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ALGEBRAS FOR DEFINABLE FAMILIES OF THEORIES

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ABSTRACT. We consider algebras associated with sentence-definable and diagram-definable subfamilies of families of theories. Topological properties and ranks for these algebras are characterized.

Keywords: family of theories, definable subfamily, algebra for definable subfamilies, rank, degree.

We define algebras associated with sentence-definable and diagram-definable subfamilies [1] as well as related characteristics connected with given families of theories. Topological properties and ranks for these algebras are characterized.

1. Preliminaries

Throughout the paper we consider families $\mathcal{T}$ of complete first-order theories of a relational language $\Sigma = \Sigma(\mathcal{T})$ and use the following terminology in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \rightleftharpoons \bigcup\limits_{i\in I} \mathcal{A}_i$ expanded by the predicates P_i is the P-*union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the P-*operator*. The structure $\mathcal{A}_P$ is called the P-*combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

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Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}' \upharpoonright P_i) \upharpoonright \Sigma(\mathcal{A}_i)$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write $\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}' \upharpoonright \bigcap\limits_{i\in I} \overline{P_i}$, maybe applying Morleyzation. Moreover, we write

$$\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$$

for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

Note that if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$, $\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_P = \mathrm{Th}(\mathcal{A}_P)$, being *$P$-combination* of T_i, which is denoted by $\mathrm{Comb}_P(T_i)_{i\in I}$.

Notice that P-combinations are represented by generalized products of structures [11].

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ being the *E-union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the *E-operator*. The structure $\mathcal{A}_E$ is also called the *E-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes. The E-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_E = \mathrm{Th}(\mathcal{A}_E)$, being *$E$-combination* of T_i, which is denoted by $\mathrm{Comb}_E(T_i)_{i\in I}$ or by $\mathrm{Comb}_E(\mathcal{T})$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the *E-representability*.

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by e-$\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-}\mathrm{Sp}(\mathcal{A}')) \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$. If structures $\mathcal{A}_i$ represent theories T_i of a family $\mathcal{T}$, consisting of T_i, $i \in I$, then the e-spectrum e-$\mathrm{Sp}(\mathcal{A}_E)$ is denoted by e-$\mathrm{Sp}(\mathcal{T})$.

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

Definition [3]. Let $\overline{\mathcal{T}}_\Sigma$ be the set of all complete elementary theories of a relational language Σ. For a set $\mathcal{T} \subset \overline{\mathcal{T}}_\Sigma$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

The operator Cl_E of E-closure can be naturally extended to the classes $\mathcal{T} \subset \overline{\mathcal{T}}$, where $\overline{\mathcal{T}}$ is the union of all $\overline{\mathcal{T}}_\Sigma$ as follows: $\mathrm{Cl}_E(\mathcal{T})$ is the union of all $\mathrm{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0 \subseteq \mathcal{T}$, where new language symbols with respect to the theories in $\mathcal{T}_0$ are empty.

For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$. Any set $\mathcal{T}_\varphi$ is called the *φ-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$, or the (φ-)*definable* subset of $\mathcal{T}$. The set $\mathcal{T}_\varphi$ is also called (*formula-* or *sentence-*)*definable* (by the sentence φ) with respect to $\mathcal{T}$, or (*sentence-*)*$\mathcal{T}$-definable*, or simply *s-definable*.

Proposition 1.1 [3]. *If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $T \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any formula $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.*

If T is an accumulation point for $\mathcal{T}$ then we also say that T is an *accumulation point* for $\mathrm{Cl}_E(\mathcal{T})$.

Theorem 1.2 [3]. *For any sets $\mathcal{T}_0, \mathcal{T}_1 \subset \overline{\mathcal{T}}$, $\mathrm{Cl}_E(\mathcal{T}_0 \cup \mathcal{T}_1) = \mathrm{Cl}_E(\mathcal{T}_0) \cup \mathrm{Cl}_E(\mathcal{T}_1)$.*

Definition [3]. Let $\mathcal{T}_0$ be a closed set in a topological space $(\mathcal{T}, \mathcal{O}_E(\mathcal{T}))$, where $\mathcal{O}_E(\mathcal{T}) = \{\mathcal{T} \setminus \mathrm{Cl}_E(\mathcal{T}') \mid \mathcal{T}' \subseteq \mathcal{T}\}$. A subset $\mathcal{T}'_0 \subseteq \mathcal{T}_0$ is said to be *generating* if $\mathcal{T}_0 = \mathrm{Cl}_E(\mathcal{T}'_0)$. The generating set $\mathcal{T}'_0$ (for $\mathcal{T}_0$) is *minimal* if $\mathcal{T}'_0$ does not contain proper generating subsets. A minimal generating set $\mathcal{T}'_0$ is *least* if $\mathcal{T}'_0$ is contained in each generating set for $\mathcal{T}_0$.

Theorem 1.3 [3]. *If $\mathcal{T}'_0$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*

(1) *$\mathcal{T}'_0$ is the least generating set for $\mathcal{T}_0$;*

(2) *$\mathcal{T}'_0$ is a minimal generating set for $\mathcal{T}_0$;*

(3) *any theory in $\mathcal{T}'_0$ is isolated by some set $(\mathcal{T}'_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}'_0)_\varphi = \{T\}$;*

(4) *any theory in $\mathcal{T}'_0$ is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.*

Notice that having the least generating set $\mathcal{T}'_0$ for a E-closed set $\mathcal{T}_0$,

$$e\text{-}\mathrm{Sp}(\mathcal{T}_0) = e\text{-}\mathrm{Sp}(\mathcal{T}'_0) = |\mathcal{T}_0 \setminus \mathcal{T}'_0|.$$

Definition [8]. Let $\mathcal{T}$ be a family of theories and T be a theory, $T \notin \mathcal{T}$. The theory T is called *$\mathcal{T}$-approximated*, or *approximated by* $\mathcal{T}$, or *$\mathcal{T}$-approximable*, or a *pseudo-$\mathcal{T}$-theory*, if for any formula $\varphi \in T$ there is $T' \in \mathcal{T}$ such that $\varphi \in T'$.

If T is $\mathcal{T}$-approximated then $\mathcal{T}$ is called an *approximating family* for T, theories $T' \in \mathcal{T}$ are *approximations* for T, and T is an *accumulation point* for $\mathcal{T}$.

An approximating family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(T)$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

It was shown in [8] that any e-minimal family $\mathcal{T}$ has unique accumulation point T with respect to neighbourhoods $\mathcal{T}_\varphi$, and $\mathcal{T} \cup \{T\}$ is also called *e-minimal*.

Following [9] we define the *rank* $\mathrm{RS}(\cdot)$ for the families of theories, similar to Morley rank [12], and a hierarchy with respect to these ranks in the following way.

For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}(\mathcal{T}) = -1$, for finite nonempty families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) = 0$, and for infinite families $\mathcal{T}$ — $\mathrm{RS}(\mathcal{T}) \geq 1$.

For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.

If α is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.

We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.

If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

If $\mathcal{T}$ is e-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Proposition 1.4 [9]. *A family $\mathcal{T}$ is e-minimal if and only if* $\mathrm{RS}(\mathcal{T}) = 1$ *and* $\mathrm{ds}(\mathcal{T}) = 1$.

Proposition 1.5 [9]. *For any family $\mathcal{T}$,* $\mathrm{RS}(\mathcal{T}) = \mathrm{RS}(\mathrm{Cl}_E(\mathcal{T}))$*, and if $\mathcal{T}$ is nonempty and e-totally transcendental then* $\mathrm{ds}(\mathcal{T}) = \mathrm{ds}(\mathrm{Cl}_E(\mathcal{T}))$.

Definition [9]. A family $\mathcal{T}$, with infinitely many accumulation points, is called *a-minimal* if for any sentence $\varphi \in \Sigma(T)$, $\mathcal{T}_\varphi$ or $\mathcal{T}_{\neg\varphi}$ has finitely many accumulation points.

Theorem 1.6 [9]. *For any family $\mathcal{T}$,* $\mathrm{RS}(\mathcal{T}) = 2$*, with* $\mathrm{ds}(\mathcal{T}) = n$*, if and only if $\mathcal{T}$ is represented as a disjoint union of subfamilies $\mathcal{T}_{\varphi_1}, \ldots, \mathcal{T}_{\varphi_n}$, for some pairwise inconsistent sentences $\varphi_1, \ldots, \varphi_n$, such that each $\mathcal{T}_{\varphi_i}$ is a-minimal.*

Definition [9]. Let α be an ordinal. A family $\mathcal{T}$ of rank α is called *α-minimal* if for any sentence $\varphi \in \Sigma(T)$, $\mathrm{RS}(\mathcal{T}_\varphi) < \alpha$ or $\mathrm{RS}(\mathcal{T}_{\neg\varphi}) < \alpha$.

Proposition 1.7 [9]. (1) *A family $\mathcal{T}$ is 0-minimal if and only if $\mathcal{T}$ is a singleton.*

(2) *A family $\mathcal{T}$ is 1-minimal if and only if $\mathcal{T}$ is e-minimal.*

(3) *A family $\mathcal{T}$ is 2-minimal if and only if $\mathcal{T}$ is a-minimal.*

(4) *For any ordinal α a family $\mathcal{T}$ is α-minimal if and only if* $\mathrm{RS}(\mathcal{T}) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}) = 1$.

Proposition 1.8 [9]. *For any family $\mathcal{T}$,* $\mathrm{RS}(\mathcal{T}) = \alpha$*, with* $\mathrm{ds}(\mathcal{T}) = n$*, if and only if $\mathcal{T}$ is represented as a disjoint union of subfamilies $\mathcal{T}_{\varphi_1}, \ldots, \mathcal{T}_{\varphi_n}$, for some pairwise inconsistent sentences $\varphi_1, \ldots, \varphi_n$, such that each $\mathcal{T}_{\varphi_i}$ is α-minimal.*

Similarly [12], for a nonempty family $\mathcal{T}$, we denote by $\mathcal{B}(\mathcal{T})$ the Boolean algebra consisting of all subfamilies $\mathcal{T}_\varphi$, where φ are sentences in the language $\Sigma(\mathcal{T})$.

Theorem 1.9 [9, 12]. *A nonempty family $\mathcal{T}$ is e-totally transcendental if and only if the Boolean algebra $\mathcal{B}(\mathcal{T})$ is superatomic.*

Proposition 1.10 [9]. *If an infinite family $\mathcal{T}$ does not have e-minimal subfamilies $\mathcal{T}_\varphi$ then $\mathcal{T}$ is not e-totally transcendental.*

Recall the definition of the Cantor–Bendixson rank. It is defined on the elements of a topological space X by induction: $\mathrm{CB}_X(p) \geq 0$ for all $p \in X$; $\mathrm{CB}_X(p) \geq \alpha$ if and only if for any $\beta < \alpha$, p is an accumulation point of the points of CB_X-rank at least β. $\mathrm{CB}_X(p) = \alpha$ if and only if both $\mathrm{CB}_X(p) \geq \alpha$ and $\mathrm{CB}_X(p) \not\geq \alpha + 1$ hold; if such an ordinal α does not exist then $\mathrm{CB}_X(p) = \infty$. Isolated points of X are precisely those having rank 0, points of rank 1 are those which are isolated in the subspace of all non-isolated points, and so on. For a non-empty $C \subseteq X$ we define $\mathrm{CB}_X(C) = \sup\{\mathrm{CB}_X(p) \mid p \in C\}$; in this way $\mathrm{CB}_X(X)$ is defined and $\mathrm{CB}_X(\{p\}) = \mathrm{CB}_X(p)$ holds. If X is compact and C is closed in X then the sup is achieved: $\mathrm{CB}_X(C)$ is the maximum value of $\mathrm{CB}_X(p)$ for $p \in C$; there are finitely many points of maximum rank in C and the number of such points is the CB_X-*degree* of C, denoted by $n_X(C)$.

If X is countable and compact then $\mathrm{CB}_X(X)$ is a countable ordinal and every closed subset has ordinal-valued rank and finite CB_X-degree $n_X(X) \in \omega \setminus \{0\}$.

For any ordinal α the set $\{p \in X \mid \mathrm{CB}_X(p) \geq \alpha\}$ is called the α-th CB-*derivative* X_α of X.

Elements $p \in X$ with $\mathrm{CB}_X(p) = \infty$ form the *perfect kernel* X_∞ of X.

Clearly, $X_\alpha \supseteq X_{\alpha+1}$, $\alpha \in \mathrm{Ord}$, and $X_\infty = \bigcap\limits_{\alpha \in \mathrm{Ord}} X_\alpha$.

Similarly, for a nontrivial superatomic Boolean algebra $\mathcal{A}$ the characteristics $\mathrm{CB}_{\mathcal{A}}(A)$, $n_{\mathcal{A}}(A)$, and $\mathrm{CB}_{\mathcal{A}}(p)$, for $p \in A$, are defined [13] starting with atomic elements being isolated points. Following [13], $\mathrm{CB}_{\mathcal{A}}(A)$ and $n_{\mathcal{A}}(A)$ are called the *Cantor–Bendixson invariants*, or CB-*invariants* of $\mathcal{A}$.

Recall that by [13, Lemma 17.9], $\mathrm{CB}_{\mathcal{A}}(A) < |A|^+$ for any infinite $\mathcal{A}$, and the following theorem holds.

Theorem 1.11 [13, Theorem 17.11]. *Countable superatomic Boolean algebras are isomorphic if and only if they have the same* CB-*invariants.*

In view of Theorem 1.9 any e-totally transcendental family $\mathcal{T}$ defines a superatomic Boolean algebra $\mathcal{B}(\mathcal{T})$, and it is easy to observe step-by-step that $\mathrm{RS}(\mathcal{T}) = \mathrm{CB}_{\mathcal{B}(\mathcal{T})}(B(\mathcal{T}))$, $\mathrm{ds}(\mathcal{T}) = n_{\mathcal{B}(\mathcal{T})}(B(\mathcal{T}))$, i.e., the pair $(\mathrm{RS}(\mathcal{T}), \mathrm{ds}(\mathcal{T}))$ consists of CB-invariants for $\mathcal{B}(\mathcal{T})$.

In particular, by Theorem 1.11, for any countable e-totally transcendental family $\mathcal{T}$, $\mathcal{B}(\mathcal{T})$ is uniquely defined, up to isomorphism, by the pair $(\mathrm{RS}(\mathcal{T}), \mathrm{ds}(\mathcal{T}))$ of CB-invariants.

By the definition for any e-totally transcendental family $\mathcal{T}$ each theory $T \in \mathcal{T}$ obtains the CB-rank $\mathrm{CB}_{\mathcal{T}}(T)$ starting with $\mathcal{T}$-isolated points T_0, of $\mathrm{CB}_{\mathcal{T}}(T_0) = 0$. We will denote the values $\mathrm{CB}_{\mathcal{T}}(T)$ by $\mathrm{RS}_{\mathcal{T}}(T)$ as the rank for the point T in the topological space on $\mathcal{T}$ which is defined with respect to $\Sigma(\mathcal{T})$-sentences.

Definition [1]. If $\mathcal{T}$ is a family of theories and Φ is a set of sentences, then we put $\mathcal{T}_\Phi = \bigcap\limits_{\varphi \in \Phi} \mathcal{T}_\varphi$ and the set $\mathcal{T}_\Phi$ is called (*type-* or *diagram-*)*definable* (by the set Φ) with respect to $\mathcal{T}$, or (*diagram-*)$\mathcal{T}$-*definable*, or simply d-*definable.*

Let Φ be closed under conjunctions. A sentence $\varphi \in \mathcal{T}_\Phi$ is called $\mathcal{T}$-*isolating*, $\mathcal{T}$-*principal* or $\mathcal{T}$-*complete* for Φ, if $\mathcal{T}_\Phi = \mathcal{T}_\varphi$. A set Φ is called $\mathcal{T}$-*isolated* or $\mathcal{T}$-*principal* if Φ contains a sentence which is $\mathcal{T}$-principal for Φ.

As noticed in [1] finite unions of d-definable sets are again d-definable. Considering infinite unions $\mathcal{T}'$ of d-definable sets $\mathcal{T}_{\Phi_i}$, $i \in I$, we can represent them by sets of formulas with infinite disjunctions $\bigvee\limits_{i\in I}\varphi_i$, $\varphi_i\in\Phi_i$. We call these unions $\mathcal{T}'$ as d_∞-*definable* sets. Since all singletons $\{T\}\subseteq\mathcal{T}$ are d-definable, each subfamily $\mathcal{T}'\subseteq\mathcal{T}$ is d_∞-definable.

2. Algebras for subfamilies of theories

A noticed in [9], for any nonempty family $\mathcal{T}$ the set of all s-definable subfamilies $\mathcal{T}_\varphi$ form a Boolean algebra $\mathcal{B}_s(\mathcal{T})$, being Lindenbaum–Tarski algebra [14, 15], with the relation $\mathcal{T}_\varphi\subseteq\mathcal{T}_\psi$, $0=\mathcal{T}_\varphi$ for an inconsistent φ, $1=\mathcal{T}=\mathcal{T}_{\forall x(x\approx x)}$, and set-theoretic operations $\mathcal{T}_\varphi\cup\mathcal{T}_\psi=\mathcal{T}_{\varphi\vee\psi}$, $\mathcal{T}_\varphi\cap\mathcal{T}_\psi=\mathcal{T}_{\varphi\wedge\psi}$, $\overline{\mathcal{T}_\varphi}=\mathcal{T}_{\neg\varphi}$.

The algebra $\mathcal{B}_s(\mathcal{T})$ is isomorphic to the quotient $\mathcal{S}(\Sigma)/\equiv_{\mathcal{T}}$ of the algebra $\mathcal{S}(\Sigma)$ on the set of all Σ-sentences, with $\Sigma=\Sigma(\mathcal{T})$ and logical operations $\vee,\wedge,\neg$. This isomorphism is defined by the rule $\mathcal{T}_\varphi\mapsto\{\psi\in\mathcal{S}(\Sigma)\mid\psi\equiv_{\mathcal{T}}\varphi\}$, where $\psi\equiv_{\mathcal{T}}\varphi\Leftrightarrow\mathcal{T}_\psi=\mathcal{T}_\varphi$.

Since every distinct theories $T_1,T_2\in\mathcal{T}$ are separated by some disjoint neighbourhoods $\mathcal{T}_\varphi$ and $\mathcal{T}_{\neg\varphi}$, all atomic elements in $\mathcal{B}(\mathcal{T})$ are singletons. Therefore applying Theorem 1.3 we have the following:

Proposition 2.1. *For any nonempty E-closed family $\mathcal{T}$ the boolean algebra $\mathcal{B}_s(\mathcal{T})$ is atomic if and only if $\mathcal{T}$ has the least/minimal generating subfamily.*

Proof. Let $\mathcal{B}_s(\mathcal{T})$ be atomic. Consider the set of all atomic elements of $\mathcal{B}_s(\mathcal{T})$ which, as noticed above, are singletons and form the subfamily $\mathcal{T}_0$ of $\mathcal{T}$ consisting of all elements of these singletons. We assert that $\mathcal{T}_0$ generates $\mathcal{T}$. Indeed, by Proposition 1.1 it suffices to show that if $T\in\mathcal{T}\setminus\mathcal{T}_0$ then for each $\varphi\in T$, $(\mathcal{T}_0)_\varphi$ is infinite. Assuming on contrary that for some $\varphi\in T$, $(\mathcal{T}_0)_\varphi$ is finite. So we can separate T from $(\mathcal{T}_0)_\varphi$ by some sentence $\psi\vdash\varphi$: $T\in\mathcal{T}_\psi$ whereas $(\mathcal{T}_0)_\psi=\emptyset$. But $\mathcal{B}_s(\mathcal{T})$ is atomic, so there is a singleton $\{T_0\}=\mathcal{T}_\chi\subseteq\mathcal{T}_\psi$, with some $\chi\vdash\psi$. So we have both $T_0\in\mathcal{T}_\psi$ and $T_0\in(\mathcal{T}_0)_\psi$ contradicting $(\mathcal{T}_0)_\psi=\emptyset$.

Since $\mathcal{T}_0$ generates $\mathcal{T}$, by Theorem 1.3 the set $\mathcal{T}_0$ is the least/minimal generating subfamily of $\mathcal{T}$.

Now we assume that $\mathcal{T}$ has the least/minimal generating subfamily $\mathcal{T}_0$. Let $\mathcal{T}_\varphi\neq\emptyset$ containing a theory T. If $T\in\mathcal{T}_0$ then there is a singleton $\mathcal{T}_\psi=\{T\}$ which is an atomic element under $\mathcal{T}_\varphi$. If $T\in\mathcal{T}\setminus\mathcal{T}_0$ then by Proposition 1.1, $\mathcal{T}_\varphi$ contains infinitely many theories in $\mathcal{T}_0$. So again $\mathcal{T}_\varphi$ has an atomic element $\mathcal{T}_\psi\subseteq\mathcal{T}_\varphi$. □

Now we extend the algebra $\mathcal{B}_s(\mathcal{T})$ till the algebra $\mathcal{B}_d(\mathcal{T})$ of all d-definable subfamilies of $\mathcal{T}$. By Properties 4 and 8 of d-definable sets [1] the algebra $\mathcal{B}_d(\mathcal{T})$ preserves the operations $\cap$ and $\cup$, whereas complements are defined only for $\mathcal{T}$-principal d-definable sets. Therefore $\mathcal{B}_d(\mathcal{T})$ contains the operations $\cap$ and $\cup$, being a partial algebra with respect to $^-$. Thus, $\mathcal{B}_d(\mathcal{T})$ form a distributive lattice, partially, for $\mathcal{T}$-principal families, with complements. Besides, by Property 9 [1], each algebra $\mathcal{B}_d(\mathcal{T})$ is atomic containing all singletons $\{T\}\subseteq\mathcal{T}$. Clearly, these atomic elements $\{T\}$ have (co-atomic) complements if and only if they are $\mathcal{T}$-isolated.

The algebra $\mathcal{B}_d(\mathcal{T})$ admits a natural extension till the algebra $\mathcal{B}_{d_\infty}(\mathcal{T})$ of all d_∞-definable subfamilies of $\mathcal{T}$, i.e., of all subfamilies of $\mathcal{T}$.

Whereas structures $\mathcal{B}_s(\mathcal{T})$ and $\mathcal{B}_{d_\infty}(\mathcal{T})$ are well known, as well as $\mathcal{B}_d(\mathcal{T})$ for finite $\mathcal{T}$ and some additional special cases, it is natural to classify structures $\mathcal{B}_d(\mathcal{T})$ in general case.

Partially answering the classification question we consider totally transcendental families $\mathcal{T}$ of small ranks. Notice that the algebras $\mathcal{B}_s(\mathcal{T})$, for these families, are atomic in view of Theorem 2.1.

Proposition 2.2. *For any nonempty family $\mathcal{T}$ the following conditions are equivalent:*

(1) $\mathrm{RS}(\mathcal{T}) = 0$ *and* $\mathrm{ds}(\mathcal{T}) = n$*;*

(2) *the Boolean algebra $\mathcal{B}_s(\mathcal{T})$ consists of 2^n elements with n atoms generating this algebra;*

(3) *$\mathcal{B}_d(\mathcal{T})$ is a Boolean algebra consisting of 2^n elements with n atoms generating this algebra.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). It suffices to note that having $\mathrm{RS}(\mathcal{T}) = 0$ and $\mathrm{ds}(\mathcal{T}) = n$, $\mathcal{T}$ consists of some theories T_i, $i = 1, \ldots, n$, whose isolating sentences generate both $\mathcal{B}_s(\mathcal{T})$ and $\mathcal{B}_d(\mathcal{T})$ with atomic elements $\{T_i\}$.

(3) $\Rightarrow$ (2) is obvious since in such a case each d-definable set is s-definable.

(2) $\Rightarrow$ (1). Since $\mathcal{B}_s(\mathcal{T})$ is finite with n atomic elements and each atomic element is a singleton, we have $|\mathcal{T}| = n$ producing $\mathrm{RS}(\mathcal{T}) = 0$ and $\mathrm{ds}(\mathcal{T}) = n$. □

Theorem 2.3. *For any nonempty E-closed family $\mathcal{T}$ and $n \in \omega \setminus \{0\}$ the following conditions are equivalent:*

(1) $\mathrm{RS}(\mathcal{T}) = 1$ *and* $\mathrm{ds}(\mathcal{T}) = n$*;*

(2) *the Boolean algebra $\mathcal{B}_s(\mathcal{T})$ is isomorphic to a direct product of n infinite Boolean algebras $\mathcal{B}_1, \ldots, \mathcal{B}_n$ each of which is generated by atomic elements;*

(3) *the algebra $\mathcal{B}_d(\mathcal{T})$ contains n new atomic elements with respect to $\mathcal{B}_s(\mathcal{T})$.*

Proof. (1) $\Rightarrow$ (2). By the assumption $\mathcal{T}$ is divided into n disjoint e-minimal s-definable subfamilies $\mathcal{T}_{\varphi_1}, \ldots, \mathcal{T}_{\varphi_n}$. By e-minimality the Boolean algebras $\mathcal{B}_s(\mathcal{T}_{\varphi_i})$ are generated by atomic elements. Taking

$$\mathcal{B}_s = \mathcal{B}_s(\mathcal{T}_{\varphi_1}) \times \ldots \times \mathcal{B}_s(\mathcal{T}_{\varphi_n})$$

we obtain the Boolean algebra representing all s-definable subfamilies of $\mathcal{T}$ as boolean combinations of (co-)finite subfamilies of $\mathcal{T}_{\varphi_i}$. Since these boolean combinations also represent elements of $\mathcal{B}_s(\mathcal{T})$, we have a natural isomorphism between $\mathcal{B}_s$ and $\mathcal{B}_s(\mathcal{T})$.

(1) $\Rightarrow$ (3). Since $\mathcal{T}$ is divided into n disjoint e-minimal s-definable subfamilies $\mathcal{T}_{\varphi_1}, \ldots, \mathcal{T}_{\varphi_n}$ there are n accumulation points for $\mathcal{T}$ and each $\mathcal{T}_{\varphi_i}$ has exactly one of these accumulation points T_i. Each T is a d-definable atomic element of $\mathcal{B}_d(\mathcal{T})$ which does not belong to $\mathcal{B}_s(\mathcal{T})$. Since all singletons $\{T\}$, for $T \in \mathcal{T} \setminus \{T_1, \ldots, T_n\}$, belong to $\mathcal{B}_s(\mathcal{T})$, $\mathcal{B}_d(\mathcal{T})$ contains n new atomic generating elements with respect to $\mathcal{B}_s(\mathcal{T})$.

(2) $\Rightarrow$ (1). If $\mathcal{B}_s(\mathcal{T})$ is isomorphic to a direct product of n infinite Boolean algebras $\mathcal{B}_1, \ldots, \mathcal{B}_n$ each of which is generated by atomic elements then it has n infinite parts, disjoint modulo $\emptyset$, being Boolean algebras such that units of these algebras can be divided only into finite and cofinite parts. It means that there are pairwise inconsistent sentences φ such that these parts correspond e-minimal s-definable families $\mathcal{T}_\varphi$. Moreover, since $\mathcal{B}_s(\mathcal{T}) \simeq \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$, each s-definable family in $\mathcal{B}_s(\mathcal{T})$ is defined by some boolean combination of sentences φ and sentences ψ isolating atomic elements. It implies that $\mathcal{T}$ is divided into n disjoint e-minimal s-definable subfamilies of $\mathcal{T}$ producing $\mathrm{RS}(\mathcal{T}) = 1$ and $\mathrm{ds}(\mathcal{T}) = n$.

(3) $\Rightarrow$ (1). By Proposition 2.2, $\mathcal{B}_d(\mathcal{T})$ is infinite. Since it contains n new atomic elements with respect to $\mathcal{B}_s(\mathcal{T})$ there are n theories in $\mathcal{T}$ which are not isolated by sentences. Since each infinite family has an accumulation point, it implies that $\mathcal{T}$ is divided into n disjoint e-minimal s-definable parts producing $\mathrm{RS}(\mathcal{T}) = 1$ and $\mathrm{ds}(\mathcal{T}) = n$. $\square$

Remark 2.4. Each algebra $\mathcal{B}_i$ in Theorem 2.3 corresponds a 1-minimal family and it is isomorphic to a union of upward directed family of finite algebras $\mathcal{B}_s(\mathcal{T})$ in Proposition 2.2 such that the cardinality of this family equals $|\mathcal{B}_i|$. Algebras $\mathcal{B}_s(\mathcal{T})$ correspond finite subfamilies $\mathcal{T}$ of given family $\mathcal{T}'$ of theories such that all theories in $\mathcal{T}$ are $\mathcal{T}'$-isolated by some sentences. Here there are n theories in $\mathcal{T}'$ and outside all $\mathcal{T}$ being non-principal ultrafilters with respect to $\mathcal{T}'$.

Remark 2.5. Similarly Remark 2.4, the class of algebras $\mathcal{B}_s(\mathcal{T})$ in Theorem 2.3 contains upward directed families whose unions produce algebras $\mathcal{B}_s(\mathcal{T}_2)$ for a-minimal, i.e., 2-minimal families $\mathcal{T}_2$, with $\mathrm{RS}(\mathcal{T}_2) = 2$ and $\mathrm{ds}(\mathcal{T}_2) = 1$. Taking direct products of n algebras $\mathcal{B}_s(\mathcal{T}_2)$ we obtain algebras $\mathcal{B}_s(\mathcal{T}_{2,n})$ for disjoint unions $\mathcal{T}_{2,n}$ of 2-minimal families $\mathcal{T}_2$, with $\mathrm{RS}(\mathcal{T}_{2,n}) = 2$ and $\mathrm{ds}(\mathcal{T}_{2,n}) = n$, $n \in \omega \setminus \{0\}$.

Now we can continue the process alternating step-by-step unions of upward directed families of ranks $< \alpha$, obtaining *α-ranked* algebras $\mathcal{B}_s(\mathcal{T}_\alpha)$ for α-minimal families $\mathcal{T}_\alpha$, and direct products of n algebras $\mathcal{B}_s(\mathcal{T}_\alpha)$, for α-minimal families $\mathcal{T}_\alpha$ obtaining their disjoint union $\mathcal{T}_{\alpha,n}$ with $\mathrm{RS}(\mathcal{T}_{\alpha,n}) = \alpha$ and $\mathrm{ds}(\mathcal{T}_{\alpha,n}) = n$, $n \in \omega \setminus \{0\}$.

Here, each step from $\mathcal{T}_\beta$, $\beta < \alpha$, to $\mathcal{T}_{\alpha,n}$ produces n new atomic elements for $\mathcal{B}_d(\mathcal{T}_{\alpha,n})$ such that these new elements have the CB-rank α and represent non-principal ultrafilters with respect to elements of ranks $< \alpha$.

Collecting the arguments above we generalize Theorem 2.3 for arbitrary e-totally transcendental E-closed family $\mathcal{T}$:

Theorem 2.4. *For any nonempty E-closed family $\mathcal{T}$, an ordinal $\alpha \geq 1$, and $n \in \omega \setminus \{0\}$, the following conditions are equivalent:*

(1) $\mathrm{RS}(\mathcal{T}) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}) = n$*;*

(2) *the Boolean algebra $\mathcal{B}_s(\mathcal{T})$ is isomorphic to a direct product of n Boolean algebras $\mathcal{B}_1, \dots, \mathcal{B}_n$ each of which is generated by elements of ranks $< \alpha$ such that each $\mathcal{B}_n$ contains infinitely many elements of each rank $\beta < \alpha$;*

(3) *the algebra $\mathcal{B}_d(\mathcal{T})$ consists of infinitely many atomic elements of each rank $\beta < \alpha$, for $\beta \geq 0$, and exactly n atomic elements of rank α, these n atomic elements correspond non-principal ultrafilters with respect to elements of ranks $< \alpha$.*

In conclusion, we notice that atomic elements in the item (3) of Theorem 2.4, being theories of rank α, can be considered as elements of the greatest complexity both in the family $\mathcal{T}$ and in the algebra $\mathcal{B}_d(\mathcal{T})$.

References

[1] N. D. Markhabatov, S. V. Sudoplatov, *Definable families of theories, related calculi and ranks*, Novosibirsk, arXiv:1901.11011v1 [math.LO], 2019.

[2] S. V. Sudoplatov, *Combinations of structures*, The Bulletin of Irkutsk State University. Series "Mathematics", **24** (2018), 65–84. MR3831622

[3] S. V. Sudoplatov, *Closures and generating sets related to combinations of structures*, Reports of Irkutsk State University. Series "Mathematics", **16** (2016), 131–144. Zbl 1402.03039

[4] S. V. Sudoplatov, *Families of language uniform theories and their generating sets*, Reports of Irkutsk State University. Series "Mathematics", **16** (2016), 62–76. Zbl 1362.03031

[5] S. V. Sudoplatov, *Combinations related to classes of finite and countably categorical structures and their theories*, Siberian Electronic Mathematical Reports, **14** (2017), 135–150. MR3622899

[6] S. V. Sudoplatov, *Relative e-spectra and relative closures for families of theories*, Siberian Electronic Mathematical Reports, **14** (2017), 296–307. MR3633262

[7] S. V. Sudoplatov, *On semilattices and lattices for families of theories*, Siberian Electronic Mathematical Reports, **14** (2017), 980–985. MR3711758

[8] S. V. Sudoplatov, *Approximations of theories*, arXiv:1901.08961v1 [math.LO], 2019.

[9] S. V. Sudoplatov, *Ranks for families of theories and their spectra*, arXiv:1901.08464v1 [math.LO], 2019.

[10] N. D. Markhabatov, S. V. Sudoplatov, *Ranks for families of all theories of given languages*, arXiv:1901.09903v1 [math.LO], 2019.

[11] S. Feferman, R. Vaught, *The first order properties of products of algebraic systems*, Fund. Math., **47** (1959), 57–103. MR0108455

[12] M. Morley, *Categoricity in Power*, Transactions of the American Mathematical Society, **114**:2 (1965), 514–538. MR0175782

[13] S. Koppelberg, *Handbook of Boolean Algebras. Vol.* 1, eds. J. D. Monk, R. Bonnet, Amsterdam: North-Holland, 1989. MR991565

[14] W. J. Blok, D. Pigozzi, *Algebraizable logics*, Memoirs of the AMS, **77**:396 (1989). MR973361

[15] P. Hinman, *Fundamentals of Mathematical Logic*, Massachusets: Wellesley, A K Peters, 2005. MR2179857

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