S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 16, стр. 542–546 (2019) УДК 515.124.2

DOI 10.33048/semi.2019.16.035 MSC 30L99, 53C23, 54D10

ON GENERALIZED JØRGENSEN INEQUALITY IN $\mathrm{SL}(2,\mathbb{C})$

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Abstract. Wang, Jiang and Cao have obtained a generalized version of the Jørgensen inequality in Proc. Indian Acad. Sci. Math. Sci., 123(2):245–251, 2013, for two generator subgroups of $\mathrm{SL}(2,\mathbb{C})$ where one of the generators is loxodromic. We prove that their inequality is strict.

Keywords: Jørgensen inequality, discreteness.

1. Introduction

The group $\mathrm{SL}(2,\mathbb{C})$ acts by Möbius transformations on the Riemann sphere $\mathbb{S}^2$ and the action extends to the three dimensional hyperbolic space $\mathbf{H}^3$ by orientation-preserving isometries. An element g in $\mathrm{SL}(2,\mathbb{C})$ is elliptic if it has a fixed point on the hyperbolic space. It is parabolic, resp. loxodromic, if it is non-elliptic and has exactly one, resp. two fixed points on the boundary $\mathbb{S}^2$. It is a classical problem to understand discreteness of subgroups of $\mathrm{SL}(2,\mathbb{C})$. The Jørgensen inequality is a celebrated result in this direction that provides necessary condition of discreteness for a two generator subgroup of $\mathrm{SL}(2,\mathbb{C})$, see [4]. Extremality of the Jørgensen inequality has been investigated in [5]. In the literature there are several generalizations of the Jørgensen inequality, and extremalities of some of those inequalities have also been investigated, e.g. [2], [3], [6], [7]. In this note we investigate extremality of one such generalized Jørgensen inequality.

In [8], Wang, Jiang and Cao have obtained a generalized version of the Jørgensen inequality for two generator subgroups of $\mathrm{SL}(2,\mathbb{C})$ where one of the generators is loxodromic. We recall their result.

Gongopadhyay, K., Mishra, M.M., Tiwari, D., On generalized Jørgensen inequality in $\mathrm{SL}(2,\mathbb{C})$.

Gongopadhyay acknowledges partial support from SERB MATRICS grant MTR/2017/000355. Tiwari is supported by NBHM-SRF.

Received September, 17, 2018, published April, 19, 2019.

Theorem WJC. [8] *Let g, h are elements in* $\mathrm{SL}(2,\mathbb{C})$ *such that g is loxodromic. Suppose that g, h are of the form: for $|\lambda| > 1$,*

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \ h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.1}$$

Let g be such that $M_g < 1$, where

$$M_g = |\lambda - 1| + |\lambda^{-1} - 1|.$$

If $\langle g, h \rangle$ is discrete and non-elementary, then

$$|abcd|^{\frac{1}{2}} \geq \frac{1 - M_g}{M_g^2}. \tag{1.2}$$

We prove that the above inequality is strict, see Theorem 2.1 below. Further, we note down a few generalized Jørgensen type inequalities which are also strict.

2. Proof of Strictness of the Inequality

Theorem 2.1. *Under the hypothesis of the above theorem, equality does not hold in* (1.2).

Доказательство. If possible suppose equality holds in (1.2), i.e. suppose

$$|abcd|^{\frac{1}{2}} = \frac{1 - M_g}{M_g^2}.$$

Let $h_1 = hgh^{-1}$. Then

$$\begin{aligned} h_1 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \begin{pmatrix} ad\lambda - bc\lambda^{-1} & -(\lambda - \lambda^{-1})ab \\ (\lambda - \lambda^{-1})cd & ad\lambda^{-1} - bc\lambda \end{pmatrix} \\ &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}. \end{aligned}$$

Note that

$$b_1 c_1 = -(\lambda - \lambda^{-1})^2 abcd = -(\lambda - \lambda^{-1})^2 bc(1 + bc). \tag{2.1}$$

$$\begin{aligned} a_1 d_1 &= (ad\lambda - bc\lambda^{-1})(ad\lambda^{-1} - bc\lambda) \\ &= (ad)^2 + (bc)^2 - abcd(\lambda^2 + \lambda^{-2}) \\ &= (ad - bc)^2 - abcd(\lambda^2 + \lambda^{-2} - 2) \\ &= 1 - (\lambda - \lambda^{-1})^2 abcd. \end{aligned}$$

This implies,

$$\begin{aligned}
|a_1 d_1| &= |1-(\lambda-\lambda^{-1})^2 abcd| \\
&\leq 1+|\lambda-1+1-\lambda^{-1}|^2|abcd| \\
&\leq 1+(|\lambda-1|+|\lambda^{-1}-1|)^2|abcd| \\
&\leq 1+M_g^2\frac{(1-M_g)^2}{M_g^4} \\
&\leq 1+\frac{(1-M_g)^2}{M_g^2} \\
&\leq \frac{M_g^2+(1-M_g)^2}{M_g^2} \\
&\leq \frac{M_g^2+(1-M_g)^2+2M_g(1-M_g)}{M_g^2}, \quad \text{since } M_g<1, \\
&\leq \frac{(M_g+1-M_g)^2}{M_g^2}.
\end{aligned}$$

That is,

$$|a_1 d_1|^{\frac{1}{2}} \leq \frac{1}{M_g}. \tag{2.2}$$

Also, we have $b_1 = -(\lambda-\lambda^{-1})ab$. Thus

$$|b_1| \leq |\lambda-1+1-\lambda^{-1}||ab| \leq M_g|ab|.$$

Similarly, $|c_1| \leq M_g|cd|$. Hence

$$|b_1 c_1| \leq M_g^2|abcd| = M_g^2.\frac{(1-M_g)^2}{M_g^4}, \text{ which implies}$$

$$|b_1 c_1|^{\frac{1}{2}} \leq \frac{1-M_g}{M_g}. \tag{2.3}$$

In particular,

$$M_g(1+|b_1 c_1|^{\frac{1}{2}}) \leq 1. \tag{2.4}$$

Now note that $\langle g, h_1\rangle$ is discrete and non-elementary. Discreteness of $\langle g, h_1\rangle$ is obvious, and if it was elementary, that would have implied that g and h had a common fixed point and hence, would have contradicted the assumption that $\langle g, h\rangle$ is non-elementary. So, applying Theorem WJC to $\langle g, h_1\rangle$, we have

$$\begin{aligned}
\frac{1-M_g}{M_g^2} &\leq |a_1 b_1 c_1 d_1|^{\frac{1}{2}} \leq |a_1 d_1|^{\frac{1}{2}}|b_1 c_1|^{\frac{1}{2}} \\
&\leq \frac{1-M_g}{M_g^2}, \quad \text{by (2.2) and (2.3).}
\end{aligned}$$

This implies,

$$|a_1 b_1 c_1 d_1|^{\frac{1}{2}} = \frac{1-M_g}{M_g^2}.$$

Next we observe that

$$\begin{aligned}\left(\frac{1-M_g}{M_g^2}\right)^2 &= |a_1b_1c_1d_1| \\ &\leq |1+b_1c_1||b_1c_1| \\ &\leq (1+|b_1c_1|)(\lambda-\lambda^{-1})^2|abcd|, \quad \text{from first equality in (2.1)} \\ &\leq (\lambda-\lambda^{-1})^2(1+|b_1c_1|)\left(\frac{1-M_g}{M_g^2}\right)^2 \\ &\leq M_g^2(1+|b_1c_1|+2|b_1c_1|^{\frac{1}{2}})\left(\frac{1-M_g}{M_g^2}\right)^2 \\ &\leq \left(M_g(1+|b_1c_1|^{\frac{1}{2}}\right)^2\left(\frac{1-M_g}{M_g^2}\right)^2 \\ &\leq \left(\frac{1-M_g}{M_g^2}\right)^2, \quad \text{by (2.4).}\end{aligned}$$

Hence we have

$$(\lambda-\lambda^{-1})^2(1+|b_1c_1|)=1. \tag{2.5}$$

Noting that

$$tr^2(g)-4=(\lambda-\lambda^{-1})^2, \quad \text{and} \quad tr[g,h_1]-2=-(\lambda-\lambda^{-1})^2b_1c_1,$$

the above equality implies that $\langle g, h_1\rangle$ satisfies equality in the classical Jørgensen inequality. By a theorem of Jørgensen and Kiikka, see [5, Theorem 2], this implies that g is either elliptic or parabolic, which is a contradiction. Hence equality can not hold in (1.2). □

Combining Theorem WJC and Theorem 2.1, we can rephrase the generalized Jørgensen inequality as follows.

Theorem 2.2. *Let g, h are elements in* SL(2, ℂ) *such that g is loxodromic. Suppose that g, h are of the form* (1.1). *Let g be such that $M_g < 1$. If*

$$|abcd|^{\frac{1}{2}} \leq \frac{1-M_g}{M_g^2}, \tag{2.6}$$

then $\langle g, h\rangle$ is either elementary or non-discrete.

2.1. **Some more inequalities.** The main idea in [8] was to embed SL(2, ℂ) into the isometry group Sp(1, 1) of the one dimensional quaternionic hyperbolic space $\mathbf{H}^1_{\mathbb{H}}$, and then use quaternionic Jørgensen inequality of Cao and Parker, see [1, Theorem 1.1]. In view of the above theorem, following arguments as used in the proof of [1, Corollary 1.2], we note the following.

Corollary 2.3. *Let g, h are elements in* SL(2, ℂ) *such that g is loxodromic. Suppose that g, h are of the form (1.1). Let g be such that $M_g < 1$. If $\langle g, h\rangle$ is discrete and non-elementary, then each of the following strict inequalities holds.*

$$|bc|^{\frac{1}{2}} > \frac{1-M_g}{M_g}. \tag{2.7}$$

$$|1+bc|^{\frac{1}{2}} > \frac{1-M_g}{M_g}. \tag{2.8}$$

$$|1+bc|+|bc| > \frac{2(1-M_g)}{M_g^2}. \tag{2.9}$$

Доказательство. If possible, suppose $|bc|^{\frac{1}{2}} \leq \frac{1-M_g}{M_g}$. Then

$$|ad|^{\frac{1}{2}}|bc|^{\frac{1}{2}} \leq (1+|bc|^{\frac{1}{2}})|bc|^{\frac{1}{2}} \leq \frac{1-M_g}{M_g^2}.$$

Using Theorem 2.2, $\langle g, h\rangle$ is either discrete or non-elementary.

If possible suppose $|1+bc|^{\frac{1}{2}} \leq \frac{1-M_g}{M_g}$. Then it follows similarly as above noting that $|bc|^{\frac{1}{2}} \leq 1+|ad|^{\frac{1}{2}}$ and $ad-bc=1$.

Finally, if $|1+bc|+|bc| \leq \frac{2(1-M_g)}{M_g^2}$, then

$$|ad|^{\frac{1}{2}}|bc|^{\frac{1}{2}} \leq \frac{1}{2}(|ad|+|bc|) \leq \frac{1-M_g}{M_g^2},$$

and the result follows from Theorem 2.2.

This completes the proof. □

In view of the results noted in this communication, the following question is natural to ask.

Question 1. *What are sharp bounds for the inequalities (1.2) and (2.7) – (2.9)?*

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