathcal{H}_j$ such that $\left|\frac{j}{n} - \sigma_a\right| < \varepsilon$, $|\ |$ is the norm of $L^2(S^{m_n})$, and $u$ is the random Kostlan–Shub–Smale polynomial. This is proved in Theorem 2.

The results described above may be treated as follows. Suppose that $n$ is large, $\frac{m}{n}$ is not very small, and $u$ is a random Kostlan–Shub–Smale polynomial. Then one have to expect that sum of harmonics from $\mathcal{H}_j$ such that $|\frac{j}{n} - \sigma_a| < \varepsilon$, where $a = \frac{m}{n}$, define a good approximation of $u$ in $L^2(S^m)$.

As usual, $f \sim g$ means that the limit of $\frac{f}{g}$ equals 1.

## 2. Preliminaries and auxiliary material

Throughout the paper, we use the decomposition (1). As a rule, $\mathcal{H}_j$ and $\mathcal{P}_n$ are treated as function spaces on the unit sphere $S^m$ in $\mathbb{R}^{m+1}$. In the sequel, $m$ (may be, with indices) denotes the dimension of the sphere. We drop it in the notation for spaces or functions in order to avoid awkward formulas. The spaces $\mathcal{H}_j$ and $\mathcal{P}_n$ are considered with two distinct Euclidean structures: the first, $\langle\ ,\ \rangle$, is induced by $L^2(S^m)$ for the invariant probability measure on $S^m$, the second, $\widetilde{\langle\ ,\ \rangle}$, is defined by

$$\widetilde{\langle x^\alpha, x^\beta\rangle} = \begin{cases} \alpha!, & \alpha = \beta \\ 0, & \alpha \neq \beta, \end{cases} \tag{3}$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{m_n})$, $\alpha! = \alpha_0! \dots \alpha_m!$, and $x^\alpha = x_0^{\alpha_0} \dots x_m^{\alpha_m}$. This inner product in $\mathcal{P}_n$ could be defined by the formula

$$\widetilde{\langle u, v\rangle} = u\left(\frac{\partial}{\partial x}\right) v \tag{4}$$

(notice that the right-hand part is constant). To the best of my knowledge, it was introduced in the the book [7] by E. Stein. This product is a very useful tool due to the evident property $\widetilde{\langle uv, w\rangle} = \widetilde{\langle u, v\left(\frac{\partial}{\partial x}\right) w\rangle}$ (for example, with $v = |x|^2$ it provides the simplest method to prove (1)).

In what follows, the objects relating to the second inner product will be distinguished by the tilde. In particular, this is true for the evaluation mappings $\mathrm{ev}, \widetilde{\mathrm{ev}} : S^m \to \mathcal{E}$ which are defined by the identities

$$u(p) = \langle \mathrm{ev}(p), u\rangle = \widetilde{\langle \widetilde{\mathrm{ev}}(p), u\rangle}, \tag{5}$$

where $p \in S^m$, $u \in \mathcal{E}$ (from now on, $\mathcal{E}$ is a non-specified finite dimensional $\mathrm{SO}(m+1)$-invariant function space on $S^m$). The spaces $\mathcal{H}_j$ are $\mathrm{SO}(m+1)$-invariant, irreducible, and pairwise non-equivalent. Hence $|\ |$ and $\widetilde{|\ |}$ are proportional on $\mathcal{H}_j$:

$$|u|^2 = \tau_{j,n} \widetilde{|u|}^2 \quad \text{for all } u \in \mathcal{H}_j. \tag{6}$$

Clearly, $|\operatorname{ev}(p)|$ and $|\widetilde{\widetilde{\operatorname{ev}}(p)}|$ are independent of $p$. We denote them by $c_{j,n}$ and $\tilde{c}_{j,n}$, respectively. The equalities (5) imply

$$\tilde{c}^2_{j,n} = \tau_{j,n} c^2_{j,n}. \tag{7}$$

Let $\mathcal{E} = \mathcal{P}_n$ and $C_n = |\operatorname{ev}(p)|$, $\tilde{C}_n = |\widetilde{\widetilde{\operatorname{ev}}(p)}|$. Since the spaces $\mathcal{H}_j$ are pairwise orthogonal, we have

$$\begin{aligned} C_n^2 &= \textstyle\sum_{j\in J_n} c^2_{j,n}, \\ \tilde{C}_n^2 &= \textstyle\sum_{j\in J_n} \tilde{c}^2_{j,n}. \end{aligned} \tag{8}$$

The following identities are well known:

$$C_n^2 = \dim \mathcal{P}_n = \binom{n+m}{n},$$

$$c^2_{j,n} = \dim \mathcal{H}_j = \frac{(m+j-2)!(m+2j-1)}{(m-1)!j!}. \tag{9}$$

The coefficient $\tau_{j,n}$ was found (in an equivalent form) in [5] (there is a direct proof in [3]):

$$\tau_{j,n} = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2^n \Gamma\left(\frac{n-j+2}{2}\right)\Gamma\left(\frac{m+n+j+1}{2}\right)}. \tag{10}$$

Euclidean norms $|\ |$ on a real vector space $\mathcal{V}$ and the Gaussian probability measures $\pi^{-\frac{d}{2}} e^{-|x|^2}\,dx$ are in one-to-one correspondence. Here $d = \dim \mathcal{V}$ and $dx$ stands for the Lebesgue measure defined by the norm $|\ |$. Let $\mathcal{U}$ be a linear subspace of $\mathcal{V}$ and $P$ be the orthogonal projection onto $\mathcal{U}$. The expectation of $|Px|^2$ for the Gaussian distribution corresponding to the norm $|\ |$ in $\mathcal{V}$ can easily be computed:

$$\mathsf{E}(|Px|^2) = \pi^{-\frac{d}{2}} \int_{\mathcal{V}} |Px|^2 e^{-|x|^2}\,dx = \frac{1}{2}\dim \mathcal{U}. \tag{11}$$

We denote the expectation in the Kostlan–Shub–Smale model by $\tilde{\mathsf{E}}$. Let $\pi_j$ be the orthogonal projection onto $\mathcal{H}_j$. Set

$$\nu_{j,n} = \frac{\tilde{\mathsf{E}}(|\pi_j u|^2)}{\tilde{\mathsf{E}}(|u|^2)}. \tag{12}$$

Clearly,

$$\sum_{j\in J_n} \nu_{j,n} = 1.$$

Due to (6), $\tilde{\mathsf{E}}(|\pi_j u|^2) = \tau_{j,n}\tilde{\mathsf{E}}(\widetilde{|\pi_{j,n} u|}^2)$. Since $2\tilde{\mathsf{E}}(\widetilde{|\pi_{j,n} u|}^2) = c^2_{j,n}$ by (11) and (9), (7) implies $2\tilde{\mathsf{E}}(|\pi_j u|^2) = \tilde{c}^2_{j,n}$ and, together with (8), $2\tilde{\mathsf{E}}(|u|^2) = \tilde{C}_n^2$. It is known that $\tilde{C}_n^2 = \frac{1}{n!}$ (see [6]; this also can be deduced from the definition above since the mapping $\widetilde{\operatorname{ev}}$ can be explicitly written: $\widetilde{\operatorname{ev}}(p)(x) = \frac{\widetilde{\langle p,x\rangle}^n}{n!}$, $x \in S^m$, due to the equivariance of the mapping $\widetilde{\operatorname{ev}}$ and the equality $\widetilde{|x_0^n|}^2 = n!$ which is the same as $\widetilde{\operatorname{ev}}(o)(x) = \frac{\widetilde{\langle o,x\rangle}^n}{n!} = \frac{x_0^n}{n!}$, where $o = (1,0,\dots,0)$). Therefore,

$$\nu_{j,n} = \frac{\tau_{j,n} c^2_{j,n}}{\tilde{C}_n^2} = \frac{n!\Gamma\left(\frac{m+1}{2}\right)(m+j-2)!(m+2j-1)}{2^n\Gamma\left(\frac{n-j+2}{2}\right)\Gamma\left(\frac{m+n+j+1}{2}\right)(m-1)!j!}. \tag{13}$$

Set $\rho_{j,n} = \frac{\nu_{j+2,n}}{\nu_{j,n}}$. A direct calculation shows that

$$\rho_{j,n} = \left(1 + \frac{m-2}{j+1}\right)\left(1 + \frac{m-2}{j+2}\right)\left(1 + \frac{4}{2j+m-1}\right)\frac{n-j}{n+m+j+1}. \tag{14}$$

## 3. Asymptotic behavior of $\nu_{j,n}$ as $n \to \infty$

Let $c^2_{a,n}(t)$ and $\tau_{a,n}(t)$ be the functions of $t$ which we get substituting $m = an$ and $j = tn$ into (9) and (10), respectively:

$$c^2_{a,n}(t) = \frac{\Gamma((a+t)n-1)((a+2t)n-1)}{\Gamma(an)\Gamma(tn+1)},$$

$$\tau_{a,n}(t) = \frac{\Gamma\left(\frac{an+1}{2}\right)}{2^n\Gamma\left(\frac{1-t}{2}n+\frac{1}{2}\right)\Gamma\left(\frac{(1+a+t)n+1}{2}\right)}.$$

In accordance with the definition of $t$, we assume $0 < t < 1$ unless the contrary is stated explicitly. The functions $\tilde{c}^2_{a,n}$, $\nu_{a,n}$, and $r_{a,n}$ may be defined by the equalities

$$\begin{cases} \tilde{c}^2_{a,n}(t) &=& \tau_{a,n}(t)c^2_{a,n}(t), \\ \nu_{a,n}(t) &=& n!\,\tilde{c}^2_{a,n}(t), \\ r_{a,n}(t) &=& \frac{\nu_{a,n}(t+\frac{2}{n})}{\nu_{a,n}(t)}, \end{cases} \tag{15}$$

as well as by substitution $m = an$, $j = tn$ into (7), (13), and (14).

Note that $\tau_{a,n}$, $c^2_{a,n}$, and $\nu_{a,n}$ are positive on $(0,1)$. These functions are also strictly concave if $n$ is sufficiently large. Indeed,

$$(\ln c^2_{a,n})''(t) = n^2\Big(\Psi'((a+t)n-1) - \Psi'((tn+1) - \frac{4}{(a+2t)n-1)^2}\Big),$$

where $\Psi'(x) = (\ln\Gamma)''(x) = \sum_{k=0}^{\infty}\frac{1}{(x+k)^2}$. If $n > \frac{2}{a}$, then $(a+t)n - 1 > tn + 1$, whence $(\ln c^2_{a,n})''(t) < 0$. The inequality $(\ln\tau_{a,n})''(t) < 0$ immediately follows from $(\ln\Gamma)''(x) > 0$. Therefore, $\ln\nu_{a,n}$ is strictly concave on $(0,1)$. Thus, we have a proof for the following lemma.

**Lemma 1.** *The functions $c^2_{a,n}$, $\tau_{a,n}$, $\tilde{c}^2_{a,n}$, and $\nu_{a,n}$ are strictly log-concave on the interval* $(0,1)$. ☐

It follows from (14) that

$$r_{a,n}(t) = \left(1 + \frac{a-\frac{2}{n}}{t+\frac{1}{n}}\right)\left(1 + \frac{a-\frac{2}{n}}{t+\frac{2}{n}}\right)\left(1 + \frac{4}{(a+2t)n-1}\right)\frac{1-t}{1+a+t+\frac{1}{n}}.$$

**Lemma 2.** *The sequence $r_{a,n}(t)$ converges to the function*

$$\varphi_a(t) = \left(1 + \frac{a}{t}\right)^2\frac{1-t}{1+a+t}$$

*as $n \to \infty$. Moreover, the approximation error $|r_{a,n}(t) - \varphi_a(t)|$ is locally uniform on $t \in (0,1)$ and $a \in (0,\infty)$ and the same is true for the convergence of the derivatives $r^{(k)}_{a,n}$ to $\varphi^{(k)}_a$ for any $k \in \mathbb{N}$.*

*Proof.* The function $r_{a,n}$ is rational of degree 4 and has four poles which lies in $(-\infty,0)$. The coefficients of $r_{a,n}$ depends continuously on $a$ and $\frac{1}{n}$. The limit points of poles as $n \to \infty$ are 0 and $-(1+a)$. This implies the locally uniform convergence of $r_{a,n}$ to $\varphi_a$ as well as the locally uniform upper bounds for the approximation error. Due to Weierstrass's theorem, the same is true for all derivatives. ☐

The straightforward computation shows that

$$\frac{r_{a,n}(t)}{\varphi_a(t)} = 1 + \frac{\varkappa_a(t)}{n} + \eta_a(n,t), \tag{16}$$

as $n \to \infty$, where

$$\varkappa_a(t) = \frac{4}{a+2t} - \frac{1}{1+a+t} - \frac{1}{a+t} - \frac{3}{t}, \tag{17}$$

and $n^2\eta_a(n,t)$ is uniformly bounded on $[\varepsilon, 1]$ for any $\varepsilon \in (0,1)$. Since

$$\left(\varkappa_a(t) + \frac{1}{1+a+t}\right)(a+t)(a+2t)t = -(3a^2 + 6at + 4t^2) < 0$$

if $a, t > 0$, we have

$$\varkappa_a(t) < -\frac{1}{2+a} < 0 \tag{18}$$

and $r_{a,n}(t) < \varphi_a(t)$ for any $t \in (0,1)$ and all sufficiently large $n$. The equation

$$\varphi_a(t) = 1$$

has the unique positive root

$$\sigma_a = \frac{\sqrt{a^2+4a} - a}{2} \tag{19}$$

since

$$\varphi_a(t) - 1 = -\frac{(2t+a)(t^2+at-a)}{t^2(t+a+1)}.$$

It is also the root of the equation $t^2 + at - a = 0$. Hence

$$\begin{cases} 1 - \sigma_a &= \frac{\sigma_a^2}{a}, \\ a + \sigma_a &= \frac{a}{\sigma_a}, \\ 1 + a + \sigma_a &= \frac{a}{\sigma_a^2}. \end{cases} \tag{20}$$

Differentiating $\varphi_a(t) - 1$, using (20) and the evident identity $(2\sigma_a + a)^2 = a(a+4)$, we get

$$\varphi'(\sigma_a) = -\frac{(2t+a)^2}{t^2(t+a+1)}\Big|_{t=\sigma_a} = -(a+4). \tag{21}$$

As a function of $a$, $\sigma_a$ is concave. Moreover, it increases, tends to 1 as $a \to \infty$, and satisfies the inequalities

$$0 < \sigma_a < 1 \tag{22}$$

for all $a > 0$. The inequality $a < \sigma_a$ holds if and only if $a < \frac{1}{2}$. Set

$$\mu_a = -\frac{\varkappa_a(\sigma_a)}{(a+4)}. \tag{23}$$

By (18), $\mu_a > \frac{1}{(a+2)(a+4)}$.

**Lemma 3.** *For any $\varepsilon > 0$ and some $A_\varepsilon > 0$, the inequality*

$$\left|1 - r_{a,n}\left(\sigma_a - \frac{\mu_a}{n}\right)\right| < \frac{A_\varepsilon}{n^2}$$

*holds for all $a > \varepsilon$ and sufficiently large $n$. Moreover, for any $\mu > 0$ such that $\mu \neq \mu_a$ the sign of $r_{a,n}\left(\sigma_a - \frac{\mu}{n}\right) - 1$ coincides with the sign of $\mu - \mu_a$ if $n$ is sufficiently large.* [1]

*Proof.* We have $r_{a,n}(\sigma_a) = \varphi_a(\sigma_a) + \frac{\varkappa_a(\sigma_a)}{n} + O(n^{-2})$ by (16), $\varphi_a(\sigma_a) = 1$, and $\varphi_a'(\sigma_a) = -(a+4)$. It follows that $\varphi_a'(\sigma_a)\frac{\mu_a}{n} = \frac{\varkappa_a(\sigma_a)}{n}$ and

$$\begin{aligned} r_{a,n}\left(\sigma_a - \frac{\mu_a}{n}\right) &= r_{a,n}(\sigma_a) - r_{a,n}'(\sigma_a)\frac{\mu_a}{n} + O(n^{-2}) \\ &= \varphi_a(\sigma_a) + \frac{\varkappa_a(\sigma_a)}{n} - r_{a,n}'(\sigma_a)\frac{\mu_a}{n} + O(n^{-2}) \\ &= 1 + \left(\varphi_a'(\sigma_a) - r_{a,n}'(\sigma_a)\right)\frac{\mu_a}{n} + O(n^{-2}) = 1 + O(n^{-2}). \end{aligned}$$

The coefficients in $O(n^{-2})$ are uniform on $a \in (\varepsilon, \infty)$ for any $\varepsilon > 0$ because the functions are analytic and the convergence of $r_{a,n}$ to $\varphi$ is uniform on any sector $|\arg(t-\varepsilon)| < \alpha$ in $\mathbb{C} \setminus (-\infty, 0]$, where $0 < \alpha < \pi$.

Replacing $\mu_a$ with $\mu$ in $r_{a,n}\left(\sigma_a - \frac{\mu_a}{n}\right)$, we add $r_{a,n}'(\sigma_a)\frac{\mu_a - \mu}{n}$ to the right-hand side of the first equality in the chain above. Since $r_{a,n}'(\sigma_a) < 0$, its sign coincides with the sign of $r_{a,n}\left(\sigma_a - \frac{\mu}{n}\right) - 1$ for large $n$. □

Let $t_{a,n}$ be the unique solution to the equation $r_{a,n}(t) = 1$ in $(0,1)$.

**Corollary 1.** *For any $a, \varepsilon > 0$ and all sufficiently large $n$ we have*

$$t_{a,n} \in \left(\sigma_a - \frac{\mu_a + \varepsilon}{n}, \sigma_a - \frac{\mu_a - \varepsilon}{n}\right).$$

*Proof.* The function $r_{a,n}(t) - 1$ change its sign in this interval if $n$ is large. □

Set

$$\left\{ \begin{array}{rcl} \xi_a(t) & = & \frac{\sqrt{a}(a+2t)}{\sqrt{(a+4)(a+t)^3 t(1-t)}}, \\ \psi_a(t) & = & \frac{(a+t)^{2(a+t)}}{a^a t^{2t}(1-t)^{1-t}(1+a+t)^{1+a+t}}, \\ \upsilon_{a,n}(t) & = & \sqrt{\frac{a+4}{\pi n}}\,\xi_a(t)\psi_a(t)^{\frac{n}{2}}. \end{array} \right. \tag{24}$$

The function $\psi_a$ admits a positive continuous extension onto $[0,1]$ but $\xi_a$ has singularities at 0 and 1 as well as $\upsilon_{a,n}$. However, $\upsilon_{a,n}$ is a good approximation of $\nu_{a,n}$ on compact intervals in $(0,1)$ for large $n$.

**Lemma 4.** *We have*

$$\lim_{n\to\infty} \frac{\nu_{a,n}(t)}{\upsilon_{a,n}(t)} = 1,$$

*as $n \to \infty$, where the convergence is locally uniform on $t \in (0,1)$ and $a \in (0,\infty)$.*

*Proof.* Since $\Gamma(x+\alpha) \sim x^\alpha \Gamma(x)$ as $x \to \infty$, (13) implies

$$\nu_{j,n} \sim \frac{2n\sqrt{m}(m+2j-1)}{j(n-j)\sqrt{n+m+j}(m+j-1)} \cdot \frac{\Gamma(n)\Gamma\left(\frac{m}{2}\right)\Gamma(m+j)}{2^n\Gamma(m)\Gamma(j)\Gamma\left(\frac{n-j}{2}\right)\Gamma\left(\frac{n+m+j}{2}\right)}$$

[1] Actually, we have the inequalities $1 < 2n^2\left(1 - r_{a,n}\left(\sigma_a - \frac{\mu_a}{n}\right)\right) < a^{-\frac{3}{2}} + 4$ for any $a > 0$ and all sufficiently large $n$. They can be proved by computation with the terms of the second order. The summand 4 is a sharp bound as $a \to \infty$, for $a^{-\frac{3}{2}}$ the same is true as $a \to 0$.

as $m, n, j \to \infty$. Substituting $m = an$ and $j = tn$, we get

$$\nu_{a,n}(t) \sim \frac{1}{n} \cdot \frac{2\sqrt{a}(a+2t)}{t(1-t)\sqrt{1+a+t}(a+t)} \cdot \frac{\Gamma(n)\Gamma\left(\frac{a}{2}n\right)\Gamma((a+t)n)}{2^n\Gamma(an)\Gamma(tn)\Gamma\left(\frac{1-t}{2}n\right)\Gamma\left(\frac{1+a+t}{2}n\right)}$$

We use the Stirling formula $\Gamma(x) \sim \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x$ to simplify the last factor. Note that all terms in it, except for $2^n$, have the form $\Gamma(kn)$. Combining the factors of the type $\sqrt{\frac{2\pi}{kn}}$ and their reciprocals, we get

$$\frac{1}{2}\sqrt{\frac{nt(1-t)(1+a+t)}{\pi(a+t)}}.$$

Its product with the first and second factors is equal to $\sqrt{\frac{a+4}{\pi n}}\,\xi_a(t)$. Computation of the powers of $\frac{n}{e}$ and 2 shows that they cancel. The remainder is

$$\left(\frac{a^{\frac{a}{2}}(a+t)^{a+t}}{a^a t^t (1-t)^{\frac{1-t}{2}}(1+a+t)^{\frac{1+t+a}{2}}}\right)^n = \psi_a(t)^{\frac{n}{2}}.$$

The results of the calculation shows that $\sqrt{\pi n}\,\nu_{a,n}(t) \sim \sqrt{a+4}\,\xi_a(t)\psi_a(t)^{\frac{n}{2}}$. It is easy to check that the transformations above are locally uniform on $t$ and $a$. □

**Lemma 5.** *The function* $\ln\psi_a$ *is strictly concave on* $(0,1)$ *and attains its maximum at* $\sigma_a$. *Moreover,*

(25) $$(\ln\psi_a)' = \ln\varphi_a,$$

(26) $$\psi_a(\sigma_a) = 1,$$

(27) $$\xi_a(\sigma_a) = 1.$$

*Proof.* Let $t > \sigma_a$ and $j_a$, $j_t$ be the least and the greatest $j$ in the set

$$J_a^t = J_n \cap [n\sigma_a, nt-2],$$

respectively. Then $\frac{\nu_{an,n}(j_t)}{\nu_{an,n}(j_a)} = \prod_{j\in J_a^t}\rho_{an,n}(j)$ and, due to (16),

$$\frac{2}{n}\ln\frac{\nu_{an,n}(j_t)}{\nu_{an,n}(j_a)} = \frac{2}{n}\sum_{j\in J_a^t}\ln r_{a,n}\left(\frac{j}{n}\right) = \frac{2}{n}\sum_{j\in J_a^t}\ln\varphi_a\left(\frac{j}{n}\right) + O(n^{-1})$$
$$= \int_{\sigma_a}^{t}\ln\varphi_a(\tau)\,d\tau + O(n^{-1}).$$

On the other hand, $\lim_{n\to\infty}\frac{2}{n}\ln\frac{\nu_{an,n}(j_t)}{\nu_{an,n}(j_a)} = \ln\psi_a(t)$ according to Lemma 4. Hence $(\ln\psi_a)'(t) = \ln\varphi_a(t)$ if $t > \sigma_a$. Since $\psi_a$ and $\varphi_a$ are real analytic on $(0,1)$, the equality (25) holds for all $t \in (0,1)$.

Clearly, $\varphi_a > 0$ and $\varphi_a' < 0$. Due to (25), $\psi_a$ is strictly concave on $(0,1)$. By definition of $\sigma_a$, we have $\varphi_a(\sigma_a) = 1$. Hence $\sigma_a$ is a critical point of $\ln\psi_a$ which is necessarily the strict maximum.

Due to the equalities (20) and (24), $\psi_a(\sigma_a)$ is a product of powers of $a$ and $\sigma_a$. A direct computation of the degrees shows that they cancel. This proves (26). Similarly, these equalities together with $a + 2\sigma_a = \sqrt{a(4+a)}$ imply (27). □

The first of the following theorems concerns the asymptotic behavior of the functions introduced above, the second deals with the rate of approximation of the coefficients $\nu_{j,n}$ by these functions.

In the statement of the theorem below, $\varkappa_a$, $\sigma_a$, $\mu_a$, $\nu_{a,n}$, and $\upsilon_{a,n}$ are defined by (17), (19), (23), (15), and (24), respectively. We assume that $\nu_{a,n}$ is a function on $(0,\infty)$ which vanishes outside $(0,1)$. Set

$$g_a(y) = \sqrt{\frac{4+a}{\pi}}\, e^{-(1+\frac{a}{4})y^2}$$

**Theorem 1.** *Let $a, \varepsilon > 0$. Then $\nu_{a,n}$ has the unique critical point $\vartheta_n \in (0,1)$ and for all sufficiently large $n$*

$$\sigma_a - \frac{\mu_a + \varepsilon}{n} < \vartheta_n < \sigma_a - \frac{\mu_a - \varepsilon - 2}{n}. \tag{28}$$

*Moreover, for any $t \in (0,1)$*

$$\nu_{a,n}(t) = \upsilon_{a,n}(t)(1 + O(n^{-1})) \tag{29}$$

*as $n \to \infty$, where $O(n^{-1})$ is locally uniform on $t \in (0,1)$ and on $a > 0$. For every $y > 0$*

$$\nu_{a,n}\left(\frac{y}{\sqrt{n}}\right) = \frac{g_a(y - \sqrt{n}\,\sigma_a)}{\sqrt{n}} + O(n^{-1}) \tag{30}$$

*as $n \to \infty$, where $O(n^{-1})$ is uniform on $y > 0$ and locally uniform on $a > 0$.*

*Proof.* Let $t_{a,n}$ be as in Corollary 1. Then $\nu_{a,n}(t_{a,n}) = \nu_{a,n}(t_{a,n} + \frac{2}{n})$. This equality implies that $\nu_{a,n}$ has a critical point

$$\vartheta_n \in \left[t_{a,n}, t_{a,n} + \frac{2}{n}\right].$$

This inclusion and Corollary 1 imply (28). Since $\ln \nu_{a,n}$ is strictly concave, $\nu_{a,n}(t)$ attains at $\vartheta_n$ its maximal value on $(0,1)$.

By (15) and (16),

$$\lim_{n\to\infty} \ln \frac{\nu_{a,n}(t + \frac{2}{n})}{\nu_{a,n}(t)} = \lim_{n\to\infty} \ln r_{a,n}(t) = \ln \varphi_a(t).$$

According to (25),

$$\lim_{n\to\infty} \frac{n}{2} \ln \frac{\psi_a(t + \frac{2}{n})}{\psi_a(t)} = (\ln \psi_a)'(t) = \ln \varphi_a(t).$$

Hence $\lim_{n\to\infty} \ln \frac{\upsilon_{a,n}(t+\frac{2}{n})}{\upsilon_{a,n}(t)} = \ln \varphi_a(t) = \lim_{n\to\infty} \ln \frac{\nu_{a,n}(t+\frac{2}{n})}{\nu_{a,n}(t)}$, where the second equality follows from (16). Moreover, (25) implies

$$\ln \frac{\psi_a(t + \frac{2}{n})}{\psi_a(t)} = \ln \varphi(t)\frac{2}{n} + \frac{1}{2}(\ln \varphi)'(t)\left(\frac{2}{n}\right)^2 + O(n^{-3})$$

and consequently

$$\ln \frac{\upsilon_{a,n}(t + \frac{2}{n})}{\upsilon_{a,n}(t)} = \ln \varphi_a(t) + \frac{2(\ln \xi_a)'(t) + (\ln \varphi_a)'(t)}{n} + O(n^{-2}).$$

Together with the equality $\xi_a(t)^2\varphi_a(t) = \frac{a(a+2t)^2}{(a+4)(a+t)t^3(1+a+t)}$, this implies

$$\lim_{n\to\infty} n \ln \frac{\upsilon_{a,n}(t+\frac{2}{n})}{\upsilon_{a,n}(t)\varphi_a(t)} = \frac{4}{a+2t} - \frac{1}{1+a+t} - \frac{1}{a+t} - \frac{3}{t} = \varkappa_a(t).$$

On the other hand, by (16) we have

$$\lim_{n\to\infty} n \ln \frac{\nu_{a,n}(t+\frac{2}{n})}{\nu_{a,n}(t)\varphi_a(t)} = \varkappa_a(t).$$

Setting $\chi_{a,n}(t) = \frac{\nu_{a,n}(t)}{\upsilon_{a,n}(t)}$, we get

$$\lim_{n\to\infty} n \ln \frac{\chi_{a,n}(t+\frac{2}{n})}{\chi_{a,n}(t)} = 0.$$

Due to (15) and (24), this implies that $\ln \frac{\chi_{a,n}(t+\frac{2}{n})}{\chi_{a,n}(t)} = \ln\left(r_{a,n}(t)\frac{\upsilon_{a,n}(t)}{\upsilon_{a,n}(t)}\right)$ is analytic on $n^{-1}$ near zero. Hence the limit of $n^2 \ln \frac{\chi_{a,n}(t+\frac{2}{n})}{\chi_{a,n}(t)}$ as $n \to \infty$ exists[2] and, moreover, it is a rational function $\omega$ that is analytic on $(0,1)$. Let $J_a^t$ be as in the proof of Lemma 5. Then

$$\ln \frac{\chi_{a,n}(t)}{\chi_{a,n}(\sigma_a)} = \sum_{j\in J_a^t} \ln \frac{\chi_{a,n}(\frac{j+2}{n})}{\chi_{a,n}(\frac{j}{n})} + O(n^{-2}) = \frac{1}{2n}\int_{\sigma_a}^t \omega(\tau)\,d\tau + O(n^{-2}).$$

This proves (29). The term $O(n^{-1})$ in (29) is locally uniform on $t$ and $a$ due to the explicit formulas for $\nu_{a,n}$ and $\upsilon_{a,n}$ above.

The proof of (30) is standard. Since $\sigma_a$ is a critical point of $\psi_a$ and $\psi_a(\sigma_a) = 1$ by (26), we have $\ln\psi_a(\sigma_a+\tau) = \alpha\tau^2 + O(\tau^3)$ as $\tau \to 0$. The equalities $\varphi_a(\sigma_a) = 1$, (21), and (25) imply

$$\alpha = \frac{1}{2}(\ln\psi_a)''(\sigma_a) = \frac{1}{2}(\ln\varphi)'(\sigma_a) = -\frac{1}{2}(a+4).$$

For $\xi_a$ we have $\ln\xi_a(\sigma_a+\tau) = \beta\tau + \gamma\tau^2 + O(\tau^3)$ by (27), where $\beta = \xi_a'(\sigma_a)$. Setting $\tau = \frac{\eta}{\sqrt{n}}$, we get

$$\ln\left(\xi_a\Big(\sigma_a + \frac{\eta}{\sqrt{n}}\Big)\psi_a\Big(\sigma_a+\frac{\eta}{\sqrt{n}}\Big)^{\frac{n}{2}}\right) = \frac{1}{2}\alpha\eta^2 + \beta\frac{\eta}{\sqrt{n}} + O(n^{-1}).$$

Hence the left-hand part of the equality above converges to $\frac{1}{2}\alpha\eta^2$ uniformly on $\eta$ on any finite interval in $\mathbb{R}$. Together with (24) and (29) this implies (30). Since $\ln\nu_{a,n}$ is concave, it decreases on the right of $\sigma_a$ and increases on the left. Hence $O(n^{-1})$ is uniform on $(0,\infty)$. □

According to the theorem above, $\nu_{j,n}$ concentrates near $\sigma_a n$. The following theorem clarifies this. In its statement, $\nu_n$ is a function on $\mathbb{R}$ that is the composition of a dilation, a shift, and the piecewise constant extension of $\sqrt{n}\,\nu_{j,n}$ onto $\mathbb{R}$, where $\nu_{j,n}$ is defined by (12):

$$\nu_n(x) = \begin{cases} \sqrt{n}\,\nu_{j,n}, & \frac{j-1}{\sqrt{n}} < x - \sqrt{n}\,\sigma_a \le \frac{j}{\sqrt{n}}, \quad j = 1,\dots,n, \\ 0, & x > \sqrt{n}(1-\sigma_a) \;\text{ or }\; x \le -\sqrt{n}\,\sigma_a. \end{cases}$$

By $\hat\nu_{a,n}$ we denote the measure $\sum_{j\in J_n}\nu_{j,n}\delta_{\frac{j}{n}}$, where $\delta_s$ is the Dirac measure at $s$. Due to the equality $\sum_{j\in J_n}\nu_{j,n} = 1$, $\hat\nu_{a,n}$ is a probability measure on $[0,1]$. Further,

[2] it is equal to $\frac{1}{6}\left(\frac{2}{(1-t)^2} - \frac{1}{t^2} + \frac{13}{(a+t)^2} + \frac{1}{(1+a+t)^2} - \frac{24}{(a+2t)^2}\right)$

$\| \, \|_1$ is the norm of $L^1(\mathbb{R})$ and $dx$ stands for the Lebesgue measure. For $\varepsilon \in (0,1)$, let $J_{\varepsilon,n}$ be the set of $j \in J_n$ such that $\frac{j}{n} \in (\sigma_a - \varepsilon, \sigma_a + \varepsilon)$ and $\pi_\varepsilon$ be the orthogonal projection onto the sum of the spaces $\mathcal{H}_j$ over $j \in J_{\varepsilon,n}$.

**Theorem 2.** *Let $a, \varepsilon > 0$ and $m_n$ be a sequence of natural numbers. Set $a_n = \frac{m_n}{n}$. Then the following assertions hold:*

(1) *if $\lim\limits_{n\to\infty} a_n = a$, then $\hat{\nu}_{a_n,n}$ converges $*$-weakly to $\delta_a$,*
(2) $\lim\limits_{n\to\infty} \|\nu_n - g_a\|_1 = 0$ *if and only if* $\lim\limits_{n\to\infty} \sqrt{n}(a_n - a) = 0$,
(3) *if $\lim\limits_{n\to\infty} a_n = a$, then for all sufficiently large $n$*

$$\frac{\mathsf{E}(|u - \pi_\varepsilon u|^2)}{\mathsf{E}(|u|^2)} < A\sqrt{n}\, q^n,$$

*where $q = \max\{\psi_a(\sigma_a - \varepsilon), \psi_a(\sigma_a + \varepsilon)\} < 1$, $A > 0$, and $A$ depends only on $a$ and $\varepsilon$.*

*Proof.* The implication (1) is evident. Due to (19), $\lim_{n\to\infty} \sqrt{n}(a_n - a) = 0$ is equivalent to $\lim_{n\to\infty} \sqrt{n}(\sigma_{a_n} - \sigma_a) = 0$. Since $O(n^{-1})$ in (30) is locally uniform, it is also equivalent to the pointwise convergence $\nu_n \to g_a$ as $n \to \infty$. Thus the part "only if" of (2) follows from Lebesgue's Dominated Convergence theorem. The part "if" is obvious.

It remains to prove (3). Let us denote $\alpha = \sigma_a - \varepsilon$, $\beta = \sigma_a + \varepsilon$ for short. According to the definitions of $\nu_{j,n}$ and $\pi_\varepsilon$,

$$\frac{\mathsf{E}(|u - \pi_\varepsilon u|^2)}{\mathsf{E}(|u|^2)} = \sum_{j \notin J_{\varepsilon,n}} \nu_{j,n} = \sum_{j \in J_n,\ j > \beta n} \nu_{a,n}\left(\frac{j}{n}\right) + \sum_{j \in J_n,\ j < \alpha n} \nu_{a,n}\left(\frac{j}{n}\right)$$

We consider only the case $j > \beta n$ because the proof can be repeated almost word-for-word if $j < \alpha n$. Moreover, we may forget that $a_n \neq a$ since the involved functions are continuous at $\alpha$ and $\beta$ and do not vanish at them for all sufficiently large $n$. Thus a perturbation of $a$ can change only the multiplicative constants.

Any smooth positive function $f$ on some interval which is strictly log-concave satisfies the inequality $f(x) < f(x_0) e^{\frac{f'(x_0)}{f(x_0)}(x - x_0)}$, $x \neq x_0$. By Lemma 1, we may apply it to $\nu_{a,n}$. Thus

$$\nu_{a,n}(t) < \nu_{a,n}(\beta) e^{\frac{\nu'_{a,n}(\beta)}{\nu_{a,n}(\beta)}(t - \beta)},$$

where $\beta < t < 1$. By Lemma 4 and Weierstrass's theorem,

$$\lim_{n\to\infty} ((\ln \nu_{a,n})'(\beta) - (\ln \upsilon_{a,n})'(\beta)) = 0.$$

It follows from (24), Lemma 4, and (25) that for all $t \in (\beta, 1)$ and sufficiently large $n$ we have

$$\nu_{a,n}(t) < \nu_{a,n}(\beta) e^{K - L(t-\beta)},$$

where $K > \ln \xi'_a(\beta)$ and $L = -\ln \varphi_a(\beta)$. Note that $\varphi_a(\beta) < 1$ since $\beta > \sigma_a$. The inequality

$$\sum_{j=0}^{\infty} e^{-\frac{2Lj}{n}} = \frac{1}{1 - e^{-\frac{2L}{n}}} < \frac{n}{2L}$$

and (24) imply the estimate $\sum_{j\in J_n,\ j>\beta n}\nu_{j,n} < B\sqrt{n}\,\psi(\beta)^{\frac{n}{2}}$, where $B$ depends only on $\beta$, holds for all sufficiently large $n$. Adding the sum over $j < \alpha n$, we get (3) with some $A > 0$. □

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