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BREZIS-MARCUS TYPE INEQUALITIES WITH LAMB CONSTANT

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Abstract. Hardy-type inequalities with an additional term are proved for compactly supported smooth functions on open convex sets in the Euclidean space. We obtain one-dimensional L_p-inequalities and their multidimensional analogs on arbitrary domains, on regular sets, on domains with θ-cone condition and on convex domains. We use Bessel's function and the Lamb constant.

Keywords: Hardy inequality, additional term, Bessel function, Lamb constant, distance function, inner radius

1. Introduction

Let Ω be an open, connected set in $\mathbb{R}^n$. By $C_0^1(\Omega)$ denote the family of continuously differentiable functions $f : \Omega \to \mathbb{R}$ with compact supports lying in Ω and denote by $\delta(x)$ the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$ of Ω, i.e.

$$\delta = \delta(x) = \operatorname{dist}(x, \partial\Omega).$$

It is know that for all convex bounded domains Ω in $\mathbb{R}^n$ and for all $f \in C_0^1(\Omega)$ the following inequalities

$$(1) \qquad \int\limits_\Omega |\nabla f(x)|^2 dx \geq \frac{1}{4} \int\limits_\Omega \frac{|f(x)|^2}{\delta(x)^2} dx$$

is valid (see for example [15], [21] and [22]). The constant 1/4 in (1) is sharp for any convex subdomain of $\mathbb{R}^n$. This inequality have been generalized and modified in many different ways (see for example [3], [9]-[13], [18]-[23] and [24]). We especially

Nasibullin, R.G., Brezis-Marcus type inequalities with Lamb constant.

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want to highlight a remarkable book by Alexander A. Balinsky, W. Desmond Evans and Roger T. Lewis "The Analysis and Geometry of Hardy's Inequality", which collected the most beautiful results on inequalities of Hardy type.

In 1997-1998 H. Brezis and M. Marcus [14] get the Hardy type inequality in a convex bounded domain Ω with an additional term, namely,

$$\int_\Omega |\nabla f(x)|^2 dx \geq \frac{1}{4}\int_\Omega \frac{|f(x)|^2}{\delta(x)^2} dx + \frac{1}{4D(\Omega)^2}\int_\Omega |f(x)|^2 dx, \forall f \in H_0^1(\Omega),$$

where $H_0^1(\Omega)$ is the closure of the family $C_0^1(\Omega)$ of smooth functions $f : \Omega \to \mathbb{R}$ with finite Dirichlet integral and supported in Ω and $D(\Omega)$ is the diameter of Ω.

H. Brezis and M. Marcus have asked whether the diameter of Ω, in the last inequality can be replaced by an expression depending on the volume $|\Omega| := Vol(\Omega)$ (see [14]). This question was recently answered in [20]. Namely, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and A. Laptev in [20] proved that for all $f \in H_0^1(\Omega)$ the following Hardy type inequality

$$\int_\Omega |\nabla f(x)|^2 dx \geq \frac{1}{4}\int_\Omega \frac{|f|^2}{\delta(x)^2} dx + \frac{1}{4}\frac{K(n)}{|\Omega|^{2/n}}\int_\Omega |f(x)|^2 dx, \tag{2}$$

holds, where $|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$ in the Euclidean space $\mathbb{R}^n$, $|\Omega|$ is the volume of the set Ω and

$$K(n) = n\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{2/n}.$$

W.D. Evans and Roger T. Lewis in [17] improved the constant in the additional term of (2). Indeed, they obtained that

$$\int_\Omega |\nabla f(x)|^2 dx \geq \frac{1}{4}\int_\Omega \frac{|f(x)|^2}{\delta(x)^2} dx + \frac{3}{2}\frac{K(n)}{|\Omega|^{2/n}}\int_\Omega |f(x)|^2 dx.$$

See also [13].

In 2004 J. Tidblom [24] proved the following inequality for functions f from the corresponding Sobolev space:

(3)

$$\int_\Omega |\nabla f(x)|^p dx \geq c_p\left(\int_\Omega \frac{|f(x)|^p}{\delta^p(x)} dx + \frac{(p-1)\sqrt{\pi}\,\Gamma(\frac{n+p}{2})}{\Gamma(\frac{p+1}{2})\,\Gamma(\frac{n}{2})}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{p}{n}}\int_\Omega |f(x)|^p dx\right),$$

where $p > 1$, Ω is a convex domain in $\mathbb{R}^n$ ($n \geq 2$), $|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$ in the Euclidean space $\mathbb{R}^n$, $|\Omega|$ is the volume of the set Ω, Γ is the gamma function of Euler and

$$c_p = \left(\frac{p-1}{p}\right)^p.$$

Note that the inequality (3) is an extension of the inequality (2) for $p = 2$ in the case of arbitrary $p > 1$. Note also that the constants in the additional term of the above inequalities depend on the diameter or the volume of a domain. In [4], [9], [10] and [18] the authors proved inequalities connected with the inradius $\delta_0(\Omega)$ defined as

$$\delta_0 = \delta_0(\Omega) = \sup_{x\in\Omega} \delta(x),$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance function to the boundary of domain.

Denote by J_ν the Bessel function:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k+\nu} k! \Gamma(k+1+\nu)}, \ \nu \geq 0.$$

Following [9] and [10] we shall call Lamb'constant the first positive root $z = \lambda_\nu\left(2(s-1)/q\right)$ of the equation

$$\frac{2(s-1)}{q} J_\nu(z) + 2z J'_\nu(z) = 0, \tag{4}$$

where $\nu \geq 0$, s and q are fixed parameters.

For example, in 2011 F.G. Avkhadiev and K.-J. Wirths [10] proved the following inequality with the sharp constants:

Assume that $s \in (1, +\infty), q \in (0, +\infty)$ *and* $\nu \in \left[0, \frac{s}{q}\right]$. *Then for any function from the closure of family smooth functions supported in* Ω *and having finite integral* $\int\limits_\Omega |\nabla f|^2 \delta^{-s+1} dx$ *the sharp inequality*

$$\int\limits_\Omega \frac{|\nabla f(x)|^2}{\delta(x)^{s-2}} dx \geq \frac{(s-1)^2 + \nu^2 q^2}{4} \int\limits_\Omega \frac{|f(x)|^2}{\delta(x)^s} dx + \frac{q^2 \lambda_\nu^2(2(s-1)/q)}{4\delta_0^q} \int\limits_\Omega \frac{|f(x)|^2}{\delta(x)^{s-q}} dx \tag{5}$$

is valid, where $\lambda_\nu^2(2(s-1)/q)$ *is the Lamb constant.*

The inequality (5) is a bridge between Hardy's inequality of the classical form and sharp estimates of the first eigenvalue $\lambda_1(\Omega)$ of the Laplacian under the Dirichlet boundary condition for n-dimensional convex domains Ω (for details, see [10] and references therein).

The aim of this article is to prove an analogue of inequality (5) with Lamb's constants for the case $p \geq 2$. Let us note that in a number of papers ([4], [5], [7]) the authors proved L_p-inequalities.

Namely, here we shall consider Hardy type inequalities of the form

$$\int\limits_\Omega \frac{|\nabla f(x)|^p}{\delta(x)^{s-p}} dx \geq c_s(\Omega) \int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^s} dx + \mu_s(\Omega) \int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^{s-q}} dx,$$

where by $c_s(\Omega)$ and μ_s we denote the best constant in this inequality.

It is know that if $s > n$ for an arbitrary open sets $\Omega \subset \mathbb{R}^n$ and $f \in C_0^1(\Omega)$

$$c_s(\Omega) = \left(\frac{s-n}{p}\right)^p, \quad \mu_s(\Omega) \geq \left(\frac{s-n}{p}\right)^p \frac{p}{(s-1)\delta_0}.$$

This was proved by F.G. Avkhadiev in [5].

In [17], for convex domains $\Omega \subset \mathbb{R}^n$ W.D. Evans and Roger T. Lewis obtained that

$$c_s(\Omega) \geq \frac{(s-1)^p}{p^p} \left(\frac{2}{D(\Omega)}\right)^{p-s} \frac{1}{B(n,p)}$$

and

$$\mu_s(\Omega) \geq \frac{(s-1)^p}{p^p} \left(\frac{2}{D(\Omega)}\right)^{p-s} \frac{(p-1)}{B(n,p)} \left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{s/n}.$$

Note that the last constants depend on the diameter and the volume of Ω.

F.G. Avkhadiev and I.K. Shafigullin (see [8]) proved that if Ω is a convex domain then the sharp constant

$$c_s(\Omega) = \left(\frac{s-1}{p}\right)^p.$$

Further assume that $s > 1, q > 0, p \in [2, \infty), \nu \in \left[0, \frac{s-1}{q}\right]$ and $\lambda_\nu(2(s-1)/q)$ is the Lamb constant. During this article we suppose that $|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$, $|\Omega|$ is the volume of the set Ω, $D(\Omega)$ is the diameter of Ω,

$$B(n, s) = \frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n+s}{2}\right)},$$

$$c_s = \frac{((s-1)^2 - \nu^2 q^2)^{p/2}}{p^p} \quad \text{and } \mu_s = \frac{p}{2}\frac{q^2\lambda_\nu^2(2(s-1)/q)}{(s-1)^2 - \nu^2 q^2}.$$

In the case $p = 2$, the following are special cases of our results. Let Ω be a convex domain in $\mathbb{R}^n$. If $s - 2 \geq 0$ and $s - q \leq 0$, then for all $f \in C_0^1(\Omega)$

$$\int\limits_\Omega \frac{|\nabla f(x)|^2}{\delta(x)^{s-2}} dx \geq n c_s \left(B(n, s) \int\limits_\Omega \frac{|f(x)|^2}{\delta(x)^s} dx + \mu_s \left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}} \int\limits_\Omega \frac{|f(x)|^2}{\delta(x)^{s-q}} dx \right)$$

and if $s - 2 < 0$ and $s - q > 0$, then

$$\int\limits_\Omega |\nabla f(x)|^2 dx \geq c_s \left(\frac{B(n, s) 2^{2-s}}{D(\Omega)^{2-s}} \int\limits_\Omega \frac{|f(x)|^2}{\delta(x)^s} dx + \frac{2^{2-q}\mu_s}{D(\Omega)^{2-q}} \left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}} \int\limits_\Omega |f(x)|^2 dx \right).$$

This paper is organized as follows. In the first part we obtain one-dimensional inequalities. The second section devoted to inequalities in the multidimensional case. We consider inequalities on arbitrary domains, on regular sets, on domains with θ-cone condition and on convex domains.

2. One-dimensional inequalities

Suppose that $q \in (0, \infty)$, $s \in (1, \infty)$ and $\nu \geq 0$. Consider the function $F_{\nu,s,q}$ defined by

$$F_{\nu,s,q}(x) = x^{(s-1)/2} J_\nu \left(\lambda(2(s-1)/q) x^{q/2}\right), \quad x \in [0, 1].$$

It can easily be checked that

$$F'_{\nu,s,q}(x) = \frac{s-1}{2} x^{\frac{s-1}{2}-1} J_\nu \left(z x^{q/2}\right) + q x^{\frac{q}{2}+\frac{s-1}{2}-1} z J'_\nu(z x^{\frac{q}{2}}),$$

$$F'_{\nu,s,q}(1) = 0,\ F_{\nu,s,q}(x) > 0, x \in (0, 1] \text{ and } F'_{\nu,s,q}(x) > 0,\ x \in (0, 1).$$

where $z = \lambda_\nu(2(s-1)/q)$ is the first positive root of the equation (4).

Further, the function $y = F_{\nu,s,q}(x)$ is a solution of the differential equation

$$x^2 y'' + (2-s) x y' + \left(\frac{(s-1)^2 - \nu^2 q^2}{4} + \frac{q^2 \lambda_\nu^2(2(s-1)/q)}{4x^{-q}}\right) y = 0. \tag{6}$$

Using the expansion for the Bessel function it is not hard to prove that

$$\lim_{x \to 0} \frac{x F'_{\nu,s,q}(x)}{F_{\nu,s,q}(x)} = \frac{s-1+\nu q}{2} \tag{7}$$

(see [9]-[11] for more information).

For an absolutely continuous function $f:[0,1]\to\mathbb{R}$ with property $f(0)=0$ and $|f'(x)|x^{(p-s)/p}\in L^p[0,1]$ we have

$$|f(x)|^p \le \left(\int_0^x |f'(t)|dt\right)^p \le \left(\int_0^x t^{\frac{s-p}{p-1}}dt\right)^{p-1}\int_0^x \frac{|f'(t)|^p}{t^{s-p}}dt$$
$$= \left(\frac{p-1}{s-1}\right)^{p-1} x^{s-1}\int_0^x \frac{|f'(t)|^p}{t^{s-p}}dt.$$

Using the last estimate and (7) we get

$$\lim_{x\to 0}\frac{|f(x)|^p}{x^{s-2}}\frac{F'_{\nu,s,q}(x)}{F_{\nu,s,q}(x)}=0. \tag{8}$$

Lemma 1. *Let $\lambda_\nu(2(s-1)/q)$ be the Lamb constant. Suppose that $p\ge 2$, $s>1$, $q\in(0,\infty)$, $\nu\in[0,(s-1)/q]$ and f be an absolutely continuous function in $[0,1]$ such that $f(0)=0$ and $|f'(x)|x^{(p-s)/p}\in L^p[0,1]$. Then*

$$\int_0^1 \frac{|f'(x)|^p}{x^{s-p}}dx \ge c_s\left(\int_0^1 \frac{|f(x)|^p}{x^s}dx+\mu_s\int_0^1\frac{|f(x)|^p}{x^{s-q}}dx\right)$$

where

$$c_s=\frac{((s-1)^2-\nu^2q^2)^{p/2}}{p^p}\qquad \textit{and}\quad \mu_s=\frac{p}{2}\frac{q^2\lambda_\nu^2(2(s-1)/q)}{(s-1)^2-\nu^2q^2}.$$

Proof. Without loss of generality it can be assumed that f is a positive and nondecreasing function. Indeed, if

$$g(x)=\int_0^x |f'(t)|dt,$$

where $f(x)=\int_0^x f'(t)dt$, and the following inequality

$$\int_a^b g^p(x)w(x)dx\le C_1\int_a^b g'^p(x)v(x)dx \tag{9}$$

holds then we have

$$\int_a^b |f(x)|^p w(x)dx\le\int_a^b g^p(x)w(x)dx\le C_1\int_a^b g'^p(x)v(x)dx=C_1\int_a^b |f'(x)|^p v(x)dx,$$

since

$$|f(x)|\le\int_0^x |f'(t)|dt=g(x),g'(x)=|f'(x)|.$$

Clearly,

$$0\le P:=\int_0^1\frac{f^{p-2}(x)}{x^{s-2}}\left(f'(x)-\frac{2}{p}\frac{y'(x)}{y(x)}f(x)\right)^2dx$$
$$=\int_0^1\frac{f^{p-2}(x)f'^2(x)}{x^{s-2}}dx-\frac{4}{p^2}\int_0^1\frac{y'(x)}{y(x)x^{s-2}}df^p(x)+\frac{4}{p^2}\int_0^1\frac{f^p(x)}{x^{s-2}}\frac{y'^2(x)}{y^2(x)}dx.$$

Integrating by parts one easily obtains

$$P=\int_0^1 \frac{f^{p-2}(x)f'^2(x)}{x^{s-2}}dx-\lim_{x\to 0}\frac{f^p(x)}{x^{s-2}}\frac{y'(x)}{y(x)}+\frac{4}{p^2}\int_0^1 f^p(x)\frac{x^2y''(x)+(2-s)xy'(x)}{y(x)x^s}dx.$$

Using the asymptotic behavior (8) and the differential equation (6) we get

$$\int_0^1 \frac{f^{p-2}(x)f'^2(x)}{x^{s-2}}dx\ge \frac{4}{p^2}\int_0^1 f^p(x)\left(\frac{(s-1)^2-\nu^2q^2}{4x^s}+\frac{q^2\lambda_\nu^2(2(s-1)/q)}{4x^{s-q}}\right)dx.$$

Consequently,

$$\frac{p^2}{(s-1)^2-\nu^2q^2}\int_0^1 \frac{f^{p-2}(x)f'^2(x)}{x^{s-2}}dx\ge \int_0^1 \frac{f^p(x)}{x^s}dx+\frac{q^2\lambda_\nu^2(2(s-1)/q)}{(s-1)^2-\nu^2q^2}\int_0^1\frac{f^p(x)}{x^{s-q}}dx.$$

Applying the elementary inequality (see [19])

$$a^{p_1}b^{p_2}\le\left(\frac{p_1a+p_2b}{p_1+p_2}\right)^{p_1+p_2}$$

to the quantities

$$a=\frac{f^p(x)}{x^s}, b=\frac{p^p}{((s-1)^2-\nu^2q^2)^{p/2}}\frac{f'^p(x)}{x^{s-p}}, p_1=1-\frac{2}{p} \text{ and } p_2=\frac{2}{p},$$

we obtain

$$\frac{p^p}{((s-1)^2-\nu^2q^2)^{p/2}}\int_0^1\frac{f'^p(x)}{x^{s-p}}dx\ge\int_0^1\frac{f^p(x)}{x^s}dx+\frac{p}{2}\frac{q^2\lambda_\nu^2(2(s-1)/q)}{(s-1)^2-\nu^2q^2}\int_0^1\frac{f^p(x)}{x^{s-q}}dx.$$

This complete the proof of Lemma 1. □

Theorem 1. *Suppose that* $0<b-a<\infty, \delta(x)=\max\{x-a,b-x\}$, $p\in[2,\infty)$, $s\in(1,\infty)$, $q\in(0,\infty)$ *and* $\nu\in[0,s/q]$. *If* $f:[a,b]\to\mathbb{R}$ *is an absolutely continuous function such that* $f(a)=f(b)=0$ *and* $|f'(x)|/\delta^{(s-p)/p}(x)\in L^p[a,b]$. *Then the following inequality*

$$\int_a^b\frac{|f'(x)|^p}{\delta^{s-p}(x)}dx\ge c_s\left(\int_a^b\frac{|f(x)|^p}{\delta^s(x)}dx+\frac{\mu_s}{\delta_0^q}\int_a^b\frac{|f(x)|^p}{\delta^{s-q}(x)}dx\right) \tag{10}$$

is valid, where $\lambda_\nu(2(s-1)/q)$ *is the Lamb constant,* $\delta_0=\frac{b-a}{2}$ *and*

$$c_s=\frac{((s-1)^2-\nu^2q^2)^{p/2}}{p^p}\quad\text{and}\quad \mu_s=\frac{p}{2}\frac{q^2\lambda_\nu^2(2(s-1)/q)}{(s-1)^2-\nu^2q^2}.$$

Proof. By the change $x=\rho t$ of variables for any constant $\rho>0$ the inequality of Lemma 1 implies that

$$\int_0^\rho\frac{|f'(x)|^p}{x^{s-p}}dx\ge c_s\left(\int_0^\rho\frac{|f(x)|^p}{x^s}dx+\frac{\mu_s}{\rho^q}\int_0^\rho\frac{|f(x)|^p}{x^{s-q}}dx\right).$$

Now apply the last inequality to functions $u(t) = f(t+a)$ and $u(t) = f(b-t)$ with $\rho = \delta_0 = \frac{b-a}{2}$. We have

$$\int_{\delta_0}^{b} \frac{|f'(x)|^p}{(b-x)^{s-p}}dx \geq c_s \left(\int_{\delta_0}^{b} \frac{|f(x)|^p}{(b-x)^s}dx + \frac{\mu_s}{\delta_0^q} \int_{\delta_0}^{b} \frac{|f(x)|^p}{(b-x)^{s-q}}dx \right) \tag{11}$$

and

$$\int_{a}^{\delta_0} \frac{|f'(x)|^p}{(x-a)^{s-p}}dx \geq c_s \left(\int_{a}^{\delta_0} \frac{|f(x)|^p}{(x-a)^s}dx + \frac{\mu_s}{\delta_0^q} \int_{a}^{\delta_0} \frac{|f(x)|^p}{(x-a)^{s-q}}dx \right). \tag{12}$$

Summing (11) and (12), we get (10). This complete the proof of Theorem 1. □

3. Inequalities in higher dimensions

3.1. **Inequalities on open sets.** Let Ω be an open subset of the Euclidean space $\mathbb{R}^n$ and let $C_0^1(\Omega)$ be the family of continuously differentiable functions $f : \Omega \to \mathbb{R}$ with compact supports lying in Ω. By $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$. For each $x \in \Omega$ and $\nu \in \mathbb{S}^{n-1}$ define

$$\tau_\nu(x) = \min\{s > 0 : x + s\nu \notin \Omega\}, \quad \rho_\nu(x) = \min(\tau_\nu(x), \tau_{-\nu}(x)),$$

$$D_\nu(x) = \tau_\nu(x) + \tau_{-\nu}(x), \quad \Omega_x = \{y \in \Omega : x + t(y-x) \in \Omega, \forall t \in [0,1]\},$$

$$\delta(x) = \inf_{\nu \in \mathbb{S}^{n-1}} \tau_\nu(x) = \operatorname{dist}(x, \partial\Omega),$$

$$D(\Omega) := \sup_{x \in \Omega, \nu \in \mathbb{S}^{n-1}} D_\nu(x).$$

The volume of Ω_x is denoted by $|\Omega_x|$. Clearly, $\Omega_x \subset \Omega$. In [16], Davies introduced the mean distance function $\rho(x)$ is defined by

$$\frac{1}{\rho(x)^2} := \int_{\mathbb{S}^{n-1}} \frac{1}{\rho_\nu(x)^2} d\omega(\nu),$$

where $d\omega(\nu)$ is the normalized measure on $\mathbb{S}^{n-1}$, i.e. $\int_{\mathbb{S}^{n-1}} d\omega(\nu) = 1$.

For general $s \in (1, \infty)$, there is the analogue (see [13], [17], [24])

$$\rho(x,s)^{-s} := \int_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu(x)^s}.$$

Obviously $\rho(x,2)^{-2} = \rho(x)^{-2}$.

The main result of this section is the following assertion.

Theorem 2. *Let Ω be a domain in $\mathbb{R}^n$ and let $\lambda_\nu(2(s-1)/q)$ be the Lamb constant. Suppose that $s \in (1,\infty), q \in (0,\infty), p \in [2,\infty)$ and $\nu \in \left[0, \frac{s-1}{q}\right]$. If $s - p \geq 0$ and $s - q \leq 0$, then for all $f \in C_0^1(\Omega)$*

$$\int_\Omega \frac{|\nabla f(x)|^p}{\delta(x)^{s-p}} dx \geq \frac{c_s}{B(n,p)} \left(\int_\Omega \frac{|f(x)|^p}{\rho(x,s)^s} dx + \mu_s \left(\frac{|\mathbb{S}^{n-1}|}{n} \right)^{\frac{q}{n}} \int_\Omega \frac{|f(x)|^p}{\delta(x)^{s-q} |\Omega_x|^{\frac{q}{n}}} dx \right).$$

If $s-p\ge 0$ and $s-q>0$, then for all $f\in C_0^1(\Omega)$

$$\int\limits_\Omega \frac{|\nabla f(x)|^p}{\delta(x)^{s-p}}dx \ge \frac{c_s}{B(n,p)}\left(\int\limits_\Omega \frac{|f(x)|^p}{\rho(x,s)^s}dx + \frac{2^{s-q}\mu_s}{D(\Omega)^{s-q}}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{q}{n}}\int\limits_\Omega \frac{|f(x)|^p}{|\Omega_x|^{\frac{q}{n}}}dx\right).$$

If $s-p<0$ and $s-q\le 0$, then for all $f\in C_0^1(\Omega)$

$$\int\limits_\Omega |\nabla f(x)|^p dx \ge \frac{c_s 2^{p-s}}{D(\Omega)^{p-s}}\left(\int\limits_\Omega \frac{|f(x)|^p}{\rho(x,s)^s}dx + \mu_s\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{q}{n}}\int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^{s-q}|\Omega_x|^{\frac{q}{n}}}dx\right).$$

If $s-p<0$ and $s-q>0$, then for all $f\in C_0^1(\Omega)$

$$\int\limits_\Omega |\nabla f(x)|^p dx \ge \frac{c_s 2^{p-s}}{D(\Omega)^{p-s}}\left(\int\limits_\Omega \frac{|f(x)|^p}{\rho(x,s)^s}dx + \frac{2^{s-q}\mu_s}{D(\Omega)^{s-q}}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{q}{n}}\int\limits_\Omega \frac{|f(x)|^p}{|\Omega_x|^{\frac{q}{n}}}dx\right).$$

Proof. By Theorem 1 we obtain

$$\int\limits_{a_\nu}^{b_\nu} \frac{|\partial_\nu f(x)|^p}{\rho_\nu^{s-p}(x)}dx \ge c_s\left(\int\limits_{a_\nu}^{b_\nu} \frac{|f(x)|^p}{\rho_\nu^s(x)}dx + \frac{p}{2\delta_0^q}\frac{q^2\lambda_\nu^2(2(s-1)/q)}{(s-1)^2-\nu^2q^2}\int\limits_{a_\nu}^{b_\nu}\frac{|f(x)|^p}{\rho_\nu^{s-q}(x)}dx\right),$$

where $\partial_\nu f, \nu\in\mathbb{S}^{n-1}$, denote the derivative of f in the direction of ν and (a_ν,b_ν) is the interval of intersection of Ω with the ray in the direction ν.

Integrating both sides of the last inequality with respect to the normalized surface measure $d\omega(\nu)$ on $\mathbb{S}^{n-1}$, we get

$$\int\limits_\Omega\int\limits_{\mathbb{S}^{n-1}} \frac{|\partial_\nu f(x)|^p}{\rho_\nu(x)^{s-p}}d\omega(\nu)dx$$

$$\ge c_s\left(\int\limits_\Omega\int\limits_{\mathbb{S}^{n-1}}\left(\frac{1}{\rho_\nu(x)^s}+\left(\frac{2}{D_\nu(x)}\right)^q\frac{\mu_s}{\rho_\nu(x)^{s-q}}\right)d\omega(\nu)|f(x)|^p dx\right),$$

where

$$c_s=\frac{((s-1)^2-\nu^2q^2)^{p/2}}{p^p} \quad \text{and } \mu_s=\frac{p}{2}\frac{q^2\lambda_\nu^2(2(s-1)/q)}{(s-1)^2-\nu^2q^2}.$$

Note that

$$|\partial_\nu f| = |\nu\cdot\nabla f| = |\nabla f||\cos(\nu,\nabla f)|$$

and

$$\int\limits_{\mathbb{S}^{n-1}} |\partial_\nu f(x)|^p d\omega(\nu) = B(n,p)|\nabla f(x)|^p,$$

where

$$B(n,p)=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n+p}{2}\right)}$$

(see [17], [24] for more information).

In his paper [24], J. Tidblom proved that

$$\int\limits_{\mathbb{S}^{n-1}}\left(\frac{2}{D_\nu(x)}\right)^q \ge \left(\frac{n|\Omega_x|}{|\mathbb{S}^{n-1}|}\right)^{-\frac{q}{n}}. \tag{13}$$

Let us consider 4 cases.

Case 1: $s-p\geq 0$ and $s-q\leq 0$. In this case, using (13) and the following inequalities

$$\frac{1}{\rho_\nu(x)^{s-p}}\leq\frac{1}{\delta^{s-p}},\qquad\frac{1}{\rho_\nu(x)^{s-q}}\geq\frac{1}{\delta^{s-q}}$$

for any $\nu\in\mathbb{S}^{n-1}$, we have

$$\int\limits_\Omega\frac{|\nabla f(x)|^p}{\delta(x)^{s-p}}dx\geq\frac{c_s}{B(n,p)}\left(\int\limits_\Omega\frac{|f(x)|^p}{\rho(x,s)^s}dx+\mu_s\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{q}{n}}\int\limits_\Omega\frac{|f(x)|^p}{\delta(x)^{s-q}|\Omega_x|^{\frac{q}{n}}}dx\right).$$

Case 2: $s-p\geq 0$ and $s-q>0$. Since

$$\frac{1}{\rho_\nu(x)^{s-p}}\leq\frac{1}{\delta^{s-p}},\qquad\frac{1}{\rho_\nu(x)^{s-p}}\geq\frac{2^{s-q}}{D(\Omega)^{s-q}}$$

and (13) it follows that

$$\int\limits_\Omega\frac{|\nabla f(x)|^p}{\delta(x)^{s-p}}dx\geq\frac{c_s}{B(n,p)}\left(\int\limits_\Omega\frac{|f(x)|^p}{\rho(x,s)^s}dx+\frac{2^{s-q}\mu_s}{D(\Omega)^{s-q}}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{q}{n}}\int\limits_\Omega\frac{|f(x)|^p}{|\Omega_x|^{\frac{q}{n}}}dx\right).$$

Case 3: $s-p<0$ and $s-q\leq 0$. Combining (13) and the following inequalities

$$\frac{1}{\rho_\nu(x)^{s-p}}\leq\frac{D(\Omega)^{p-s}}{2^{p-s}},\qquad\frac{1}{\rho_\nu(x)^{s-q}}\geq\frac{1}{\delta^{s-q}}\text{ for any }\nu\in\mathbb{S}^{n-1}$$

we get

$$\int\limits_\Omega|\nabla f(x)|^p dx\geq\frac{c_s 2^{p-s}}{D(\Omega)^{p-s}}\left(\int\limits_\Omega\frac{|f(x)|^p}{\rho(x,s)^s}dx+\mu_s\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{q}{n}}\int\limits_\Omega\frac{|f(x)|^p}{\delta(x)^{s-q}|\Omega_x|^{\frac{q}{n}}}dx\right).$$

Case 4: $s-p<0$ and $s-q>0$. In the last case, since

$$\frac{1}{\rho_\nu(x)^{s-p}}\leq\frac{D(\Omega)^{p-s}}{2^{p-s}}\qquad\text{and}\qquad\frac{1}{\rho_\nu(x)^{s-p}}\geq\frac{2^{s-q}}{D(\Omega)^{s-q}}$$

it follows that

$$\int\limits_\Omega|\nabla f(x)|^p dx\geq\frac{c_s 2^{p-s}}{D(\Omega)^{p-s}}\left(\int\limits_\Omega\frac{|f(x)|^p}{\rho(x,s)^s}dx+\frac{2^{s-q}\mu_s}{D(\Omega)^{s-q}}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{q}{n}}\int\limits_\Omega\frac{|f(x)|^p}{|\Omega_x|^{\frac{q}{n}}}dx\right).$$

This complete the proof of theorem 2. □

3.2. **Inequalities on regular sets.** Recall that

$$\frac{1}{\rho(x)^2}:=\int\limits_{\mathbb{S}^{n-1}}\frac{1}{\rho_\nu(x)^2}d\omega(\nu),$$

where $d\omega(\nu)$ is the normalized measure on $\mathbb{S}^{n-1}$.

We say that a domain $\Omega\subset\mathbb{R}^n$ is regular if there exists a finite constant $m(\Omega)>0$ such that

$$\delta(x)\leq\rho(x)\leq m(\Omega)\delta(x)\qquad\forall x\in\Omega.$$

We shall call $m(\Omega)$ a regularity constant for the domain Ω (see [16], [25] for information).

In [16], [25] sufficient conditions for regularity are obtained. For example, E.B. Davies [16] get the following sufficient condition: *The region $\Omega \subseteq \mathbb{R}^n$ is regular if there exists a constant $m(\Omega)$ such that*

$$|\{y \in \Omega : |y-a| < r\}| \geq 2m(\Omega)r^2$$

for all $a \in \partial\Omega$ and all $r > 0$.

In [25], A.M. Tukhvatullina proved a sufficient condition of regularity for multidimensional domains in $\mathbb{R}^n$, $n \geq 2$. Some examples of regular domains were considered in [25]. In particular, annuli with radii R_1 and R_2, when $R_2 \geq R_1/5$, and balls with removed spherical sector are examples of regular domains.

Let us remember that $|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$, $|\Omega|$ is the volume of the set Ω, $D(\Omega)$ is the diameter of Ω,

$$\Omega_x = \{y \in \Omega : x + t(y-x) \in \Omega, \forall t \in [0,1]\},$$

$$B(n,s) = \frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n+s}{2}\right)},$$

$$c_s = \frac{((s-1)^2 - \nu^2 q^2)^{p/2}}{p^p} \quad \text{and } \mu_s = \frac{p}{2}\frac{q^2\lambda_\nu^2(2(s-1)/q)}{(s-1)^2 - \nu^2 q^2}.$$

The following theorem holds.

Theorem 3. *Let Ω be a regular domain in $\mathbb{R}^n$, $m(\Omega)$ be a regularity constant for the domain Ω and let $\lambda_\nu(2(s-1)/q)$ is the Lamb constant. Suppose that $s \geq 2, q > 0, p \geq 2$ and $\nu \in \left[0, \frac{s-1}{q}\right]$. If $s - p \geq 0$ and $s - q \leq 0$, then for all $f \in C_0^1(\Omega)$*

$$\int\limits_\Omega \frac{|\nabla f(x)|^p}{\delta(x)^{s-p}}dx \geq \frac{c_s}{B(n,p)}\left(\frac{1}{m(\Omega)^s}\int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^s}dx + \mu_s\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}}\int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^{s-q}}dx\right).$$

If $s - p \geq 0$ and $s - q > 0$, then for all $f \in C_0^1(\Omega)$

$$\int\limits_\Omega \frac{|\nabla f(x)|^p}{\delta(x)^{s-p}}dx \geq \frac{c_s}{B(n,p)}\left(\frac{1}{m(\Omega)^s}\int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^s}dx + \frac{2^{s-q}\mu_s}{D(\Omega)^{s-q}}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}}\int\limits_\Omega |f(x)|^p dx\right).$$

If $s - p < 0$ and $s - q \leq 0$, then for all $f \in C_0^1(\Omega)$

$$\int\limits_\Omega |\nabla f(x)|^p dx \geq \frac{c_s 2^{p-s}}{D(\Omega)^{p-s}}\left(\frac{1}{m(\Omega)^s}\int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^s}dx + \mu_s\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}}\int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^{s-q}}dx\right).$$

If $s - p < 0$ and $s - q > 0$, then for all $f \in C_0^1(\Omega)$

$$\int\limits_\Omega |\nabla f(x)|^p dx \geq \frac{c_s 2^{p-s}}{D(\Omega)^{p-s}}\left(\frac{1}{m(\Omega)^s}\int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^s}dx + \frac{2^{s-q}\mu_s}{D(\Omega)^{s-q}}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}}\int\limits_\Omega |f(x)|^p dx\right).$$

Proof. Since the function $f(t) = t^{s/2}$ is convex when $s \geq 2$ and $t > 0$, we can use Jensen's inequality to get

$$\int\limits_{\mathbb{S}^{n-1}} \frac{1}{\rho_\nu(x)^s}d\omega(\nu) \geq \left(\int\limits_{\mathbb{S}^{n-1}} \frac{1}{\rho_\nu(x)^2}d\omega(\nu)\right)^{s/2}.$$

Consequently, for regular domains we have

$$\frac{1}{\rho(x,s)^s} := \int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu(x)^s} \geq \frac{1}{m(\Omega)^s \delta(x)^s}.$$

It is obvious that $|\Omega_x| \leq |\Omega|$. Therefore, the inequalities in the statement of the theorem follow from theorem 2. $\square$

Example 1. Let Ω_0 be concentric circles with radii R_1 and R_2, when $R_2 \geq R_1/5$. It is proved in [25] that $m(\Omega_0) = 2\sqrt{12}$. Consequently, if $s - p \geq 0$ and $s - q \leq 0$, then for all $f \in C_0^1(\Omega_0)$

$$\int\limits_{\Omega_0} \frac{|\nabla f(x)|^p}{\delta(x)^{s-p}} dx \geq \frac{c_s}{B(2,p)} \left(\frac{1}{48^{s/2}} \int\limits_{\Omega_0} \frac{|f(x)|^p}{\delta(x)^s} dx + \mu_s \left(\frac{1}{R_1^2 - R_2^2} \right)^{\frac{q}{2}} \int\limits_{\Omega_0} \frac{|f(x)|^p}{\delta(x)^{s-q}} dx \right).$$

Example 2. Let Ω_1 be a ball with removed spherical sector. Let R be radius of the ball. Consider the cone that correspond to the removed spherical sector. By α denote the cone angle, i.e., the angle between the rim of the cap and the direction to the middle of the cap as seen from the sphere center. In [25] it was shown that

$$m(\Omega_1) = \frac{2\sqrt{7}}{\sin \frac{\alpha}{4}}.$$

Consequently, if $s - p \geq 0$ and $s - q \leq 0$, then for all $f \in C_0^1(\Omega_1)$

$$\int\limits_{\Omega_1} \frac{|\nabla f(x)|^p}{\delta(x)^{s-p}} dx \geq \frac{c_s}{B(3,p)} \left(\left(\frac{\sin \frac{\alpha}{4}}{2\sqrt{7}} \right)^s \int\limits_{\Omega_1} \frac{|f(x)|^p}{\delta(x)^s} dx + \mu_s \left(\frac{1}{R^3 \cos^2 \frac{\alpha}{4}} \right)^{\frac{q}{3}} \int\limits_{\Omega_1} \frac{|f(x)|^p}{\delta(x)^{s-q}} dx \right).$$

Above we use that $|\mathbb{S}^2| = 4\pi$ and $|\Omega_1| = \frac{4}{3}\pi R^3 \cos \frac{\alpha}{4}$.

3.3. **Domains with θ-cone condition.** The boundary $\partial\Omega$ is said a θ-cone condition if every $x \in \partial\Omega$ is the vertex of circular cone C_x of semi angle θ which lies entirely in $\mathbb{R}^n \setminus \Omega$ (see [13]).

Assume that $|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$, $|\Omega|$ is the volume of the set Ω, $D(\Omega)$ is the diameter of Ω, $h = h\left(\frac{1}{2}\sin\theta\right)$,

$$B(n,s) = \frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n+s}{2}\right)},$$

$$c_s = \frac{((s-1)^2 - \nu^2 q^2)^{p/2}}{p^p} \quad \text{and } \mu_s = \frac{p}{2} \frac{q^2 \lambda_\nu^2 (2(s-1)/q)}{(s-1)^2 - \nu^2 q^2}.$$

The following theorems holds.

Theorem 4. *Let $\partial\Omega$ satisfies a θ-cone condition and let $\lambda_\nu(2(s-1)/q)$ is the Lamb constant. Suppose that $s \in (1,\infty), q \in (0,\infty), p \in [2,\infty)$ and $\nu \in \left[0, \frac{s-1}{q}\right]$. If $s - p \geq 0$ and $s - q \leq 0$, then for all $f \in C_0^1(\Omega)$*

$$\int\limits_{\Omega} \frac{|\nabla f(x)|^p}{\delta(x)^{s-p}} dx \geq \frac{c_s}{B(n,p)} \left(\frac{h}{2^s} \int\limits_{\Omega} \frac{|f(x)|^p}{\delta(x)^s} dx + \mu_s \left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|} \right)^{\frac{q}{n}} \int\limits_{\Omega} \frac{|f(x)|^p}{\delta(x)^{s-q}} dx \right).$$

If $s-p\geq 0$ and $s-q>0$, then

$$\int\limits_{\Omega}\frac{|\nabla f(x)|^p}{\delta(x)^{s-p}}dx \geq \frac{c_s}{B(n,p)}\left(\frac{h}{2^s}\int\limits_{\Omega}\frac{|f(x)|^p}{\delta(x)^s}dx+\frac{2^{s-q}\mu_s}{D(\Omega)^{s-q}}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}}\int\limits_{\Omega}|f(x)|^p dx\right).$$

If $s-p<0$ and $s-q\leq 0$, then for all $f\in C_0^1(\Omega)$

$$\frac{D(\Omega)^{p-s}}{2^{p-s}}\int\limits_{\Omega}|\nabla f(x)|^p dx \geq c_s\left(\frac{h}{2^s}\int\limits_{\Omega}\frac{|f(x)|^p}{\delta(x)^s}dx+\mu_s\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}}\int\limits_{\Omega}\frac{|f(x)|^p}{\delta(x)^{s-q}}dx\right).$$

If $s-p<0$ and $s-q>0$, then for all $f\in C_0^1(\Omega)$

$$\int\limits_{\Omega}|\nabla f(x)|^p dx \geq \frac{2^{p-s}c_s}{D(\Omega)^{p-s}}\left(\frac{h}{2^s}\int\limits_{\Omega}\frac{|f(x)|^p}{\delta(x)^s}dx+\frac{2^{s-q}\mu_s}{D(\Omega)^{s-q}}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}}\int\limits_{\Omega}|f(x)|^p dx\right).$$

Proof. By $h(\alpha)$ denote the solid angle subtended at the origin by a ball of radius $\alpha<1$, whose centre is at a distance 1 from the origin. If $\partial\Omega$ satisfies a θ-cone condition, then for all $x\in\Omega$

$$\frac{1}{\rho(x,s)^s}\geq\frac{h\left(\frac{1}{2}\sin\theta\right)}{2^s\delta(x)^s}.$$

For more information we refer to the book [13], p. 86.

Using the last estimates and theorem 2, we get the inequalities in the statement of the theorem. □

3.4. **Inequalities in convex sets.** Let Ω be convex domain. It is know that for convex domains

$$\rho(x;s)^{-s}:=\int\limits_{\mathbb{S}^{n-1}}\frac{d\omega(\nu)}{\rho_\nu(x)^s}\geq\frac{B(n,s)}{\delta(x)^s},$$

where

$$B(n,s)=\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n+s}{2}\right)}.$$

Moreover, for the case of convex domains $|\Omega_x|=|\Omega|$.Taking into account theorem 2 we obtain the following theorem.

Theorem 5. *Let Ω be a convex domain in $\mathbb{R}^n$ and let $\lambda_\nu(2(s-1)/q)$ be the Lamb constant. Suppose that $s\in(1,\infty), q\in(0,\infty), p\in[2,\infty)$ and $\nu\in\left[0,\frac{s-1}{q}\right]$. If $s-p\geq 0$ and $s-q\leq 0$, then for all $f\in C_0^1(\Omega)$*

$$\int\limits_{\Omega}\frac{|\nabla f(x)|^p}{\delta(x)^{s-p}}dx \geq c_s\left(\frac{B(n,s)}{B(n,p)}\int\limits_{\Omega}\frac{|f(x)|^p}{\delta(x)^s}dx+\frac{\mu_s}{B(n,p)}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}}\int\limits_{\Omega}\frac{|f(x)|^p}{\delta(x)^{s-q}}dx\right).$$

If $s-p\geq 0$ and $s-q>0$, then for all $f\in C_0^1(\Omega)$

$$\int\limits_{\Omega}\frac{|\nabla f(x)|^p}{\delta(x)^{s-p}}dx \geq \frac{c_s}{B(n,p)}\left(B(n,s)\int\limits_{\Omega}\frac{|f(x)|^p}{\delta(x)^s}dx+\frac{2^{s-q}\mu_s}{D(\Omega)^{s-q}}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}}\int\limits_{\Omega}|f(x)|^p dx\right).$$

If $s-p<0$ and $s-q\le 0$, then for all $f\in C_0^1(\Omega)$

$$\frac{D(\Omega)^{p-s}}{2^{p-s}}\int\limits_{\Omega}|\nabla f(x)|^p dx \ge c_s\left(B(n,s)\int\limits_{\Omega}\frac{|f(x)|^p}{\delta(x)^s}dx+\mu_s\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}}\int\limits_{\Omega}\frac{|f(x)|^p}{\delta(x)^{s-q}}dx\right).$$

If $s-p<0$ and $s-q>0$, then for all $f\in C_0^1(\Omega)$

$$\int\limits_{\Omega}|\nabla f(x)|^p dx \ge \frac{2^{p-s}c_s}{D(\Omega)^{p-s}}\left(B(n,s)\int\limits_{\Omega}\frac{|f(x)|^p}{\delta(x)^s}dx+\frac{2^{s-q}\mu_s}{D(\Omega)^{s-q}}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{q}{n}}\int\limits_{\Omega}|f(x)|^p dx\right).$$

Taking $s=q=p$ and $\nu=0$ in theorem 5 gives the following corollary.

Corollary 1. *Let Ω be a convex domain in $\mathbb{R}^n$ and let $\lambda_\nu(2(p-1)/p)$ be the Lamb constant. If $p\in[2,\infty)$, then for all $f\in C_0^1(\Omega)$*

$$\int\limits_{\Omega}|\nabla f(x)|^p dx \ge c_p\left(\int\limits_{\Omega}\frac{|f(x)|^p}{\delta(x)^p}dx+\frac{p^3\lambda_0^2(2(p-1)/p)}{2B(n,p)(p-1)^2}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{p}{n}}\int\limits_{\Omega}|f(x)|^p dx\right).$$

where $c_p=\left(\frac{p-1}{p}\right)^p$.

Let now we compare the constant in the last inequality with the constant in (3). More precisely, we shall show that

$$\frac{p^3\lambda_0^2(2(p-1)/p)}{2(p-1)^2B(n,p)}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{p}{n}}\ge\frac{(p-1)}{B(n,p)}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{p}{n}}.$$

In [10], F.G. Avkhadiev and K.-J. Wirths proved that the Lamb's constant $z=\lambda(s)$ as a function is a positive and monotonic increasing function of the variable $s>0$. Moreover, for any $s\in(0,\infty)$ it is known that

$$\lim_{s\to 0}\lambda_0(s)=0<\lambda_0(s)<j_0=\lim_{s\to\infty}\lambda_0(s)\quad\text{and}\quad\lambda_0(2)\ge\lambda_0(1)=0.94\ldots.$$

Obviously, if $p\ge 2$, then

$$1<4\lambda_0(1)\le\frac{p^3\lambda_0^2(2(p-1)/p)}{2(p-1)^3}.$$

Consequently, the constant in the inequality of corollary 1 large than the constant in (3).

3.5. **Sharpness of the constant.** Note that the both constants in the inequality of Lemma 1 are sharp, when $\nu\ge 0$ and $p=2$ (see [10] for more information). At the same time, we do not know about sharpness of constants in the case $p>2$.

In the following assertion we shall prove that the constant $((s-1)^2-\nu^2q^2)^{p/2}/p^p$ in Lemma 1 is sharp in the case $\nu=0$.

Lemma 2. *If $s>1, p\ge 2$ and $q>0$ then for any $\varepsilon>0$ there exist function f_ε that satisfy the conditions of Lemma 1 and the following inequality*

$$\int\limits_0^1\frac{|f_\varepsilon'(x)|^p}{x^{s-p}}dx\le$$

$$\frac{((s-1)^2+4\varepsilon)^{p/2}}{p^p}\int\limits_0^1 \frac{|f_\varepsilon(x)|^p}{x^s}dx + (s-1)^{p-2}\frac{q^2\lambda_0^2(2(s-1)/q)}{2p^{p-1}}\int\limits_0^1 \frac{|f_\varepsilon(x)|^p}{x^{s-q}}dx.$$

Proof. Let $\varepsilon > 0$ and let $f_\varepsilon(x) = t^{(s-1+\varepsilon/(s-1))/p}$. Without loss of generality we suppose that $\varepsilon \geq 1$. Straightforward computations give that

$$\int\limits_0^1 \frac{|f'_\varepsilon(x)|^p}{x^{s-p}}dx = \left(s-1+\frac{\varepsilon}{s-1}\right)^p \frac{s-1}{p^p\varepsilon} < \left((s-1)^2+4\varepsilon\right)^{p/2}\frac{s-1}{p^p\varepsilon}$$

$$= \frac{\left((s-1)^2+4\varepsilon\right)^{p/2}}{p^p}\int\limits_0^1 \frac{|f_\varepsilon(x)|^p}{x^s}dx,$$

which implies the Lemma 2. □

4. Remark

In the proof of the inequalities of Theorem 1 we do not use the restriction $\nu > (s-1)/q$. In this case we have the following theorems.

Theorem 6. *Suppose that* $0 < b-a < \infty, \delta(x) = \max\{x-a, b-x\}$, $s \in (1,\infty)$, $q \in (0,\infty)$ *and* $\nu > (s-1)/q$. *If* $f : [a,b] \to \mathbb{R}$ *is an absolutely continuous function such that* $f(a) = f(b) = 0$ *and* $f'(x)/\delta^{(s-p)/p}(x) \in L^p[a,b]$, *then for all even* $p \geq 2$ *the following inequality*

$$\frac{p^p}{(\nu^2q^2-(s-1)^2)^{p/2}}\int\limits_a^b \frac{|f'(x)|^p}{\delta^{s-p}(x)}dx + (p-1)\int\limits_a^b \frac{|f(x)|^p}{\delta^s(x)}dx$$

$$\geq \frac{p}{2\delta_0^q}\frac{q^2\lambda_\nu^2(2(s-1)/q)}{\nu^2q^2-(s-1)^2}\int\limits_a^b \frac{|f(x)|^p}{\delta^{s-q}(x)}dx$$

is valid, where $\lambda_\nu(2(s-1)/q)$ *is the Lamb constant and* $\delta_0 = \frac{b-a}{2}$.

Letting $s \to 1$ gives the following

Corollary 2. *Suppose that* $0 < b-a < \infty, \delta(x) = \max\{x-a, b-x\}$, $p \in [2,\infty)$, $q \in (0,\infty)$ *and* $\nu > 0$. *If* $f : [a,b] \to \mathbb{R}$ *is an absolutely continuous function such that* $f(a) = f(b) = 0$ *and* $f'(x)/\delta^{(1-p)/p}(x) \in L^p[a,b]$, *then for all even* $p \geq 2$ *the following inequality*

$$\frac{p^p}{\nu^pq^p}\int\limits_a^b \frac{|f'(x)|^p}{\delta^{1-p}(x)}dx + (p-1)\int\limits_a^b \frac{|f(x)|^p}{\delta(x)}dx \geq \frac{p}{2\delta_0^q}\frac{q^2{j'}_\nu^2}{\nu^2q^2}\int\limits_a^b \frac{|f(x)|^p}{\delta^{1-q}(x)}dx$$

holds, where j'_ν *is the first positive zero of the derivative* J'_ν *of Bessel's function* J_ν *and* $\delta_0 = \frac{b-a}{2}$.

Using the inequalities of theorem 6 and corollary 2 we can prove multidimensional inequalities like above.

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References

[1] F.G. Avkhadiev, *A geometric description of domains whose Hardy constant is equal to 1/4*, Izvestiya: Mathematics, **78**:5 (2014), 855–876. MR3308642

[2] F.G. Avkhadiev, *Hardy-Rellich inequalities in domains of the Euclidean space*, J. Math Anal. Appl., **442** (2016), 469–484. MR3504010

[3] F.G. Avkhadiev, *Integral inequalities in domains of hyperbolic type and their applications*, Sb. Math., **206**:12 (2015), 1657–1681. MR3438572

[4] F.G. Avkhadiev, *Hardy type inequalities in higher dimensions with explicit estimate of constants*, Lobachevskii J. Math., **21** (2006), 3–31. MR2220697

[5] F.G. Avkhadiev, *Hardy-type inequalities on planar and spatial open sets*, Proceedings of the Steklov Institute of Mathematics, **255**:1 (2006), 2–12. Zbl 1351.42024

[6] F.G. Avkhadiev and R.G. Nasibullin, *Hardy-type inequalities in arbitrary domains with finite inner radius*, Siberian Mathematical Journal, **55**:2 (2014), 239–250. MR3237329

[7] F.G. Avkhadiev, R.G. Nasibullin and I. K. Shafigullin, *Hardy-type inequalities with power and logarithmic weights in domains of the Euclidean space*, Russian Mathematics, **55**:9 (2011), 76–79. Zbl 1236.26019

[8] F.G. Avkhadiev and I. K. Shafigullin, *Sharp estimates of Hardy constants for domains with special boundary properties*, Russian Mathematics, **58**:2 (2014), 58–61. MR3254462

[9] F.G. Avkhadiev and K.-J. Wirths, *Unified Poincaré and Hardy inequalities with sharp constants for convex domains*, Z. Angew. Math. Mech., **87** (2007), 632–642. MR2354734

[10] F.G. Avkhadiev and K.-J. Wirths, *Sharp Hardy-type inequalities with Lamb's constants*, Bull. Belg. Math. Soc. Simon Stevin, **18** (2011), 723–736. MR2907615

[11] F.G. Avkhadiev and K.-J. Wirths, *Weighted Hardy inequalities with sharp constants*, Lobachevskii J. Math., **31** (2010), 1–7. MR2610299

[12] F.G. Avkhadiev and K.-J. Wirths, *On the best constants for the Brezis-Marcus inequalities in balls*, J. Math. Analysis and Applications, **396**:2 (2012), 473–480. MR2961240

[13] A.A. Balinsky , W.D. Evans and R.T. Lewis, *The Analysis and Geometry of Hardy's Inequality*, Universitext, Springer, Heidelberg – New York – Dordrecht – London, 2015. MR3408787

[14] H. Brezis and M. Marcus, *Hardy's inequality revisited*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **25**:1–2 (1997), 217–237. MR1655516

[15] E. B. Davies, *The Hardy constant*, Quart. J. Math. Oxford (2), **46**:4 (1995), 417–431. MR1366614

[16] E.B. Davies, *Spectral Theory and Differential Operators*, Cambridge: Cambridge Univ.Press., Cambridge Studies in Advanced Mathematics, **42**, 1995. MR1349825 .

[17] W.D. Evans and R.T. Lewis, *Hardy and Rellich inequalities with remainders*, J. Math. Inequal., **1**:4 (2007), 473–490. MR2408402

[18] S. Filippas, V.G. Maz'ya and A. Tertikas, *On a question of Brezis and Marcus*, Calc. Var. Partial Differential Equations, **25**:4 (2006), 491–501. MR2214621

[19] G.H. Hardy, J.E. Littlewood and G. Polya, *Inequalities*, Cambridge: Cambridge Univ. Press, 1973.

[20] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and A. Laptev, *A geometrical version of Hardy's inequality*, J. Funct. Anal., **189**:2 (2002), 539–548. MR1892180

[21] M. Marcus, V.J. Mizel and Y. Pinchover, *On the best constants for Hardy's inequality in R^n*, Trans. Amer. Math. Soc., **350** (1998), 3237–3250. MR1458330

[22] T. Matskewich and P.E. Sobolevskii, *The best possible constant in a generalized Hardy's inequality for convex domains in R^n*, Nonlinear Anal., **28** (1997), 1601-1610. MR1431208

[23] R.G. Nasibullin and A.M. Tukhvatullina, *Hardy type inequalities with logarithmic and power weights for a special family of non-convex domains*, Ufa Mathematical Journal, **5**:2 (2013), 43–55. MR3430775

[24] J. Tidblom, *A geometrical version of Hardy's inequality for $W_0^{1,p}(\Omega)$*, Proc. Amer. Math. Soc., **132** (2004), 2265–2271. MR2052402

[25] A.M. Tukhvatullina, *Hardy type inequalities for a special family of non-convex domains*, Physics and mathematics, Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki, **153**:1 (2011), 211–220. MR3151535

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