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MINIMUM SUPPORTS OF EIGENFUNCTIONS IN BILINEAR FORMS GRAPHS

E.V. SOTNIKOVA

Abstract. In this paper we study eigenfunctions corresponding to the minimum eigenvalue of bilinear forms graphs. Our main goal is to find eigenfunctions with the supports (non-zero positions) of minimum cardinality. For bilinear forms graphs of diameter $D = 2$ over a prime field we prove that there exist eigenfunctions with the support achieving the weight distribution bound. We also provide an explicit construction of such functions. For bilinear forms graphs of diameter $D \geq 3$ we show the non-existance of eigenfunctions with supports achieving the weight distribution bound.

Keywords: bilinear forms graph, eigenfunctions, minimum supports, distance-regular graphs.

1. Introduction

The problems connected to eigenvalues of graphs attract a lot of attention since eigenfunctions of graphs can represent and also generalize many interesting combinatorial objects (including but not limited to equitable partitions, perfect codes, combinatorial designs etc.). A nice introduction to the general theory describing these connections can be found in [13, 12].

One of the interesting questions is the following. Suppose we fix a graph together with some of its eigenvalues and consider all the eigenfunctions corresponding to this eigenvalue — what minimum number of non-zero values can an eigenfunction have, in other words, what is a minimum support of this eigenfuncion? If an induced subgraph is built on the non-zero vertices, what kind of graph is obtained?

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These questions were considered for several families of graphs by different researchers:

- Grassman graphs were originally considered in terms of finding the minimum null-designs by James [3] and Cho [5, 6]. The similar result in the context of eigenfunctions was obtained in the paper [12]
- Hamming graph $H(n, q)$ was intensively studied in a serie of works [10, 11, 14, 15]. It is worth noting that in the latter paper a more general problem is considered, namely, what is the minimum support of the function that belongs to the direct sum of some eigenspaces corresponding to subsequent eigenvalues.
- Johnson graphs $J(n, m)$ was explored in [16]
- Doob graphs were considered in [17]
- Paley graphs were studied in [18]
- Cubical distance-regular graphs were explored in [19]

The current work continues the research in this direction and focus on the study of biliniear forms graph with its eigenfunctions. The paper has the following structure. Section 2 gives basic definitions and necessary notation. We also recall what a weight distibution bound is and explicitely calculate the weight distribution values corresponding to the minimum eigenvalue of a bilinear forms graph. Section 3 is dedicated to a family of strongly regular bilinear forms graphs and contains the first main result of the paper: bilinear forms graphs of diameter $D = 2$ over a prime field have eigenfunctions with supports achieving the weight distribution bound. We also provide an explicit construction of such eigenfunctions. In Section 4 using the connection between bilinear forms graph and a Grassmann graph we prove the second main result of the paper about non-existance of eigenfunctions with minimum supports achieving the weight distribution bound in the case when the diameter is at least 3.

2. Preliminaries

The bilinear forms graph $\mathrm{Bil}_q(n, m)$ is a distance-regular graph with the vertex set V consisting of all $n \times m$ matrices over a finite field $\mathbb{F}_q$ and two vertices being adjacent when their matrix difference has a rank 1. For more information about distance-regular graphs the reader is referred to a classic book [4].

A distance $d(x, y)$ between two vertices x and y is the length of the shortest path in a graph that connects them. The greatest distance between any pair of vertices is called the diameter D of a graph. For a fixed vertex x the set of all vertices at distance i from x is denoted by $G_i(x) = \{y \in V \mid d(x, y) = i\}$. We will write $x \sim y$ if $d(x, y) = 1$.

A function $f : V \to \mathbb{R}$ that is not constantly zero and satisfies the equation

$$\theta f(u) = \sum_{v \sim u} f(v) \quad \forall u \in V$$

is called an *eigenfunction* of a graph corresponding to an eigenvalue θ of its adjacency matrix. A support $\mathrm{Supp}(f)$ of a function f is defined as follows $\mathrm{Supp}(f) = \{v \in V \mid f(v) \neq 0\}$. We are interested in finding the supports of minimum cardinality.

Recall that a Gaussian binomial coefficient is defined as follows:

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \prod_{i=0}^{m-1} \frac{q^{n-i} - 1}{q^{i+1} - 1}$$

Weight distribution bound. In case of distance-regular graphs, the lower bound for the size of an eigensupport can be obtained from the weight distribution of its eigenfunction. We will give a brief introduction, for more details the reader is reffered to [9] and [12].

Suppose we have a graph $G = (V, E)$ and an eigenfunction $f^\theta : V \to \mathbb{R}$ corresponding to an eigenvalue θ. Choose some vertex v such that $f^\theta(v) \neq 0$. Without loss of generality we can consider $f(v) = 1$. Let

$$W_i^v(f^\theta) = \sum_{u \in G_i(v)} f^\theta(u) \quad \text{where } i = 0, \ldots, D$$

The set $\{W_0^v(f^\theta), \ldots, W_D^v(f^\theta)\}$ is called the weight distribution of an eigenfunction f with respect to a vertex v. It is known that in case of graph G being distance-regular, the values of weight distribution do not depend on the choice of a vertex v and can be derived from the intersection array of G and the corresponding eigenvalue. Thus we can define

$$W_i^v(f^\theta) = W_i(\theta) \quad \forall v \in V, \forall f^\theta$$

Moreover, the following recurrence takes place:

$$W_0(\theta) = 1$$
$$W_1(\theta) = \theta$$
$$W_i(\theta) = \frac{\theta W_{i-1}(\theta) - b_{i-2} W_{i-2}(\theta) - a_{i-1} W_{i-1}(\theta)}{c_i}, \text{ where } i = 2, \ldots, D$$

Next well known lemma gives a lower bound on the eigensupport cardinality in terms of weight distribution values (see [11], for example):

Lemma 1. *For the cardinality of a support* $\mathrm{Supp}(f^\theta)$ *the following estimation is true:* $|\mathrm{Supp}(f^\theta)| \geq \sum_{i=0}^{D} \lceil |W_i(\theta)| \rceil$.

Cliques, Delsarte cliques. A clique C is a subset of vertices of a graph G such that any two distinct vertices in a clique are adjacent. It is well known that cliques in distance-regular graphs cannot be of an arbitrary size and the following bound takes place:

$$|C| \leq 1 - k/\theta_D,$$

where θ_D is the minimum eigenvalue of a graph (see [2] and [4], Proposition 4.4.6). If clique consists of exactly $1 - k/\theta_D$ vertices, it is called a Delsarte clique. Graph G is called a Delsarte clique graph if every edge lies in the same number of Delsarte cliques. We will briefly introduce the necessary definitions, more details the reader can find in [7].

Suppose G is a non-complete distance-regular graph of degree k with the smallest eigenvalue θ and let $\mathcal{C}$ be a non-empty set of cliques with cardinalities equal to $s+1$. A pair $(G, \mathcal{C})$ is called a $(k, s, n_\mathcal{C})$ pair if each edge of a graph G is contained exactly in $n_\mathcal{C}$ cliques from $\mathcal{C}$. If in addition a set $\mathcal{C}$ consists of Delsarte cliques, the pair $(G, \mathcal{C})$ is called a *Delsarte pair* with parameters $(k, s, n_\mathcal{C})$ and the set $\mathcal{C}$ is called a *Delsarte set* with parameters $(s, n_\mathcal{C})$. If a graph G contains a full set $\mathcal{C}$ of Delsarte cliques with parameters (s, n), then G is a *Delsarte clique graph* with parameters (k, s, n). Note that $k, s \geq 2$ and $n_\mathcal{C}, n \geq 1$.

Known results. In this work we actively use the results from [12]; so it will be convenient to provide them here:

Theorem 1. (see Theorem 2 from [12]) *Let G be a distance-regular graph with the minimum eigenvalue θ_D. Assume that for some collection S of cliques (G,S) is a Delsarte pair. Then a function f over a vertex set of G is an eigenfunction with the eigenvalue θ_D if and only if for every clique C from S it holds $\sum_{x\in C} f(x) = 0$.*

Before proceeding to the next result we recall what a (bi)trade is. Suppose $(G,\mathcal{C})$ is a $(k,s,n_{\mathcal{C}})$ pair and T_0, T_1 — are two mutually disjoint set of vertices. If every clique from $\mathcal{C}$ either intersects with each of T_0 and T_1 in exactly one vertex or does not intersect with both of them, we will call (T_0,T_1) a clique-bitrade, or a $\mathcal{C}$-bitrade. A set T_0 is called a $\mathcal{C}$-trade if together with some other set T_1 it gives rise to a $\mathcal{C}$-bitrade (T_0,T_1). To avoid confusion, it is important to note that an alternative notation is also used in the literature, when a pair (T_0,T_1) is defined to be a trade and each set T_i is called a trade partner, or leg.

Theorem 2. (see Theorem 3 from [12]) *Let G be a distance-regular graph of degree k. Let (G,S) be a (k,s,m) pair. Let $T=(T_0,T_1)$ be a pair of disjoint independent sets of vertices of G. Then the following are equivalent:*

1. *T is an S-bitrade meeting the w.d. bound.*
2. *The function $f_T = \begin{cases} (-1)^i & x\in T_i \\ 0 & else \end{cases}$ is an eigenfunction of G meeting the w.d.bound with an eigenvalue $-k/s$.*
3. *The subgraph G_T induced on the vertex set $T_0\cup T_1$ is a regular isometric subgraph with degree k/s.*

Applying preliminaries to bilinear forms graph. Now we will calculate the weight distribution bound for a bilinear forms graph $\mathrm{Bil}_q(n,m)$. The parameters of its intersection array are well known [4]:

$$b_i = \frac{q^{2i}(q^{m-i}-1)(q^{n-i}-1)}{q-1}$$

$$c_i = \frac{q^{i-1}(q^i-1)}{q-1}$$

$$a_i = \frac{(q^m+q^n-q^i-q^{i-1}-1)(q^i-1)}{q-1}$$

Bilinear forms graph has the minimum eigenvalue $\theta_D = -\left[{D \atop 1}\right]$ (see [4], Corollary 8.4.2).

Lemma 2. *For a bilinear forms graph $\mathrm{Bil}_q(n,m)$ the values of the weight distribution of any eigenfunction corresponding to the minimum eigenvalue θ_D are as follows:*

$$W_0 = 1 \quad W_1 = -\frac{q^m-1}{q-1}$$

$$W_i = -W_{i-1}\cdot\frac{q^{i-1}}{q^i-1}\cdot(q^{m-i+1}-1)$$

Proof. We will prove the equality using the induction on i. First we directly check it for W_2:

$$W_2 = \frac{\theta_D W_1 - b_0 W_0 - a_1 W_1}{c_2}$$
$$= \frac{\left(-\frac{q^m-1}{q-1}\right)^2 - \frac{(q-1)(q^m-1)(q^n-1)}{(q-1)^2} + (q^m+q^n-q-2)\cdot\frac{q^m-1}{q-1}}{\frac{q(q^2-1)}{q-1}}$$
$$= \frac{(q^m-1)(q^m-1+(q-1)(-q^n+1+q^m+q^n-q-2)}{(q-1)^2 q(q+1)}$$
$$= \frac{(q^m-1)}{(q-1)}\cdot\frac{q}{q^2-1}\cdot(q^{m-1}-1)$$

Now suppose the statement is true for W_i, we will prove it for W_{i+1}.

$$W_{i+1} = \frac{\theta_D W_i - b_{i-1}W_{i-1} - a_i W_i}{c_{i+1}}$$
$$= \frac{-\frac{q^m-1}{q-1} - \frac{q^{2(i-1)}(q^{m-i+1}-1)(q^{n-i+1}-1)}{q-1}W_{i-1} - \frac{(q^m+q^n-q^i-q^{i-1}-1)(q^i-1)}{q-1}W_i}{\frac{q^i(q^{i+1}-1)}{q-1}}$$
$$= -\frac{W_i}{q^i(q^{i+1}-1)}\cdot\Big[q^m - 1 - \frac{(q^{m-i+1}-1)(q^{n-i+1}-1)q^{2(i-1)}(q^i-1)(q-1)}{q^{i-1}(q-1)(q^{m-i+1}-1)}$$
$$+q^{m+i}+q^{n+i}-q^{2i}-q^{2i-1}-q^i-q^m-q^n+q^i+q^{i-1}+1\Big]$$
$$= -\frac{W_i}{q^i(q^{i+1}-1)}\cdot\big[(q^{n-i+1}-1)(q^{2i-1}-q^{i-1})$$
$$+q^{m+i}+q^{n+i}-q^{2i}-q^{2i-1}-q^n+q^{i-1}\big]$$
$$= -W_i\cdot\frac{q^i}{q^{i+1}-1}\cdot(q^{m-i}-1)$$

This completes the proof. □

Hence we can write an explicit form:

$$W_i = (-1)^i\prod_{j=1}^{i} q^{j-1}\frac{q^{m-i+1}-1}{q^i-1} = (-1)^i\cdot q^{i(i-1)/2}\begin{bmatrix} m \\ i \end{bmatrix}_q$$

From here we obtain the weight distribution bound:

$$\sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_q\cdot q^{i(i-1)/2}.$$

Note that the weight distribution for bilinear forms graph coincides with that of Grassmann graph. We will use this connection in Section 4.

Regarding being a Delsarte clique graph: Since a bilinear forms graph is edge-transitive and contains a Delsarte clique, it is a Delsarte clique graph (see Proposition 3.2 in [7]).

Finding a minimum support: In case of $D = 2$ and θ_D from Theorem 2 we obtain that if a weight distribution bound is achieved then the minimum support induces a complete bipartite subgraph $K_{q+1,q+1}$.

3. Minimum Support

In this work we consider eigenfunctions corresponding to the smallest eigenvalue. We start with a case of $D = 2$.

The structure of the section is as follows:

- Local structure
- Adjacencies between vertices
- Independent set as one part of a complete bipartite graph
- Reconstruction of the second part
- Partial solution: explicit construction

Local structure. It is known that the local structure of a bilinear forms graph, i.e. $G_1(U)$ for an arbitrary vertex U, is the $(q-1)$-clique extension of the $\begin{bmatrix} n \\ 1 \end{bmatrix}_q \times \begin{bmatrix} m \\ 1 \end{bmatrix}_q$-lattice. The $a \times b$ lattice graph is the Cartesian product of two path graphs with a and b vertices respectively.

The q-clique extension $\widetilde{G}$ of a graph $G = (V, E)$ is obtained by replacing each vertex x with a clique $C_x = \{x_1, \ldots, x_q\}$ of cardinality q. And any two vertices x_i and y_j from distinct cliques C_x and C_y are adjacent if and only if their origins x and y are adjacent.

Figure 1 presents a lattice graph. The vertices belonging to the same row (or column) are adjacent. Figure 2 shows a $(q-1)$-clique-extension. The circles represent the cliques of cardinality $q-1$. And again the lines and columns present the adjacencies.

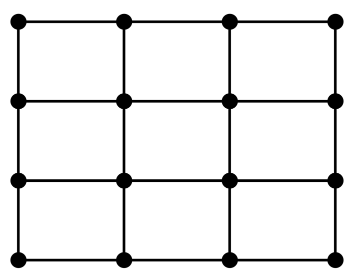

Fig. 1. 4×4 lattice graph

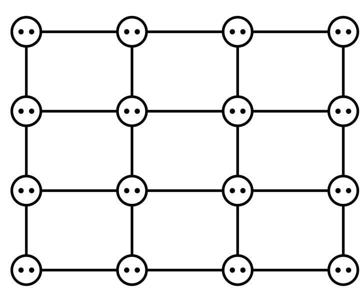

Fig. 2. 2-clique extension

Now will explicitly describe the local structure of an arbitrary vertex U. Let $\{e_i \mid e_i \in \mathbb{F}_q^m\}$ be the set of vectors with the first non-zero element equal to 1. We will refer to these vectors as canonical directions and note that there are $\begin{bmatrix} m \\ 1 \end{bmatrix}_q$ of them. Next we introduce the following equivalence classes $K(\delta)$ in $\mathbb{F}_{q^n}^*$: let $\{\delta_i \mid \delta_i \in \mathbb{F}_{q^n}^*\}$ be the set of column-vectors with the first non-zero element equal to 1. Denote $K(\delta_i) = \{a_t \cdot \delta_i \mid a_t \in \mathbb{F}_q^*\}$. The total number of classes is equal to $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$. We will refer to δ_i as the canonical representative of the equivalence class $K(\delta_i)$. Now we can easily get the following.

Lemma 3. *Let G be a bilinear forms graph $\mathrm{Bil}_q(n,m)$. For any arbitrary vertex U, all its adjacencies can be presented in the following form:*

$$U \sim U + a_t \cdot \delta_j \cdot e_i$$

Canonical directions e_i correspond to the columns, equivalence classes $K(\delta_i)$ correspond to the rows of a $\begin{bmatrix} n \\ 1 \end{bmatrix}_q \times \begin{bmatrix} m \\ 1 \end{bmatrix}_q$-lattice graph, and a_t correspond to the $(q-1)$-clique.

A bilinear forms graph $\mathrm{Bil}_q(n,m)$, with $m \leq n$, has two types of maximal cliques:

- Cliques of size $q^n = (q-1)\begin{bmatrix} n \\ 1 \end{bmatrix} + 1$ — correspond to fixed e_i, i.e. columns together with the original vertex U. The total number of such cliques is $\begin{bmatrix} m \\ 1 \end{bmatrix}_q$.
- Cliques of size $q^m = (q-1)\begin{bmatrix} m \\ 1 \end{bmatrix} + 1$ — correspond to fixed δ_i, i.e. rows together with the original vertex U. The total number of such cliques is $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$.

Strongly-regular case. In case of $D = 2$ and θ_D from Theorem 2 we obtain that if the weight distribution bound is achieved then a minimum support is represented with a complete bipartite subgraph $K_{q+1,\,q+1}$. Since a bilinear forms graph is distance-transitive, we can fix any arbitrary vertex U from a desired minimum support and consider the distance distribution with respect to this vertex U. In order to obtain a complete bipartite graph $K_{q+1,\,q+1}$, we need to find a set B of $q+1$ vertices in $G_1(U) = L_1$ and a set A of q vertices in $G_2(U) = L_2$ with a condition that each set is independent and any vertex from A is adjacent to all vertices from B.

The following lemma describes any independent set in L_1.

Lemma 4. *Consider a bilinear forms graph $\mathrm{Bil}_q(n,m)$, $m \leq n$. Let $\{e_i\}$ be the set of its canonical directions, and let $\{\delta_i\}$ be the set of canonical representatives. For an arbitrary vertex U, any independent set $\{V_k \mid k = 1..s\}$ in $G_1(U)$ can be represented as follows:*

$$V_k = U + a_{t_k} \cdot \delta_{j_k} \cdot e_{i_k},$$

where if $k \neq l$ then $e_{i_k} \neq e_{i_l}$, $\delta_{j_k} \neq \delta_{j_l}$, and $a_{t_k} \in \mathbb{F}_q^*$ can be chosen in an arbitrary way.

Proof. Follows from Lemma 3. □

Lemma 5. *Let $G = (V,E)$ be a bilinear forms graph $\mathrm{Bil}_q(n,m)$ and let f be an eigenfunction of G corresponding to some eigenvalue θ, with the support $\{V_1, \ldots, V_s\}$. Then the following holds:*

- *For an arbitrary vertex $W \in V$, the set $\{V_1 + W, \dots, V_s + W\}$ also is the support of some eigenfunction f' corresponding to the same eigenvalue θ, where $f(V_i) = f'(V_i + W)$.*
- *For an arbitrary non-zero $\gamma \in \mathbb{F}_q^n$, the set $\{\gamma V_1, \dots, \gamma V_s\}$ is the support of some eigenfunction f' corresponding to the same eigenvalue θ, where $f(V_i) = f'(\gamma V_i)$.*

Proof. The ranks of differences do not change under an affine transformation. □

Remark: Note that both Lemma 4 and Lemma 5 hold for bilinear forms graphs $\mathrm{Bil}_q(n, m)$ of arbitrary diameter D.

Having this lemma, from now we can consider the eigenfunctions supports reduced to the all-zero vertex. With respect to all-zero matrix, L_1 consists of matrices of rank 1 and L_2 consists of matrices or rank 2. Moreover without loss of generality we can suppose that our desired support contains some fixed vertice from L_1.

Structure of L_2. Let a_1 be a generating element of the multiplicative group $\mathbb{F}_q^*$. Denote $a_0 = 0$; $a_2 = a_1^2; \dots; a_{q-2} = a_1^{q-2}$; $a_{q-1} = a_1^{q-1} = 1$. Let us fix the following enumeration of the canonical directions e_i:

$$e_* = [0,\, 1], \quad e_0 = [1,\, 0], \quad e_1 = [1,\, a_1], \quad \dots, \quad e_{q-1} = [1, a_{q-1}]$$

Denote $\gamma_{t,j} = a_t \delta_j$. Vertices of L_2 can be presented as $\gamma_{t,j} \cdot e_0 + \sigma \cdot e_*$, where $\sigma \notin K(\gamma_{t,j})$.

Adjacencies between L_1 and L_2. Now we focus on the adjacencies between L_1 and L_2.

Lemma 6. *For any fixed $\gamma_{t_s,j_s} \in \mathbb{F}_q^*$, the vertex $\gamma_{t_s,j_s} \cdot e_s \in L_1$ has only the following neighbours in L_2:*

(1) *If $s \in \{0, \dots, q-1\}$ the adjacencies are:*
   - *$\gamma_{t_s,j_s} \cdot e_s \sim \gamma_{t_s,j_s} \cdot e_0 + \sigma \cdot e_*$, for any $\sigma \notin K(\gamma_{t_s,j_s})$. The total number of such neighbours is $q^2 - q$.*
   - *$\gamma_{t_s,j_s} \cdot e_s \sim \gamma_{t,j} \cdot e_0 + ((a_s - a_k)\gamma_{t_s,j_s} + a_k \gamma_{t,j}) \cdot e_*$, for any $\gamma_{t,j} \notin K(\gamma_{t_s,j_s})$ and for any $a_k \neq a_s$. The total number of such neighbours is $(q^2 - q)(q - 1)$.*

(2) *If $s = *$ the adjacencies are*
   - *$\gamma_{t_s,j_s} \cdot e_s \sim \gamma_{t,j} \cdot e_0 + (a_k \gamma_{t,j} + \gamma_{t_s,j_s}) \cdot e_*$, for any $\gamma_{t,j} \notin K(\gamma_{t_s,j_s})$ and for any $a_k \in \mathbb{F}_q$. The total number of such neighbours is $(q^2 - q)q$.*

Proof. (1) In case $s \in \{0, \dots, q-1\}$, for the neighbours of the first type the adjacency is obvious. Since we can take any $\sigma \notin K(\gamma_{t_s,j_s})$, the number of neighbours of this type is $q^2 - q$. Now we will prove that $\gamma_{t_s,j_s} \cdot e_s \nsim a_t \gamma_{t_s,j_s} \cdot e_0 + \sigma \cdot e_*$ for any $a_t \neq 1$ and for any $\sigma \notin K(\gamma_{t_s,j_s})$. Suppose the adjacency holds for some σ_s. Thus for some $a_k \in \mathbb{F}_q^*$ we have the equality $a_k(1 - a_t)\gamma_{t_s,j_s} = (\sigma_s - a_s \gamma_{t_s,j_s})$ Hence, $\sigma_s \in K(\gamma_{t_s,j_s})$. A contradiction.

Consider the remaining vertices of L_2. Suppose $\gamma_{t_s,j_s} \cdot e_s \sim \gamma_{t,j} \cdot e_0 + \sigma \cdot e_*$ for some $\gamma_{t,j} \notin K(\gamma_{t_s,j_s})$ and $\sigma \notin K(\gamma_{t,j})$. Then for some $a_k \in \mathbb{F}_q$ the equality $a_k(\gamma_{t_s,j_s} - \gamma_{t,j}) = a_s \gamma_{t_s,j_s} - \sigma$ holds. Thus $\sigma = (a_s - a_k)\gamma_{t_s,j_s} + a_k \gamma_{t,j}$. To fulfil the requirement $\sigma \notin K(\gamma_{t,j})$ we add the restriction $a_k \neq a_s$. Thus the number of neighbours of this type is $(q^2 - q)(q - 1)$. Since we checked all vertices from L_2 there are no other adjacencies.

(2) In case $s = *$, we can directly check that the described adjacency holds. Since the total number of the neighbours of this type is $q(q^2 - q) = b_1$, there does not exist any other neighbours in L_2.

□

Reconstruction. This section is dedicated to the problem of reconstructing of the second half of complete bipartite graph corresponding to the minimum support of the eigenfunction, when the independent set in L_1 is fixed. According to Lemma 4, any independent set in L_1 can be represented as $\{a_{t_k}\delta_{j_k} \cdot e_{i_k}\}$. For the purpose of convenience, we denote $\lambda_k = a_{t_k}\delta_{j_k}$.

Lemma 7. *Under the notation above, for any maximum independent set A in L_1, where*

$$A = \{\lambda_* e_*;\ \lambda_0 e_0,\ \ldots,\ \lambda_{q-1}e_{q-1}\},$$

if there exists an independent set B' in L_2 such that any vertex from B' is adjacent to all vertices from A, then the set B' consists of the following vertices

$$B' = \{\lambda_0 e_0 + \sigma_{k_0} e_*;\ \lambda_1 e_0 + \sigma_{k_1} e_*\ \ldots;\ \lambda_{q-1} e_0 + \sigma_{k_{q-1}} e_*\},$$

for some appropriate set $\{\sigma_{k_i}\}$, where $\sigma_{k_i} \notin K(\lambda_i)$

Proof. The statement follows from Lemma 6. □

Lemma 7 together with Lemma 6 give us the following system
$\lambda_* e_* \sim \lambda_j e_0 + (a_{k_j}\lambda_j + \lambda_*)e_* \quad \forall a_{k_j} \in \mathbb{F}_q \quad \forall j \in \{0, \ldots, q-1\}$

$$\lambda_s e_s \sim \left[\begin{array}{lll} \lambda_j e_0 + \sigma e_*, & \forall \sigma \notin K(\lambda_s); & \text{if } j = s \\ \lambda_j e_0 + ((a_{t_s} - a_{k_{s,j}})\lambda_s + a_{k_{s,j}}\lambda_j)e_*, & a_{k_{s,j}} \in \mathbb{F}_q, a_{k_{s,j}} \neq a_{t_s} & \text{if } j \neq s \end{array}\right.$$

In other words, to describe all the minimum supports we need to find the sets $\{a_{k_j}\}$ and $\{a_{k_{s,j}}\}$ such that the following equality holds:

$$\sigma_{k_j} = a_{k_j}\lambda_j + \lambda_* = (a_s - a_{k_{s,j}})\lambda_s + a_{k_{s,j}}\lambda_j, \quad \forall \lambda_s \neq \lambda_j, a_{k_{s,j}} \neq a_s$$

Explicit construction (partial solution). Even though we still cannot provide the full solution to the system above, the partial solution exists and can be constructed explicitely. This result is presented in the following statement

Theorem 3. *Consider a bilinear forms graph $\mathrm{Bil}_p(2,2)$ over a prime field. Let a_1 be a generating element of the multiplicative group $\mathbb{F}_p^*$. Denote $a_0 = 0$; $a_2 = a_1^2$; $\ldots$; $a_{p-2} = a_1^{p-2}$; $a_{p-1} = a_1^{p-1} = 1$. Choose $\delta \in \mathbb{F}_p$, such that $\delta \neq -\xi^2$ for all $\xi \in \mathbb{F}_p$. Denote $b_i = \frac{1}{a_i^2\delta+1}$. The independent set*

$$A = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_*,\ b_0 \begin{bmatrix} 1 \\ a_0\delta \end{bmatrix} e_0,\ \ldots,\ b_{p-1} \begin{bmatrix} 1 \\ a_{p-1}\delta \end{bmatrix} e_{p-1} \right\}$$

together with

$$B = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} e_*;\ b_0 \begin{bmatrix} 1 \\ a_0\delta \end{bmatrix} e_0 + b_0 \begin{bmatrix} -a_0 \\ 1 \end{bmatrix} e_*;\ \ldots;\ b_{p-1} \begin{bmatrix} 1 \\ a_{p-1}\delta \end{bmatrix} e_0 + b_{p-1} \begin{bmatrix} -a_{p-1} \\ 1 \end{bmatrix} e_* \right\}$$

form a minimum eigensupport as two parts of a complete bipartite graph $K_{p+1,\,p+1}$.

Proof. We need to check the ranks of the vertices, the independence of the sets and the adjacencies between the set A and the set B.

- Ranks validation:
The set A consists of vertices of rank 1. The set B has the all-zero vertex $U = \begin{bmatrix} 0 \\ 0 \end{bmatrix} e_*$ of rank 0. Any other vertex $V_i \in B$ has rank 2, since $\begin{bmatrix} -a_i \\ 1 \end{bmatrix} \notin K\Big(\begin{bmatrix} 1 \\ a_i\delta \end{bmatrix}\Big)$ for any $i = 0, \dots, q-1$.
- Independence of the sets:
The independence of the set A follows from Lemma 4 since $\begin{bmatrix} 1 \\ a_i\delta \end{bmatrix} = \delta_i$ up to some enumeration. Now consider the set B. It is obvious that $U \nsim V_i$ for all $i = 0, \dots, q-1$. We need to prove that $V_i \nsim V_j$ for $i \neq j$. Consider the rank of their difference:
$$\operatorname{rank}(V_i - V_j) = \operatorname{rank}\Big(\begin{bmatrix} b_i - b_j \\ b_i a_i \delta - b_j a_j \delta \end{bmatrix} e_0 + \begin{bmatrix} -b_i a_i + b_j a_j \\ b_i - b_j \end{bmatrix} e_*\Big)$$
By direct calculations we check that rank is 2.
- Adjacencies between the sets A and B:

Suppose $V_i = b_i \begin{bmatrix} 1 \\ a_i\delta \end{bmatrix} e_0 + b_i \begin{bmatrix} -a_i \\ 1 \end{bmatrix} e_*$ and $U_j = b_j \begin{bmatrix} 1 \\ a_j\delta \end{bmatrix} e_j$ are the arbitrary vertices of the sets B and A correspondingly, where $0 \leq i, j \leq p-1$. Consider their difference:
$$V_i - U_j = b_i \begin{bmatrix} 1 \\ a_i\delta \end{bmatrix} e_0 + b_i \begin{bmatrix} -a_i \\ 1 \end{bmatrix} e_* - b_j \begin{bmatrix} 1 \\ a_j\delta \end{bmatrix} e_0 - b_j a_j \begin{bmatrix} 1 \\ a_j\delta \end{bmatrix} e_* =$$
$$= \begin{bmatrix} b_i - b_j \\ \delta(b_i a_i - b_j a_j) \end{bmatrix} e_0 + \begin{bmatrix} -b_i a_i - b_j a_j \\ b_i - b_j a_j^2 \delta \end{bmatrix} e_*$$
Since the following equation holds
$$\begin{bmatrix} b_i - b_j \\ \delta(b_i a_i - b_j a_j) \end{bmatrix} = \begin{bmatrix} -b_i a_i - b_j a_j \\ b_i - b_j a_j^2 \delta \end{bmatrix} \cdot \frac{\delta(a_i - a_j)}{1 + \delta a_i a_j}$$
and $\frac{\delta(a_i - a_j)}{1+\delta a_i a_j} \in \mathbb{F}_p$ we conclude that $\operatorname{rank}(V_i - U_j) = 1$, thus $V_i \sim U_j$ for $0 \leq i, j \leq p-1$.

Now consider the remaining vertex $U_* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_*$ from A.
$$V_i - U_* = \begin{bmatrix} b_i \\ \delta b_i a_i \end{bmatrix} e_0 + \begin{bmatrix} -b_i a_i \\ b_i - 1 \end{bmatrix} e_*$$
Since $\begin{bmatrix} b_i \\ \delta b_i a_i \end{bmatrix} \cdot a_i = \begin{bmatrix} -b_i a_i \\ b_i - 1 \end{bmatrix}$ we obtain that $\operatorname{rank}(V_i - U_*) = 1$, therefore $V_i \sim U_*$. The remaining vertex $V_* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} e_*$ from B is obviously adjacent to all the vertices from A.

This completes the proof.

□

Thus we obtain the complete bipartite graph $K_{p+1,p+1}$ that gives rise to the minimum support of the eigenfunction corresponding to the minimum eigenvalue of the bilinear forms graph $\mathrm{Bil}_p(2,2)$ over a prime field $\mathbb{F}_p$. Using Lemma 5 we can obtain more eigenfunctions with the minimum support. However the construction described above does not provide the full characterization of the eigenfunctions with the minimum support.

4. Bilinear forms graphs with $D \geq 3$

Now we consider the bilinear forms graphs $\mathrm{Bil}_q(n,m)$ with larger diameter, i.e. $D \geq 3$. In this Section we will prove that there does not exist eigenfunctions with minimum supports achieving the weight distribution bound. In order to this we will use the connection between bilinear forms graphs and Grassmann graphs.

Grassmann graph introduction. The Grassmann graph $J_q(n{+}m, m)$ is a distance-regular graph with the vertex set consisting of all m-dimensional subspaces of a vector space of dimension $(n+m)$ over $\mathbb{F}_q$. Two vertices are adjacent whenever the corresponding subspaces intersect in a $(m-1)$-dimensional subspace (cf. [4]). The Grassmann graph has the diameter $D = \min(n,m)$ and the minimum eigenvalue $\theta_D = -\begin{bmatrix} D \\ 1 \end{bmatrix}_q$. The local structure of the Grassmann graph is the q-clique extension of a $\begin{bmatrix} n \\ 1 \end{bmatrix}_q \times \begin{bmatrix} m \\ 1 \end{bmatrix}_q$ lattice.

The bilinear forms graph $\mathrm{Bil}_q(n,m)$ with $m \leq n$ can be considered as a subgraph of the Grassman graph $J_q(n+m,m)$ as follows: given a fixed subspace W of dimension n, all m-spaces U such that $U \cap W = 0$ are the vertices of $\mathrm{Bil}_q(n,m)$ (see [4]). If we return to a matrix-notation, this embedding can be described in the following way. Let $\{M_i\}$ be the set of vertices of $\mathrm{Bil}_q(n,m)$. And let H be some fixed non-degenerate matrix of size $m \times m$. Consider the extended matrices $\overline{M_i}$ constructed as below:

$$\overline{M_i} = \begin{bmatrix} H \\ M_i \end{bmatrix}$$

It is easy to check that $M_i \sim M_j \Leftrightarrow \overline{M_i} \sim \overline{M_j}$. Since H is a non-degenerate matrix, the linear span of the columns of $\overline{M_i}$ yields the m-dimensional space. For example, if we take H to be the identity matrix, then all the vertices of the Grassmann graph $J_q(n+m,m)$ can be represented as the linear span of the columns of the matrices $L \in \mathcal{L}$, where $\mathcal{L}$ is the set of all $(n+m) \times m$ matrices L such that L^T is in a reduced row echelon form. Let $\omega_i = (0,\ldots,\underset{i}{1},\ldots,0)^T$ and $W = <\{\omega_i\}>$, where $m+1 \leq i \leq n+m$. Then W is an n-dimensional space such that $L \cap W = 0 \Leftrightarrow L = \overline{M_i}$ for some vertex M_i from a bilinear forms graph.

There are two types of maximal cliques in the Grassmann graph:

- The m-subspaces that contain a fixed $(m-1)$-subspace. The cardinality of such a clique is $\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q = q\begin{bmatrix} n \\ 1 \end{bmatrix} + 1$. These cliques correspond to the columns (see Figure 3 for reference). The total number of such cliques is $\begin{bmatrix} m \\ 1 \end{bmatrix}_q$
- The m-subspaces that lie in a fixed $(m+1)$-subspace. The cardinality of such a clique is $\begin{bmatrix} m+1 \\ 1 \end{bmatrix}_q = q\begin{bmatrix} m \\ 1 \end{bmatrix} + 1$. These cliques correspond to the rows (see Figure 3 for reference). The total number of such cliques is $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$.

If $n = m$ both types of cliques are Delsarte cliques.

$\mathrm{Bil}_q(n,m)$ as a subgraph of the Grassmann graph.

Lemma 8. *Delsarte cliques of bilinear forms graph* $\mathrm{Bil}_q(n,m)$ *are embedded in Delsarte cliques of a Grassman graph* $J_q(n{+}m,m)$ *in the sense that for any Delsarte*

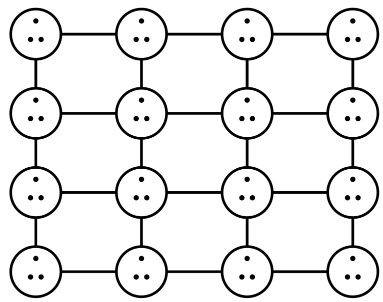

Fig. 3. 3-clique extension

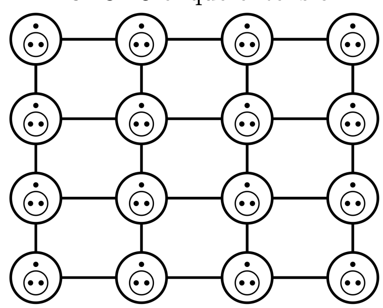

Fig. 4. Embedding illustration

cliques C and $\widehat{C}$ of a bilinear forms graph and the Grassmann graph correspondingly, either $C \subset \widehat{C}$ or $C \cap \widehat{C} = \emptyset$.

Proof. The proof follows from the graphs local structures representation. □

Figure 4 illustrates the embedding described above. The following statement is crucial for the proof of the main result.

Corollary 1. *Suppose f is an eigenfunction of a bilinear forms graph $\mathrm{Bil}_q(n,m)$ corresponding to the minimum eigenvalue θ_D. Then $\widehat{f}$ is an eigenfunction of the Grassmann graph $J_q(n+m,m)$, where*

$$\widehat{f}(\overline{M}) = \begin{cases} f(M), & \text{if } M \in V(\mathrm{Bil}_q(n,m)) \\ 0, & \text{else} \end{cases}$$

Proof. Since f is an eigenfunction corresponding to the minimum eigenvalue, from Theorem 1 we obtain that for every Delsarte clique C of a bilinear forms graph $\mathrm{Bil}_q(n,m)$ it holds $\sum\limits_{x \in C} f(x) = 0$. Lemma 8 shows that any Delsarte clique $\widehat{C}$ of the Grassmann graph $J_q(n+m,m)$ either contains a Delsarte clique C of a bilinear forms graph $\mathrm{Bil}_q(n,m)$ or does not intersect with it. Hence for any Delsarte clique $\overline{C}$ it holds $\sum\limits_{x \in \widehat{C}} \widehat{f} = 0$. Thus from Theorem 1 function $\widehat{f}$ is an eigenfunction corresponding to the minimum eigenvalue of the Grassmann graph. □

Now we need to jump into the theory of quadratic and bilinear forms. As for notations and definitions we basically follow Cho [5] and James (Chapter 18 of [3]) with slight modifications according to [12].

- Let V be a $2m$-dimensional vector space with a basis $\{u_1, \ldots, u_m, v_1, \ldots, v_m\}$ over a finite field $\mathbb{F}_q$
- For any vectors $f = (x_1, \ldots, x_m, y_1, \ldots, y_m)$ and $f' = (x'_1, \ldots, x'_m, y'_1, \ldots, y'_m)$ from the space V, a bilinear form $<,>: V \times V \to \mathbb{F}_q$ is defined as follows:

$$<x_1u_1 + \ldots + x_mu_m + y_1v_1 + \ldots + y_mv_m,$$
$$x'_1u_1 + \ldots + x'_mu_m + y'_1v_1 + \ldots + y'_mv_m> = \sum_{i=1}^{m}(x_iy'_i + x'_iy_i),$$

thus its Gram matrix with respect to the chosen basis of V is $\begin{pmatrix} \mathbf{0} & E_m \\ E_m & \mathbf{0} \end{pmatrix}$, where $\mathbf{0}$ and E_m — are all-zeros matrix and identity matrix correspondingly.
- We also define a special function $\phi : V \to \mathbb{F}_q$ that will help us to deal with the case of $\mathrm{char}(\mathbb{F}_q) = 2$:

$$\phi(x_1u_1 + \ldots + x_mu_m + y_1v_m + \ldots + y_mv_m) = \sum_{i=1}^{m} x_iy_i$$

- For a subspace $W < V$ define

$$\mathrm{rad}\, W = \{w \in W \mid \phi(w) = 0 \text{ and } <w, w'> = 0 \text{ for all } w' \in W\}$$

- A subspace W is called totally isotropic if $W = \mathrm{rad}\, W$
- A quadratic form $Q : \mathbb{F}_q^n \to \mathbb{F}_q$ is said to be non-degenerate if its kernel $\{x \mid Q(x + y) = Q(y)\, \forall y \in \mathbb{F}_q^n\} = 0$ is zero.

Now we can reformulate Theorem 2 from [5] in our terms:

Theorem 4. *Let $J_q(N, m)$ be the Grassmann graph, $N \geq 2m$. Suppose f is an eigenfunction of the Grassmann graph corresponding to its minimum eigenvalue with the condition that* $\mathrm{Supp}(f)$ *is of minimum cardinality. Then the non-zeros of the function f correspond to the maximal totally isotropic subspaces of a $2m$-dimensional space, equipped with a bilinear form B with a Gram matrix* $\begin{pmatrix} \mathbf{0} & E_m \\ E_m & \mathbf{0} \end{pmatrix}$ *up to the equivalence.*

Remark: Note that using Theorem 4 together with Theorem 5 from [13] without loss of generality we can further consider the totally isotropic spaces with respect to a quadratic form $Q = x_1y_1 + \ldots + x_my_m$.

We also need several known results which we provide here for the paper to be self-contained.

Theorem 5. (Chevalley, [1]) *Let n, d_1, $\ldots$, d_r be positive integers such that $d_1 + \ldots + d_r < n$. For each $1 \leq i \leq r$, let $P_i(t_1, \ldots, t_n) \in \mathbb{F}_q[t_1, \ldots, t_n]$ be a polynomial of total degree d_i with zero constant term, i.e. $P_i(0, \ldots, 0) = 0$. Then there exists $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n \setminus \{(0, \ldots, 0)\}$ such that $P_1(x) = \ldots = P_r(x) = 0$.*

A well-known corollary from Theorem 5 is:

Corollary 2. *Let U be a vector space with a dimension not less than* 3 *over a finite field $\mathbb{F}_q$. Then any non-degenerate quadratic form Q on this space is isotropic, i.e. there is a non-zero vector on which the form evaluates to zero.*

The next theorem that we need is the following ([8], Proposition 8.11).

Theorem 6. *Let ϕ be a non-degenerate quadratic form on a vector space V over a finite field F. Then every totally isotropic subspace of V is contained in a totally isotropic subspace of dimension equal to the Witt index of ϕ.*

Now we are finally ready to prove the main result of this section.

Theorem 7. *Let $\mathrm{Bil}_q(n,m)$ be a bilinear forms graph of diameter $D \geq 3$. Then the minimum support of an eigenfunction corresponding to the minimum eigenvalue does not achieve the weight distribution bound.*

Proof. Suppose the opposite. Without loss of generality, we can consider $n \geq m \geq 3$. Let f be an eigenfunction with the support achieving the weight distribution bound. Under the notation of Corollary 1, $\widehat{f}$ is an eigenfunction of the Grassmann graph $J_q(n+m,m)$. Since the weight distribution bounds coincide for these graphs, $\widehat{f}$ is an eigenfunction with the minimum support. According to Theorem 4, the non-zeros of $\widehat{f}$ correspond to the maximal isotropic spaces of a non-degenerate quadratic form Q. Since f is an eigenfunction of $\mathrm{Bil}_q(n,m)$, we conclude that there exists a subspace W of dimension n such that it trivially intersects with all the maximal isotropic subspaces. Consider the values of the quadratic form Q on this space W. From Corollary 2, since $n \geq 3$, there exists a non-zero vector $w \in W$ such that $Q(w)=0$. By definition of quadratic form, $Q\mid_{<w>}=0$. Thus $<w>$ is a 1-dimensional totally isotropic space. According to Theorem 6, $<w>$ is contained in a maximal totally isotropic subspace. A contradiction. □

Theorem 7 shows that for bilinear forms graphs of diameter $D \geq 3$ there do not exist eigenfunctions corresponding to θ_D with the minimum supports that achieve the weight distribution bound. This leads to an open question: what is the lower bound for the cardinality of the minimum support of such functions?

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Evgeniia Vadimovna Sotnikova
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
E-mail address: lucernavesper@gmail.com