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KAPLAN'S PENALTY OPERATOR IN APPROXIMATION OF A DIFFUSION-ABSORPTION PROBLEM WITH A ONE-SIDED CONSTRAINT

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Abstract. We consider the homogeneous Dirichlet problem for the nonlinear diffusion-absorption equation with a one-sided constraint imposed on diffusion flux values. The family of approximate solutions constructed by means of Alexander Kaplan's integral penalty operator is studied. It is shown that this family converges weakly in the first-order Sobolev space to the solution of the original problem, as the small regularization parameter tends to zero. Thereafter, a property of uniform approximation of solutions is established in Hölder's spaces via systematic study of structure of the penalty operator.

Key words: penalty method, p-Laplace operator, diffusion-absorption equation, one-sided constraint

Introduction

In this article, we consider the problem of description of a stationary nonlinear diffusion-absorption process in a bounded continuum Ω of d-dimensional space of independent variables $\boldsymbol{x}$. Dimension $d \in \mathbb{N}$ is given arbitrarily. The vector-function of diffusion flux $\boldsymbol{J}$ is given by either Fick's or Fourier's law and nonlinearly depends on the gradient of the sought scalar function $u = u(\boldsymbol{x})$. This dependence is power-law. More precisely, the divergence of diffusion flux is p-Laplacian of u, i.e., $\operatorname{div}_x \boldsymbol{J} = \Delta_p u := \operatorname{div}_x(|\nabla_x u|^{p-2}\nabla_x u)$. The scalar absorption function is

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also nonlinear and is defined by the formula $a(u) = |u|^{p-2}u$.[1] From the physical viewpoint, the sought function u may have the sense of concentration of some component of matter or the sense of temperature of a matter continuum. Also u may stand for deformation in equilibrium problems in hyper-elasticity. In general, the equations of the form $-\Delta_p u + \alpha(u) = f$ (with a nonlinear function $\alpha = \alpha(u)$) arise in mathematical modelling in rheology, glaciology, radiation of heat, and plastic moulding; in description of Brownian motion and even in game theory (see mathematical *tug-of-war games* in [8, 17, 20]).

A great amount of publications is devoted to various questions about well-posedness and qualitative properties of solutions to such equations and the vast theory is constructed as the result (see, for example, monographs and surveys [1,2,15,16,25]). In the present work, a problem with constraint $|\nabla_x u| \leq 1$ is studied. In turn, the presence of the constraint leads to the mathematical formulation in the form of variational inequality or, equivalently, in the form of the minimization problem for the functional $F(u) = \dfrac{1}{p}\displaystyle\int_\Omega \big(|\nabla_x u|^p + |u|^p\big)\,d\boldsymbol{x} - \int_\Omega f u\, d\boldsymbol{x}$ on the convex closed set $\{u\colon\ |\nabla_x u| \leq 1\}$.[2] Here f is a given density of distributed external sources in molecular diffusion or distributed external mass forces in hydrodynamics and hyper-elasticity.

Penalty methods take place among widely used methods for fruitful study of the above stated questions on existence and qualitative properties of solutions. Besides, penalty methods are constructive ones in the sense that they provide approximation of solutions of original problems with the help of families of solutions to perturbed unconstrained minimization problems. The penalty method is exactly the method that we use in this paper. With the help of the integral version of Alexander Kaplan's penalty function, for an arbitrarily given and small $\delta > 0$ we succeed to obtain refined approximation in $C(\Xi^\delta)$ for the solution of the nonlinear diffusion-absorption problem with the constraint on the diffusion flux, with Ξ^δ being a closed set in Ω such that the Lebesgue measure of $\Omega\setminus\Xi^\delta$ is less than δ. This article is somewhat close to other studies devoted to description of diffusion-convection-absorption problems with one-sided constraints [3,4] and it is a natural continuation of the works [9, 21, 22].

1. Formulation and well-posedness of the problem

In the present work, we consider the following homogeneous Dirichlet problem for the diffusion-absorption equation with a one-sided constraint on the diffusion flux.

Problem D-A. *In a bounded domain $\Omega \subset \mathbb{R}^d$ with the smooth boundary $\partial\Omega$, it is necessary to find a function $u = u(\boldsymbol{x})$ satisfying the equation*

$$-\operatorname{div}_x \boldsymbol{J} + |u|^{p-2}u = f, \tag{1.1a}$$

[1]It is clear from the further outline in the article that, on the strength of the first Sobolev embedding theorem [19, Ch. I, Th. 1.1], all results of the article can be naturally extended to the case when $a(u) = |u|^{q-2}u$ with any $q \in (1, pd/(d-p))$ for $d > p$ and any $q \in (1, +\infty)$ for $d \leq p$.

[2]By analogy with the theory of hyper-elasticity, this functional can be referred to as *the stored p-energy* [15], [24, Sec. 17.3]. Within the framework of this theory, the identity $|\nabla_x u| = 1$ is the yield criterion.

where the diffusion flux $\boldsymbol{J}$ *is defined by the formulas*

$$\text{(1.1b)}\quad \boldsymbol{J} \in \partial\Phi(\nabla_x u),\ \Phi(\boldsymbol{\tau}) = \frac{1}{p}\int_\Omega Q(\boldsymbol{\tau}(\boldsymbol{x}))d\boldsymbol{x},\ Q(\boldsymbol{\tau}) = \begin{cases} |\boldsymbol{\tau}|^p & \textit{for} \quad |\boldsymbol{\tau}| \le 1, \\ +\infty & \textit{for} \quad |\boldsymbol{\tau}| > 1, \end{cases}$$

and the homogeneous Dirichlet condition

$$\text{(1.1c)}\qquad u|_{\partial\Omega} = 0.$$

In the formulation of Problem D-A, $p \in (1, +\infty)$ is a given constant exponent, $f \in W^{-1,p'}(\Omega)$ is a given functional, and $W^{-1,p'}(\Omega)$ $(p^{-1} + (p')^{-1} = 1)$ is the dual space to the Sobolev space $W_0^{1,p}(\Omega)$. The norm in $W_0^{1,p}(\Omega)$ is standardly defined by the formula

$$\|\phi\|_{W_0^{1,p}(\Omega)} = \Big(\int_\Omega \big[|\phi(\boldsymbol{x})|^p + |\nabla_x \phi(\boldsymbol{x})|^p\big]\, d\boldsymbol{x}\Big)^{1/p}.$$

Relations (1.1b) mean that $\boldsymbol{J}$ is an element of the subdifferential $\partial\Phi$ of the functional Φ: $\boldsymbol{\tau} \mapsto \dfrac{1}{p}\displaystyle\int_\Omega Q(\boldsymbol{\tau}(\boldsymbol{x}))d\boldsymbol{x}$ at the point $\boldsymbol{\tau} = \nabla_x u$. Notice that Φ is a Gâteaux-differentiable mapping on the set

$$M := \{\boldsymbol{\tau}:\ \Omega \mapsto \mathbb{R}^d \text{ are measurable vector-functions such that } |\boldsymbol{\tau}(\boldsymbol{x})| \le 1 \text{ almost everywhere in } \boldsymbol{x} \in \Omega\} \subset L^p(\Omega)^d,$$

and its Gâteaux derivative $\Phi'(\boldsymbol{\tau})$ is defined by the formula

$$\langle \Phi'(\boldsymbol{\tau}), \boldsymbol{v}\rangle = \int_\Omega |\boldsymbol{\tau}(\boldsymbol{x})|^{p-2}\boldsymbol{\tau}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})d\boldsymbol{x} \quad \forall\, \boldsymbol{v} \in L^p(\Omega)^d.$$

Notation 1. *Here and further in the article by* $\langle\cdot,\cdot\rangle$ *the duality brackets are defined, i.e.,* $\langle\Psi, \psi\rangle$ *is the value of a functional* $\Psi \in \mathcal{V}^*$ *on an element* $\psi \in \mathcal{V}$*, where* $\mathcal{V}$ *is a reflexive Banach space and* $\mathcal{V}^*$ *is the dual space to* $\mathcal{V}$*.*

On the strength of the well-known equivalency property in subdifferential calculus [6, Ch. I, Sec. 5], we conclude that $\partial\Phi(\boldsymbol{\tau})$ is defined on M and that $\partial\Phi(\boldsymbol{\tau}) = \{\Phi'(\boldsymbol{\tau})\}$. Also remark that Φ is not a proper function on $L^p(\Omega)^d \setminus M$ and therefore $\partial\Phi(\boldsymbol{\tau}) = \emptyset$ for $\boldsymbol{\tau} \in L^p(\Omega)^d \setminus M$. This observation leads to the following formulation.

Definition 1. *By the* weak generalized solution (w.g.s.) *of Problem D-A we call a function* $u \in W_0^{1,p}(\Omega)$ *satisfying the bound*

$$\text{(1.2a)}\qquad |\nabla_x u| \le 1 \ \textit{almost everywhere in } \Omega$$

and the variational inequality

$$\text{(1.2b)}\qquad \int_\Omega \big[|\nabla_x u|^{p-2}\nabla_x u \cdot \nabla_x(\varphi - u) + |u|^{p-2}u(\varphi - u)\big]\, d\boldsymbol{x} \ge \langle f, \varphi - u\rangle$$

for any test function $\varphi \in W_0^{1,p}(\Omega)$ *such that* $|\nabla_x \varphi| \le 1$ *almost everywhere in* Ω*.*

Notation 2. *Introduce the notation for two solution-dependent*[3] *domains in* Ω*:*

$$\Omega_- = \{\boldsymbol{x} \in \Omega:\ |\nabla_x u(\boldsymbol{x})| < 1\}, \quad \Omega_1 = \Omega \setminus \overline{\Omega}_-.$$

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[3]Notice that the w.g.s. of Problem D-A exists and is unique. See Proposition 2.

Remark 1. *Since M is a convex set containing the origin $\boldsymbol{x} = (0, 0, \dots, 0)$,* (1.2b) *is equivalent to equation*[4]

$$-\operatorname{div}_x(|\nabla_x u|^{p-2}\nabla_x u) + |u|^{p-2}u = f \quad \text{in } \Omega_- \tag{1.3}$$

in the sense of distributions. In Ω_1, (1.2b) *does not imply* (1.3)*, in general.*

Notation 3. *Introduce notation for the class of sought functions u,*

$$\mathcal{M} := \{\varphi \in W_0^{1,p}(\Omega)\colon\ |\nabla_x \varphi| \le 1 \text{ almost everywhere in } \Omega\}. \tag{1.4}$$

Validity of the next two assertions directly follows from the well-known facts in convex analysis and theory of variational inequalities for monotonous operators [6, Ch. I, Prop. 5.5; Ch. II, Prop. 2.1], [14, Ch. III, Th. 1.4].

Proposition 1. *Function $u \in \mathcal{M}$ is a w.g.s. to Problem D-A in the sense of Definition 1 if and only if it is a solution to the minimization problem on $\mathcal{M}$ for the functional*

$$F(\varphi) = \frac{1}{p}\int_\Omega (|\nabla_x \varphi|^p + |\varphi|^p)\, d\boldsymbol{x} - \langle f, \varphi\rangle,$$

i.e., if

$$F(u) = \inf_{\varphi \in \mathcal{M}} F(\varphi). \tag{1.5}$$

Proposition 2. *Whenever $f \in W^{-1,p'}(\Omega)$, Problem D-A has a unique w.g.s. in the sense of Definition 1.*

Remark 2. *The set $\mathcal{M}$ consists of rather regular functions. By the second Sobolev embedding theorem [19, Ch. I, Th. 1.2], one has that if $\varphi \in \mathcal{M}$ then $\varphi \in W_0^{1,\infty}(\Omega)$ and the operator of natural embedding of $\mathcal{M}$ into $W_0^{1,\infty}(\Omega)$ is continuous. Furthermore, by the first Sobolev embedding theorem [19, Ch. I, Th. 1.1], one has that if $\varphi \in \mathcal{M}$ then φ has a continuous extension on $\overline{\Omega}$. In particular, for any $\boldsymbol{x}_0 \in \partial\Omega$ one has that $\operatorname{ess\,lim}_{\substack{\boldsymbol{x} \to \boldsymbol{x}_0 \\ \boldsymbol{x} \in \Omega}} \varphi(\boldsymbol{x}) = 0$ for **any** sequence $\{\boldsymbol{x} \to \boldsymbol{x}_0,\ \boldsymbol{x} \in \Omega\}$.*

2. Kaplan's penalty function

In applications, it is often useful to find solutions to problems of the form (1.5) approximately via solutions of problems of unconstrained optimization. Penalty methods give ways to implement this approach. These methods have a long history and they constitute a large and powerful theory. The basics of this theory can be found, for example, in [7, 11, 14, 16]. Also, these methods are fruitful for studies of topics on refined regularity of solutions to optimization problems with nonlinear constraints (see [16, Ch. 3, Sec. 5.5]).

Let us briefly recall that, for a closed convex nonempty subset $\mathcal{K}$ of a reflexive Banach space $\mathcal{V}$, by a *penalty operator associated with* $\mathcal{K}$ we call any operator β satisfying the following properties:

$$\beta\colon\ \mathcal{V} \mapsto \mathcal{V}^* \quad \text{is monotonous, bounded, and semicontinuous,}$$
$$\{w\colon\ w \in \mathcal{V},\ \beta(w) = 0\} = \mathcal{K}.$$

[4] Recall that (1.3) is a nonlinear degenerate equation for $p > 2$ and a nonlinear singular equation for $p \in (1, 2)$. For $p = 2$, it is a semilinear equation with the convenient Laplace operator $\Delta_2 \equiv \Delta_x$.

Let us consider the minimization problem for some Gâteaux-differentiable strictly convex functional $\Psi\colon \mathcal{V} \mapsto \mathbb{R}$:

$$\text{Find} \quad u \in \mathcal{K} \quad \text{such that} \quad \Psi(u) = \inf_{w\in\mathcal{K}} \Psi(w). \tag{2.1}$$

Problem

$$\Psi'(u_\varepsilon) + \frac{1}{\varepsilon}\beta(u_\varepsilon) = 0, \quad u_\varepsilon \in \mathcal{K} \quad (\varepsilon > 0), \tag{2.2}$$

where β is a penalty operator associated with $\mathcal{K}$, is called the *penalized problem associated with problem* (2.1). Suppose additionally that the norms defined in $\mathcal{V}$ and $\mathcal{V}^*$ are strictly convex.[5] On the strength of the above made assumptions and the well-known results from [16, Ch. III, Sec. 5], penalty operators exist, problem (2.2) is uniquely solvable for any fixed $\varepsilon \in (0,1)$, and the sequence $\{u_\varepsilon\}$ of its solutions converges weakly in $\mathcal{V}$ to the unique solution $u \in \mathcal{K}$ of problem (2.1), as $\varepsilon \to 0$.

In general, the penalty operator β can be defined in different ways, the question about how to choose the 'best possible' operator is somewhat fuzzy, as well as an answer to it. In the present article, in order to construct approximate solutions to Problem D-A, we apply the penalty function which was originally introduced by Alexander Kaplan [11] and then has been widely implemented in study of nonlinear problems of variational calculus with constraints [10, 12, 13, 22].

Definition 2. *[11, Ch. III, §3.4, Formula (3.44)]. Functional $\Phi_\varepsilon^{(t)}\colon W_0^{1,p}(\Omega) \mapsto \mathbb{R}$ defined by the formula*

$$\Phi_\varepsilon^{(t)}(\varphi) = \frac{1}{\varepsilon p}\int_\Omega \Big(|\nabla_x\varphi|^p - 1 + \sqrt{(|\nabla_x\varphi|^p - 1)^2 + \varepsilon^{2+t}}\Big)\,d\boldsymbol{x}, \quad \varepsilon > 0,\ t \geq 0$$

is called Alexander Kaplan's integral penalty function *associated with the set $\mathcal{M}$, defined by formula* (1.4).

Notation 4. *Introduce notation for the interior of the set $\mathcal{M}$:*

$$\mathit{int}\,\mathcal{M} := \{\,\varphi \in \mathcal{M}\colon\ \exists\,\delta = \mathit{const} \in (0,1) \text{ such that } |\nabla_x\varphi|^p \leq 1 - \delta \text{ for almost all } \boldsymbol{x} \in \Omega\}.$$

Proposition 3. *Family $\{\Phi_\varepsilon^{(t)}\}_{\varepsilon\in(0,1)}$ has the following properties.*

(i) $\Phi_\varepsilon^{(t)}$ *are convex functionals.*

(ii) $\Phi_\varepsilon^{(t)}(\varphi) \underset{\varepsilon\to0}{\longrightarrow} 0 \quad \forall\,\varphi \in \mathrm{int}\,\mathcal{M}$.

(iii) $\Phi_\varepsilon^{(t)}(\varphi) \underset{\varepsilon\to0}{\longrightarrow} +\infty \quad \forall\,\varphi \in W_0^{1,p}(\Omega)\setminus\mathcal{M}$.

(iv) *For fixed $\varepsilon > 0$ and $t \geq 0$ functional $\Phi_\varepsilon^{(t)}$ is Gâteaux-differentiable. Its derivative $(\Phi_\varepsilon^{(t)})'\colon W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$ is defined by the formula*

$$\langle(\Phi_\varepsilon^{(t)})'(\varphi), \psi\rangle = \frac{1}{\varepsilon}\int_\Omega \Big(1 + \frac{|\nabla_x\varphi|^p - 1}{\sqrt{(|\nabla_x\varphi|^p - 1)^2 + \varepsilon^{2+t}}}\Big)|\nabla_x\varphi|^{p-2}\nabla_x\varphi\cdot\nabla_x\psi\,d\boldsymbol{x} \qquad \forall\,\varphi,\psi \in W_0^{1,p}(\Omega).$$

[5] Note that the canonical norms in $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ $(p \in (1,+\infty))$ are strictly convex.

Proof. Justification of assertions (i)–(iii) is fulfilled analogously to the proof of these assertions for algebraic finite (non-integral) Alexander Kaplan's penalty function, keeping track of considerations in [9], [11, Ch. III, Sec. 3.4]. Assertion (iv) is verified by the direct calculation of the Gâteaux derivative. □

Remark 3. *We call operator* $\beta_\varepsilon^{(t)} := \varepsilon(\Phi_\varepsilon^{(t)})'$ Alexander Kaplan's approximate penalty operator *and we call the problem*

$$F'(u_\varepsilon) + \frac{1}{\varepsilon}\beta_\varepsilon^{(t)}(u_\varepsilon) = 0, \quad u_\varepsilon \in W_0^{1,p}(\Omega), \quad \varepsilon > 0 \tag{2.3}$$

Alexander Kaplan's penalized problem *associated with Problem D-A.*

As well as for the 'abstract' problem (2.2), solution of (2.3) is understood in the generalized sense.

Definition 3. *Function* $u_\varepsilon \in W_0^{1,p}(\Omega)$ *is called a* weak generalized solution (w.g.s.) *of problem* (2.3) *if it satisfies the integral equality*

$$\begin{aligned}(2.4)\quad &\int_\Omega \big(|\nabla_x u_\varepsilon|^{p-2}\nabla_x u_\varepsilon \cdot \nabla_x\varphi + |u_\varepsilon|^{p-2}u_\varepsilon\varphi\big)\,d\boldsymbol{x}\\ &+\frac{1}{\varepsilon}\int_\Omega\Big(1+\frac{|\nabla_x u_\varepsilon|^p - 1}{\sqrt{(|\nabla_x u_\varepsilon|^p-1)^2+\varepsilon^{2+t}}}\Big)|\nabla_x u_\varepsilon|^{p-2}\nabla_x u_\varepsilon\cdot\nabla_x\varphi\,d\boldsymbol{x} = \langle f,\varphi\rangle\\ &\qquad\qquad\qquad\forall\,\varphi\in W_0^{1,p}(\Omega).\end{aligned}$$

Remark 4. *Strictly speaking, operator* $\beta_\varepsilon^{(t)}$ *itself is not a penalty operator associated with the set* $\mathcal{M}$. *Furthermore, for any* $\varphi \in W_0^{1,p}(\Omega)$ *one has that*

$$\beta_\varepsilon^{(t)}(\varphi) \underset{\varepsilon\to 0}{\longrightarrow} \beta^{(t)}(\varphi) \text{ uniformly in } W^{-1,p'}(\Omega),$$

where $\beta^{(t)}$ *is defined by the formula*

(2.5)

$$\langle\beta^{(t)}(\varphi),\psi\rangle = \int_{\{\boldsymbol{x}:\ |\nabla_x\varphi|=1\}} \nabla_x\varphi\cdot\nabla_x\psi\,d\boldsymbol{x} + \int_{\{\boldsymbol{x}:\ |\nabla_x\varphi|>1\}} 2|\nabla_x\varphi|^{p-2}\nabla_x\varphi\cdot\nabla_x\psi\,d\boldsymbol{x} \qquad \forall\,\varphi,\psi\in W_0^{1,p}(\Omega).$$

Denote $\mathcal{M}_0 := \{\varphi\in W_0^{1,p}(\Omega):\ |\nabla_x\varphi| < 1$ *almost everywhere in* $\Omega\}$. *From* (2.5) *it is clear that* $\{\varphi\in W_0^{1,p}(\Omega):\ \beta^{(t)}(\varphi)=0\} = \mathcal{M}_0$. *Thus the limiting penalty operator is associated with the unclosed in* $W_0^{1,p}(\Omega)$ *convex set* $\mathcal{M}_0$. *Obviously, the set* $\mathcal{M}$ *is its closure in* $W_0^{1,p}(\Omega)$.

In view of this remark, the limiting passage as $\varepsilon\to 0$ in (2.3) is more delicate than the similar one in the classical monograph [16, Ch. 3, Th. 5.2]. The next section is devoted to the proof of solvability of problem (2.3) for fixed $\varepsilon > 0$ and to justification of the limiting passage in this problem as $\varepsilon \to 0$.

3. Proof of solvability of Problem D-A by means of operator $\beta_\varepsilon^{(t)}$

The first main result of the article is as follows.

Theorem 1. (i) *For any fixed* $\varepsilon\in(0,1)$ *and for any given* $f\in W^{-1,p'}(\Omega)$ *there is a unique w.g.s.* u_ε *to problem* (2.3) *in the sense of Definition 3.*

(ii) $u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u$ *weakly in* $W_0^{1,p}(\Omega)$, *where* u *is the w.g.s. of Problem D-A.*

Let us introduce one useful notation and then turn to verification of Theorem 1.

Notation 5. *Define operator* A*:* $W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$ *by the formula*

$$A(\varphi) := F'(\varphi) + f \quad \forall\, \varphi \in W_0^{1,p}(\Omega),$$

or, equivalently,

$$\langle A(\varphi), \psi \rangle = \int_\Omega \big(|\nabla_x \varphi|^{p-2} \nabla_x \varphi \cdot \nabla_x \psi + |\varphi|^{p-2} \varphi \psi \big)\, d\boldsymbol{x} \quad \forall\, \varphi, \psi \in W_0^{1,p}(\Omega).$$

Proof. **(1)** In order to justify assertion (i), it is sufficient to verify that the operator

$$\Big(\varphi \mapsto A(\varphi) + \frac{1}{\varepsilon} \beta_\varepsilon^{(t)}(\varphi) \Big) : \; W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$$

is bounded, strictly monotonous, semicontinuous, and coercive [16, Ch. 2, Sec. 2, Th. 2.1]. *Coerciveness* is understood in the sense of the limiting relation

$$\frac{\Big\langle A(\varphi) + \dfrac{1}{\varepsilon} \beta_\varepsilon^{(t)}(\varphi), \varphi \Big\rangle}{\|\varphi\|_{W_0^{1,p}(\Omega)}} \to +\infty \quad \text{for } \|\varphi\|_{W_0^{1,p}(\Omega)} \to +\infty.$$

Also recall that an operator mapping from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ is called *bounded*, if it maps bounded sets in $W_0^{1,p}(\Omega)$ into bounded sets in $W^{-1,p'}(\Omega)$; an operator $\mathcal{A}$: $W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$ is called *semicontinuous* if

the function $\lambda \mapsto \langle \mathcal{A}(\varphi + \lambda \psi), \zeta \rangle$ is continuous from $\mathbb{R}$ into $\mathbb{R}$, $\forall\, \varphi, \psi, \zeta \in W_0^{1,p}(\Omega)$;

and an operator $\mathcal{A}$: $W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$ is called *monotonous* if

$$\langle \mathcal{A}(\varphi) - \mathcal{A}(\psi), \varphi - \psi \rangle \geq 0 \quad \forall\, \varphi, \psi \in W_0^{1,p}(\Omega).$$

If this inequality is strict for all $\varphi \neq \psi$ then $\mathcal{A}$ is called *strictly monotonous*.

All four properties (i.e., boundedness, strict monotonicity, semicontinuity, and coercivity) are verified for the mapping $\varphi \mapsto A(\varphi) + \dfrac{1}{\varepsilon} \beta_\varepsilon(\varphi)$ directly by means of Hölder's, Young's, and Cauchy — Schwarz' inequalities. Also notice that the mapping $\varphi \mapsto F(\varphi) + \langle f, \varphi \rangle$ is a strictly convex functional and $\Phi_\varepsilon^{(t)}$ is a convex functional (see assertion (ii) in Proposition 3 and [16, Ch. 2, Sec. 1, Prop. 1.1]).

Thus, assertion (i) in Theorem 1 is proved. Now we pass to the limit and rigorously justify the limiting relation in (2.4), as $\varepsilon \to 0$.

(2) Insert u_ε into (2.4) on the place of φ, which is legal. This yields the energy identity

(3.1)

$$\int_\Omega \big(|\nabla_x u_\varepsilon|^p + |u_\varepsilon|^p \big) d\boldsymbol{x} + \frac{1}{\varepsilon} \int_\Omega \Big(1 + \frac{|\nabla_x u_\varepsilon|^p - 1}{\sqrt{(|\nabla_x u_\varepsilon|^p - 1)^2 + \varepsilon^{2+t}}} \Big) |\nabla_x u_\varepsilon|^p \, d\boldsymbol{x} = \langle f, u_\varepsilon \rangle.$$

Estimating in the right-hand side by means of Cauchy — Schwarz' and Young's inequalities, we arrive at the energy inequality

$$\|u_\varepsilon\|^p_{W_0^{1,p}(\Omega)} \leq \|f\|^{p'}_{W^{-1,p'}(\Omega)} \tag{3.2}$$

and the elementary bound for the penalty operator

(3.3)

$$0 < \langle \beta_\varepsilon^{(t)}(u_\varepsilon), u_\varepsilon \rangle \equiv \int_\Omega \Big(1 + \frac{|\nabla_x u_\varepsilon|^p - 1}{\sqrt{(|\nabla_x u_\varepsilon|^p - 1)^2 + \varepsilon^{2+t}}} \Big) |\nabla_x u_\varepsilon|^p \, d\boldsymbol{x} \leq \frac{\varepsilon}{p'} \|f\|^{p'}_{W^{-1,p'}(\Omega)}.$$

Now rewriting (2.3) in the form

$$\beta_\varepsilon^{(t)}(u_\varepsilon) = -\varepsilon F'(u_\varepsilon) \tag{3.4}$$

and using (3.2) and Cauchy — Buniakovskii's and Cauchy — Schwarz' inequaities, we derive the estimate on the norm of $\beta_\varepsilon^{(t)}(u_\varepsilon)$:

$$\langle \beta_\varepsilon^{(t)}(u_\varepsilon), \varphi\rangle \le \varepsilon(1 + C^2_{sob}(\Omega))\|f\|_{W^{-1,p'}(\Omega)}\|\varphi\|_{W_0^{1,p}(\Omega)} \quad \forall\, \varphi \in W_0^{1,p}(\Omega),$$

that is,

$$\|\beta_\varepsilon^{(t)}(u_\varepsilon)\|_{W^{-1,p'}(\Omega)} \le \varepsilon(1 + C^2_{sob}(\Omega))\|f\|_{W^{-1,p'}(\Omega)}. \tag{3.5}$$

Here $C_{sob}(\Omega)$ is the constant from either the first (in the case $p \le d$) or the second (in the case $p > d$) Sobolev embedding theorem (see [19, Ch. I, Sec. 1.2]). It depends merely on the exponent p and on geometry of the domain Ω.

Alongside (3.5) we establish the bound

$$\|A(u_\varepsilon)\|_{W^{-1,p'}(\Omega)} \le C^2_{sob}(\Omega)\|f\|_{W^{-1,p'}(\Omega)}. \tag{3.6}$$

(3) On the strength of (3.2) and (3.6), extracting a proper subsequence from $\{\varepsilon \to 0\}$, if necessary,[6] we conclude that

$$u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u \quad \text{weakly in} \quad W_0^{1,p}(\Omega), \tag{3.7}$$

$$A(u_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \chi \quad \text{weakly* in} \quad W^{-1,p'}(\Omega), \tag{3.8}$$

with some limiting u and χ.

Using Lebesgue's dominated convergence theorem [23, theorem 1.4.48] as $\varepsilon \to 0$, from (3.3) we derive that $u \in \mathcal{M}$.

Let $v \in \operatorname{int}\mathcal{M}$. Take $\varphi := v$ in (2.4) and combine the result with (3.1) to get

$$\langle A(u_\varepsilon) - f, v - u_\varepsilon\rangle = \frac{1}{\varepsilon}\,\langle \beta_\varepsilon^{(t)}(v) - \beta_\varepsilon^{(t)}(u_\varepsilon), v - u_\varepsilon\rangle - \frac{1}{\varepsilon}\,\langle \beta_\varepsilon^{(t)}(v), v - u_\varepsilon\rangle. \tag{3.9}$$

Here we added and subtracted the term $\frac{1}{\varepsilon}\,\langle \beta_\varepsilon^{(t)}(v), v - u_\varepsilon\rangle$ in the right-hand side.

Since $v \in \operatorname{int}\mathcal{M}$, by the definition we have $|\nabla_x v|^p \le 1 - \delta$ for almost all $\boldsymbol{x} \in \Omega$ with some constant $\delta \in (0,1)$. By simple technical considerations we establish the estimate

$$\begin{aligned} 0 < 1 + \frac{|\nabla_x v|^p - 1}{\sqrt{(|\nabla_x v|^p - 1)^2 + \varepsilon^{2+t}}} &= \\ = \frac{\varepsilon^{2+t}}{(|\nabla_x v|^p - 1)^2 + \varepsilon^{2+t} + (1 - |\nabla_x v|^p)\sqrt{(|\nabla_x v|^p - 1)^2 + \varepsilon^{2+t}}} &\le \\ \le \frac{\varepsilon^{2+t}}{\delta^2 + \varepsilon^{2+t} + \delta\sqrt{\delta^2 + \varepsilon^{2+t}}} \le \frac{\varepsilon^{2+t}}{2\delta^2} \quad &\text{almost everywhere in } \Omega. \end{aligned} \tag{3.10}$$

On the strength of (3.2) and (3.10), from the definition of penalty operator $\beta_\varepsilon^{(t)}$ (see Remark 3 and assertion (iv) in Proposition 3) we derive that

$$\frac{1}{\varepsilon}\langle \beta_\varepsilon^{(t)}(v), v - u_\varepsilon\rangle = o(\varepsilon^{1+t-\theta}) \text{ as } \varepsilon \to 0 \quad \forall\, \theta \in (0,1),\ \forall\, v \in \operatorname{int}\mathcal{M}. \tag{3.11}$$

[6] We keep index ε for the extracted subsequence.

Since $\beta_\varepsilon^{(t)}$ is monotonous, we have that $\frac{1}{\varepsilon}\langle\beta_\varepsilon^{(t)}(v)-\beta_\varepsilon^{(t)}(u_\varepsilon),v-u_\varepsilon\rangle\geq 0$. On the strength of (3.11), this inequality and (3.9) yield that

$$\langle A(u_\varepsilon)-f,v-u_\varepsilon\rangle\geq-\frac{1}{\varepsilon}\langle\beta_\varepsilon^{(t)}(v),v-u_\varepsilon\rangle. \tag{3.12}$$

In turn, due to (3.7) and (3.8), this gives

$$\langle\chi-f,v\rangle\geq\limsup_{\varepsilon\to0}\langle A(u_\varepsilon),u_\varepsilon\rangle-\langle f,u\rangle. \tag{3.13}$$

Notice that $v:=\frac{u}{\lambda}$ with any $\lambda>1$ is an admissible test function for (3.13), since $\frac{u}{\lambda}\in\operatorname{int}\mathcal{M}$ in this case. Now passing to the limit as $\lambda\to1+0$, from (3.13) we derive

$$\langle\chi-f,u\rangle\geq\limsup_{\varepsilon\to0}\langle A(u_\varepsilon),u_\varepsilon\rangle-\langle f,u\rangle,$$

i.e.,

$$\langle\chi,u\rangle\geq\limsup_{\varepsilon\to0}\langle A(u_\varepsilon),u_\varepsilon\rangle. \tag{3.14}$$

Recall from item (1) of this proof that A is bounded monotonous and semicontinuous. Hence it is *pseudo-monotonous* in the sense that the following property holds true [16, Ch. 2, Defn. 2.1 and Th. 2.5].

$$\begin{aligned}&\text{From the condition}\\ &w_j\underset{j\to\infty}{\longrightarrow}w \text{ weakly in } W_0^{1,p}(\Omega) \text{ and } \limsup_{j\to\infty}\langle A(w_j),w_j-v\rangle\leq0,\\ &\text{it follows that}\quad \liminf_{j\to\infty}\langle A(w_j),w_j-v\rangle\geq\langle A(w),w-v\rangle\quad\forall\,v\in W_0^{1,p}(\Omega).\end{aligned} \tag{3.15}$$

From (3.14) and (3.15) it follows that $\liminf_{\varepsilon\to0}\langle A(u_\varepsilon),u_\varepsilon-v\rangle\geq\langle A(u),u-v\rangle$. On the strength of (3.11) and (3.12), this yields that $\langle f,u-v\rangle\geq\langle A(u),u-v\rangle$ for all $v\in\operatorname{int}\mathcal{M}$, and hence for all $v=\varphi\in\mathcal{M}$ due to denseness of $\operatorname{int}\mathcal{M}$ in $\mathcal{M}$. This means that (1.2b) is valid. Therefore u is the w.g.s. of Problem D-A.

Theorem 1 is proved. □

Remark 5. (On convergence of energies.) *Since $\{u_\varepsilon\}_{\varepsilon\to0}$ is a minimizing sequence and $\langle A(v),v\rangle=\|v\|^p_{W_0^{1,p}(\Omega)}$ $\forall\,v\in W_0^{1,p}(\Omega)$, the following limiting relation for the stored p-energies holds true:*

$$\frac{1}{p}\|u_\varepsilon\|^p_{W_0^{1,p}(\Omega)}-\langle f,u_\varepsilon\rangle\underset{\varepsilon\to0}{\longrightarrow}\frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}-\langle f,u\rangle. \tag{3.16}$$

Evidently, in the case $p=2$ from (3.16) *it follows that*

$$u_\varepsilon\underset{\varepsilon\to0}{\longrightarrow}u \text{ strongly in } W_0^{1,2}(\Omega). \tag{3.17}$$

Also, analogously to [16, Ch. 3, Sec. 5.3] we establish the limiting relation

$$\langle A(u_\varepsilon)-A(u),u_\varepsilon-u\rangle\underset{\varepsilon\to0}{\longrightarrow}0. \tag{3.18}$$

Let us notice that properties (3.16)–(3.18) *do not depend on a choice of penalty operator.*[7]

[7] The relations (3.16)–(3.18) hold true for any admissible penalty operator.

In the next (concluding) section, we find out rather subtle and nonstandard features of the family $\{u_\varepsilon\}$ that have place precisely thanks to the construction of Alexander Kaplan's penalty function.

4. Property of uniform approximation

At first, we establish one important geometric property that leads further to the result on uniform convergence of the family $\{u_\varepsilon\}$, as $\varepsilon \to 0$. We introduce into consideration some sets in Ω depending on the solution of problem (2.3).

Notation 6. *For $\varepsilon, \delta \in (0,1)$ denote*

$$\Omega_-(\varepsilon) := \{\boldsymbol{x} \in \Omega\colon\ |\nabla_x u_\varepsilon(\boldsymbol{x})| < 1\}, \quad \Omega_+(\varepsilon) := \{\boldsymbol{x} \in \Omega\colon\ |\nabla_x u_\varepsilon(\boldsymbol{x})| \geq 1\},$$
$$\Omega_-(\varepsilon,\delta) := \{\boldsymbol{x} \in \Omega\colon\ |\nabla_x u_\varepsilon(\boldsymbol{x})|^p \leq 1-\delta\},$$
$$\Omega_1(\varepsilon,\delta) := \{\boldsymbol{x} \in \Omega\colon\ 1-\delta < |\nabla_x u_\varepsilon(\boldsymbol{x})|^p < 1\}.$$

Remark 6. *Obviously, the sets $\Omega_+(\varepsilon)$, $\Omega_-(\varepsilon,\delta)$, and $\Omega_1(\varepsilon,\delta)$ do not intersect pairwise, and $\Omega_+(\varepsilon) \cup \Omega_-(\varepsilon,\delta) \cup \Omega_1(\varepsilon,\delta) = \Omega$ for all $\varepsilon, \delta \in (0,1)$.*

Notation 7. *In the formulation of Theorem 2 and further in the text by* meas $\mathcal{Q}$ *we denote the Lebesgue measure of a Lebesgue-measurable set $\mathcal{Q}$.*

Theorem 2. *There exists an increasing by inclusion sequence*[8] *of closed sets $\{E^{\varepsilon_l}\}$, $\varepsilon_l \underset{l\to\infty}{\longrightarrow} 0$, such that*

(i) $E^{\varepsilon_l} \subset \Omega_-(\varepsilon_l, \varepsilon_l^{1+t/2})$, $l = 1,2,3,\ldots$,

(ii) $\operatorname{meas}(\Omega \setminus E^{\varepsilon_l}) \leq \varepsilon_{l-1}$. *In particular,* $\operatorname{meas}\Big(\Omega \setminus \bigcup_{l=1}^{\infty} E^{\varepsilon_l}\Big) = 0$.

Proof. For all $\varepsilon \in (0,1)$ due to (3.3) we have that

(4.1)
$$\frac{\varepsilon}{p'}\|f\|^{p'}_{W^{-1,p'}(\Omega)} \geq \int_{\Omega_+(\varepsilon)} \Big(1 + \frac{|\nabla_x u_\varepsilon|^p - 1}{\sqrt{(|\nabla_x u_\varepsilon|^p-1)^2 + \varepsilon^{2+t}}}\Big)|\nabla_x u_\varepsilon|^p\, d\boldsymbol{x} \geq \operatorname{meas}\,\Omega_+(\varepsilon),$$

$$\frac{\varepsilon}{p'}\|f\|^{p'}_{W^{-1,p'}(\Omega)} \geq \int_{\Omega_-(\varepsilon)} \Big(1 - \frac{1-|\nabla_x u_\varepsilon|^p}{\sqrt{(|\nabla_x u_\varepsilon|^p-1)^2 + \varepsilon^{2+t}}}\Big)|\nabla_x u_\varepsilon|^p\, d\boldsymbol{x} = \tag{4.2}$$
$$= \int_{\Omega_-(\varepsilon)} \frac{\varepsilon^{2+t}|\nabla_x u_\varepsilon|^p\, d\boldsymbol{x}}{(1-|\nabla_x u_\varepsilon|^p)^2 + \varepsilon^{2+t} + (1-|\nabla_x u_\varepsilon|^p)\sqrt{(1-|\nabla_x u_\varepsilon|^p)^2+\varepsilon^{2+t}}}.$$

From (4.2), for all $\varepsilon, \delta \in (0,1)$ the estimate

$$\frac{\varepsilon}{p'}\|f\|^{p'}_{W^{-1,p'}(\Omega)} \geq$$
$$\geq \int_{\Omega_1(\varepsilon,\delta)} \frac{\varepsilon^{2+t}|\nabla_x u_\varepsilon|^p\, d\boldsymbol{x}}{(1-|\nabla_x u_\varepsilon|^p)^2 + \varepsilon^{2+t} + (1-|\nabla_x u_\varepsilon|^p)\sqrt{(1-|\nabla_x u_\varepsilon|^p)^2+\varepsilon^{2+t}}} \geq$$
$$\geq \int_{\Omega_1(\varepsilon,\delta)} \frac{\varepsilon^{2+t}(1-\delta)\, d\boldsymbol{x}}{\delta^2 + \varepsilon^{2+t} + \delta\sqrt{\delta^2+\varepsilon^{2+t}}}$$

---

[8] The increase in the inclusion means that $E^{\varepsilon_n} \subset E^{\varepsilon_m}$ for $m > n$.

follows. Since δ and ε are mutually independent, we may take $\delta = \varepsilon^{1+t/2}$. Continue estimating, taking into account that $1-\delta = 1-\varepsilon^{1+t/2} \geq \dfrac{1}{2}$ for $\varepsilon \leq 2^{-2/(2+t)}$, to get

$$\frac{\varepsilon}{p'}\|f\|^{p'}_{W^{-1,p'}(\Omega)} \geq \frac{\operatorname{meas}\big(\Omega_1(\varepsilon,\varepsilon^{1+t/2})\big)}{4+2\sqrt{2}}. \tag{4.3}$$

In view of Remark 6 due to (4.1) and (4.3) we conclude that

$$\operatorname{meas}\big(\Omega\setminus\Omega_-(\varepsilon,\varepsilon^{1+t/2})\big) \leq \frac{(5+2\sqrt{2})\|f\|^{p'}_{W^{-1,p'}(\Omega)}}{p'}\,\varepsilon \quad \forall\,\varepsilon\in(0,2^{-2/(2+t)}). \tag{4.4}$$

Introduce the characteristic functions of the sets Ω and $\Omega_-(\varepsilon,\varepsilon^{1+t/2})$:

$$\mathbf{1}[\Omega](\boldsymbol{x}) \equiv 1 \text{ for } \boldsymbol{x}\in\Omega,$$
$$\mathbf{1}[\Omega_-(\varepsilon,\varepsilon^{1+t/2})](\boldsymbol{x}) := \begin{cases} 1 & \text{for} \quad \boldsymbol{x}\in\Omega_-(\varepsilon,\varepsilon^{1+t/2}),\\ 0 & \text{for} \quad \boldsymbol{x}\in\Omega\setminus\Omega_-(\varepsilon,\varepsilon^{1+t/2}).\end{cases}$$

Estimate (4.4) means that

$$\mathbf{1}[\Omega_-(\varepsilon,\varepsilon^{1+t/2})] \underset{\varepsilon\to0}{\longrightarrow} \mathbf{1}[\Omega] \quad \text{in measure in } \Omega. \tag{4.5}$$

Hence by Riesz's theorem [18, Ch. IV, Sec. 3, Th. 4] there exists $\varepsilon_k\to0$ $(k\to\infty)$ such that

$$\mathbf{1}[\Omega_-(\varepsilon_k,\varepsilon_k^{1+t/2})] \underset{k\to\infty}{\longrightarrow} \mathbf{1}[\Omega] \quad \text{almost everywhere in } \Omega. \tag{4.6}$$

On the strength of Egorov's theorem [18, Ch. IV, Sec. 3, Th. 5], for any $\gamma>0$ there exists such measurable set $\Omega_\gamma\subset\Omega$ that

$$\operatorname{meas}(\Omega\setminus\Omega_\gamma) < \gamma, \tag{4.7}$$

$$\mathbf{1}[\Omega_-(\varepsilon_k,\varepsilon_k^{1+t/2})] \underset{k\to\infty}{\longrightarrow} \mathbf{1}[\Omega] \quad \text{uniformly on } \Omega_\gamma. \tag{4.8}$$

Choose some value $\gamma=\varepsilon_{k_1}\in\{\varepsilon_k\}_{k=1,2,\dots}$ and set $\Omega_{\varepsilon_{k_1}}$ according to Egorov's theorem. Due to (4.8) and the fact that functions $\mathbf{1}[\Omega_-(\varepsilon_k,\varepsilon_k^{1+t/2})]$ and $\mathbf{1}[\Omega]$ attain values 0 and 1 only, there exists $K_1\in\mathbb{N}$ such that

$$\mathbf{1}[\Omega_-(\varepsilon_k,\varepsilon_k^{1+t/2})] = 1 \text{ on } \Omega_{\varepsilon_{k_1}} \quad \forall\,k\geq K_1 \tag{4.9}$$

Take $k_2:=\max\{K_1,k_1+1\}$, $E^{\varepsilon_{k_2}}:=\Omega_{\varepsilon_{k_1}}$. By the construction, we have $E^{\varepsilon_{k_2}}\subset\Omega_-(\varepsilon_{k_2},\varepsilon_{k_2}^{1+t/2})$ and $\operatorname{meas}(\Omega\setminus E^{\varepsilon_{k_2}})\leq\varepsilon_{k_1}$. Further take $\gamma=\varepsilon_{k_2}$ and repeat the procedure. This means, build $E^{\varepsilon_{k_3}}\subset\Omega_-(\varepsilon_{k_3},\varepsilon_{k_3}^{1+t/2})$ such that $E^{\varepsilon_{k_2}}\subset E^{\varepsilon_{k_3}}$ (due to (4.9)), $\operatorname{meas}(\Omega\setminus E^{\varepsilon_{k_3}})\leq\varepsilon_{k_2}$. Repeating this process, we build the sequence

$$E^{\varepsilon_{k_l}}\subset\Omega_-(\varepsilon_{k_l},\varepsilon_{k_l}^{1+t/2}),\quad E^{\varepsilon_{k_{l-1}}}\subset E^{\varepsilon_{k_l}},\quad \operatorname{meas}(\Omega\setminus E^{\varepsilon_{k_l}})\leq\varepsilon_{k_{l-1}},\quad \varepsilon_{k_l}\underset{l\to\infty}{\longrightarrow}0. \tag{4.10}$$

Relations (4.10) are precisely the assertions (i) and (ii) of the theorem with $\varepsilon_l:=\varepsilon_{k_l}$. Theorem 2 is proved. □

The next theorem is the second main result of the article.

Theorem 3. (On uniform convergence.) *For any $\delta > 0$ there exist a value $\varepsilon_0 = \varepsilon_0(\delta) \in (0,1)$ and a closed set $\Xi^\delta \subset \Omega$,* meas $\Xi^\delta >$ meas $\Omega - \delta$, *such that $u_\varepsilon \in C^{0+\vartheta}(\Xi^\delta)$ $\forall\, \vartheta \in [0,1)$, $\forall\, \varepsilon \in (0,\varepsilon_0]$ and*

$$u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u \text{ in } C(\Xi^\delta) \text{ uniformly in } C(\Xi^\delta). \tag{4.11}$$

Notation 8. *By $C^{0+\vartheta}(\Xi^\delta)$ in the formulation of Theorem 3 we standardly denote the space of Hölder-continuous functions on set Ξ^δ with Hölder's exponent ϑ. For $\vartheta = 0$ we set $C^{0+0}(\Xi^\delta) := C(\Xi^\delta)$.*

Proof. By Theorem 2 we can take E^{ε_l} with $\varepsilon_{l-1} \le \delta$ as the set Ξ^δ. On the strength of the second Sobolev embedding theorem, we obtain that the family $\{u_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is uniformly bounded in $C^{0+\vartheta}(\Xi^\delta)$ $(\forall\, \vartheta \in [0,1))$. Hence it is uniformly bounded and equicontinuous in $C(\Xi^\delta)$. On the other hand, on the strength of (3.7) and the first Sobolev embedding theorem, we have that $u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u$ weakly in $L^q(\Xi^\delta)$ for any $q \in [1,\infty)$. From this, due to the Ascoli — Arcel theorem [5, Sec. 7.5.7], it follows that (4.11) holds true. It remains to notice that (4.11) holds ***for any*** sequence $\varepsilon \to 0$, since the weak generalized solutions of Problem D-A and problem (2.3) are unique.

Theorem 3 is proved. □

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