REMARKS ON OSTROVSKY’S THEOREM

ALEXANDER V. OSIPOV

Abstract. In this paper we prove that the condition ‘one-to-one’ of the continuous open-resolvable mapping is necessary in the Ostrovsky theorem (Theorem 1 in [4]). Also we get that the Ostrovsky problem ([6], Problem 2) (Is every continuous open-$LC_n$ function between Polish spaces piecewise open for $n = 2, 3, \ldots$?) has a negative solution for each $n > 1$.

Keywords: open-resolvable function, open function, resolvable set, open-$LC_n$ function, piecewise open function, scatteredly open function.

1. Introduction

By theorem of Jayne and Rogers a function $f : X \mapsto Y$ between Polish spaces is $\Delta_0^2$-measurable iff it is piecewise continuous (see [2, 3]).

The generalization of theorem of Jayne and Rogers to multi-valued functions raises some topological problems for single-valued functions (see [6], Problem 1 and 2).

In the following definitions we will suppose that $X$ is a subspace of the Cantor set $\mathbb{C}$.

A function $f : X \mapsto Y$ is called piecewise open if $X$ admits a countable, closed and disjoint cover $\mathcal{V}$, such that for each $V \in \mathcal{V}$ the restriction $f|_V$ is open.

Recall, that a subset $E$ of a metric space $X$ is resolvable [1], if for each nonempty closed in $X$ subset $F$ we have $\text{cl}_X(F \cap E) \cap \text{cl}_X(F \setminus E) \neq E$.

If $E \subset X$ is resolvable, then $E$ is $\Delta_0^2$-set in $X$ and vice versa if the space $X$ is Polish (= separable complete metrizable).

Recall that a subset of $X$ is $LC_n$-set if it can be written as union of $n$ locally closed in $X$ sets (a set is locally closed if it is an intersection of an open and a closed set). Every $LC_n$-set (constructible) set is resolvable.
A mapping $f$ is open if it maps open sets onto open ones. More generally, for $n \in \omega$ a mapping $f$ is said to be open-resolvable (open-$LC_n$) if $f$ maps open set onto resolvable ($LC_n$-set) ones.

A piecewise open function $f : X \to Y$ is called scatteredly open if, in addition, the cover $\mathcal{V}$ is scattered, that is: for every nonempty subfamily $T \subset \mathcal{V}$ there is a clopen set $G \subset X$ such that $\mathcal{T}_G = \{ T \in T : T \subset G \}$ is a singleton and $T \cap G = \emptyset$ for every $T \in T \setminus \mathcal{T}_G$.

2. Main result

A.V. Ostrovsky proved the following result.

**Theorem 1.** (Theorem 1 in [4]) Let $X$ and $Y$ be subspaces of the Cantor set $\mathbb{C}$, and $f : X \to Y$ a continuous bijection. If the image under $f$ of every open set in $X$ is resolvable in $Y$, then $f$ is scatteredly open, and, hence, $f$ is scattered homeomorphism.

**Theorem 2.** (Proposition 3.2 in [5]) Every continuous open-$LC_1$ function $X \to Y$ onto a metrizable crowded space $Y$ is open.

In ([6], Problem 2) A.V. Ostrovsky posed the following

**Problem.** Is every continuous open-$LC_n$ function between Polish spaces piecewise open for $n = 2, 3, \ldots$?

We prove that

- the condition ‘one-to-one’ of the mapping $f$ in Theorem 1 is necessary.
- Ostrovsky’s problem has a negative solution for $n = 2$ (hence for each $n > 1$).

**Example.** Let $\mathbb{C}$ be the Cantor set such that $\mathbb{C} \subset [0, 1]$. As usually, we start by deleting the open middle third $\left( \frac{1}{3}, \frac{2}{3} \right)$ from the interval $[0, 1]$, leaving two segments: $P_1 = C_0 = \left[ 0, \frac{1}{3} \right) \cup \left( \frac{2}{3}, 1 \right]$. Next, the open middle third of each of these remaining segments is deleted, leaving four segments: $P_2 = C_{00} \cup C_{01} \cup C_{20} \cup C_{21} = \left[ 0, \frac{1}{3} \right) \cup \left( \frac{2}{3}, \frac{1}{3} \right) \cup \left\{ \frac{1}{3} \right\} \cup \left[ \frac{1}{3}, 1 \right)$. This process is continued ad infinitum, where the $n$th set is $P^n = \left( \frac{1}{3} \right) \cup \left( \frac{2}{3} \right) \cup P_{n-1}$ for $n \geq 1$, and $P_0 = \left[ 0, 1 \right]$.

The Cantor ternary set contains all points in the interval $[0, 1]$ that are not deleted at any step in this infinite process:

$$\mathbb{C} := \bigcap_{n=1}^{\infty} P_n.$$  

Consider the clopen base $\mathcal{B} := \{ C_n = \bigcap_{i=1}^{k} s_i : s_i \in \{ 0, 1 \}, i \in [k] \}$ on $\mathbb{C}$. We enumerate $\mathcal{B} = \{ B_n : n \in \omega \}$ as $B_1 = C_0 \cap \mathbb{C}$, $B_2 = C_2 \cap \mathbb{C}$, $B_4 = C_{02} \cap \mathbb{C}$, ..., $B_n = C_{s(n)} \cap \mathbb{C}$, ..., where $s(n)$ is the binary representation of the number $n + 1$ without the first digit and the digit 1 must be replaced by 2. Consider the countable dense set $\{ b_n : n \in \omega \}$ such that $b_n \in B_n$, $b_n \neq b_m$ ($n \neq m$) for $n, m \in \omega$.

Let us fix the countable dense set $\{ (a_n, b_n) : n \in \omega \}$ in $\mathbb{C} \times \mathbb{C}$ such that $a_n \neq a_m$ for $n \neq m$, and for each $n$ pick $a_{n,i} \mapsto a_n$ such that $a_{n,i} \neq a_m$, $a_{n,i} \neq a_{m,j}$ for $\langle n, i \rangle \neq \langle m, j \rangle$, and $|a_{n,i} - a_n| < \frac{1}{n}$.

Let $X = \mathbb{C} \setminus \bigcup_{n,i} \{ a_{n,i} \} \times B_n$. Note that $X$ is a $G_\delta$-set in $\mathbb{C} \times \mathbb{C}$. It follows that $X$ is a Polish space.
Let \( \pi|X : X \to C \) be a restriction to \( X \) of the projection \( \pi : C \times C \to C \) onto the first coordinate. Note that \( \pi(X) = C \) because of \( \text{diam} C > \text{diam} B_n \) for any \( n \in \omega \).

Suppose \( X = \bigcup_{n \in \omega} X_n \) is a countable union of closed subsets \( X_n \) of \( X \). By the Baire Category Theorem [1], there is \( X_m \) such that \( V = \text{Int} X_m \neq \emptyset \) because otherwise \( \bigcap_{n=1}^{\infty} (X \setminus X_n) \) is not a dense set in \( X \).

Since the set \( \{(a_n, b_n) : n \in \omega \} \) is dense in \( X \), there are \( n' \in \omega \) and \( W \in B \) such that the point \((a_{n'}, b_{n'}) \in ((W \times B_{n'}) \cap X) \subseteq V \). Since the set \( \{(a_n, b_n) : n \in \omega \} \) is dense in \((W \times B_{n'}) \cap X\), choose \( n'' \in \omega \) such that \( n'' > n' \) and \((a_{n''}, b_{n''}) \in ((W \times B_{n''}) \cap X) \subseteq (W \times B_{n''}) \cap X \). Then \( \pi|X_m : X_m \to \pi(X_m) \) is not open at \((a_{n''}, b_{n''})\) because of \( \pi((W \times B_{n''}) \cap X) \) does not contain \( \{(a_{n''}, i) : i \in \omega \} \) and hence it is not an open subset of \( \pi(X_m) \). Therefore \( \pi|X \) is not piecewise open and hence it is not scatteredly open.

Let \( U \subseteq C \times C \) be open. We have to check that \( \pi(U \cap X) \subseteq \Delta_0^0 \).

Construct for every point \((a, b) \in U \cap X \) sets \( W(a) \) and \( B(b) \) such that
- \( a \in W(a) \in B \), \( b \in B(b) \in B \) and \((W(a) \times B(b)) \cap X \subseteq U \).
- If \( a \neq a_m \) for any \( m \in \omega \), then \( \pi((W(a) \times B(b)) \cap X) = W(a) \).
- If \( a = a_m \) for some \( m \in \omega \), then \( \pi((W(a) \times B(b)) \cap X) = W(a) \) \( \{a_m, j : j \in \omega \} \) for some subsequence \( \{a_{m, i} : j \in \omega \} \subseteq \{a_{m, i} : i \in \omega \} \).

Case 1. Let \( a \neq a_m \) and \( a \neq a_{m, i} \) for any \( m, i \in \omega \). One can choose \( W, (B(b) = B_{n'} \in B \) such that \( a \in W, b \in B(b), (W \times B(b)) \cap X \subseteq U \). Since \( a \neq a_m \), for any \( m \in \omega \), then there exists \( W(a) \in B \) such that \( a \in W(a) \subseteq W \) and \( W(a) \cap \{a_{i, j} : j \in \omega \} \subseteq \{a_{m, i} : i \in \omega \} \).

Case 2. Let \( a = a_m \) for some \( m, i \in \omega \). One can choose \( W, (B(b) = B_{n'} \in B \) such that \( a \in W, b \in B(b), (W \times B(b)) \cap X \subseteq U \).

If \( m > n' \) then there exists \( a \in W(a) \in B \) such that \( a \in W(a) \subseteq W \) and \( W(a) \cap \{a_{i, j} : j \in \omega \} \subseteq \{a_{m, i} : i \in \omega \} \).

Case 3. Let \( a = a_m \) for some \( m \in \omega \). Analogously to Case 1, we can choose \( B(b) \in B \) such that \( B(b) \cap B_n \neq \emptyset \) for all \( n > n' > m \), and \( W(a) \in B \) can choose such that \( W(a) \cap \{a_{i, j} : j \in \omega \} \subseteq \{a_{m, i} : i \in \omega \} \).

Then \( W(a) \cap \pi((W(a) \times B(b)) \cap X) \subseteq \{a_{m, i} : i \in \omega \} \) and hence \( \pi((W(a) \times B(b)) \cap X) = W(a) \) \( \{a_{m, i} : j \in \omega \} \subseteq W_a \cup \{a_m \} \) where \( W_a = W(a) \setminus \{a_m \} \cup \{a_{m, i} : j \in \omega \} \) is open in \( C \).

Thus

\[
\pi(U \cap X) = \bigcup_{(a, b) \in U \cap X} \pi((W(a) \times B(b)) \cap X) = \bigcup_{(a, b) \in U \cap X, a \neq a} W(a) \cup \bigcup_{(a, b) \in U \cap X, a = a_m} W(a) \cup \{a_m \}.
\]

By definition of the clopen base \( B \), \( \pi(U \cap X) = S \cup D \) where

\[
S = \bigcup_{(a, b) \in U \cap X, a \neq a_m} W(a) \cup \bigcup_{(a, b) \in U \cap X, a = a_m} W(a)
\]
is an open set in $C$ and $D = \{a_{m_k} : k \in \omega\}$ is a discrete in itself set such that $S \cap D = \emptyset$. Indeed, by Case 3, for every $a_{m_k} \in D$ there is $W(a_{m_k}) \in \mathcal{B}$ such that $a_{m_k} \in W(a_{m_k})$ and $W(a_{m_k}) \cap \{a_{m_i} : i \in \omega, i \neq k\} = \emptyset$. It follows that $D$ is discrete in itself. Hence $\pi(U \cap X)$ is $\Delta^0_2$. Since $\pi(X) = C$ is Polish, the mapping $\pi|X$ is continuous open-resolvable.

Note that $\pi(U \cap X) = S \cup \left( \bigcup_{(a,b) \in U \cap X} W((a)) \right) \cap D$. It follows that $\pi(U \cap X)$ is $LC_2$-set and hence $\pi|X$ is open-$LC_2$.

References


Alexander Vladimirovich Osipov
Ural Federal University,
Ural State University of Economics
Krasovskii Institute of Mathematics and Mechanics,
16, S.Kovalevskay str.,
Yekaterinburg, 620990, Russia
E-mail address: oab@list.ru