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UNIQUE DETERMINATION OF CONFORMAL TYPE
FOR DOMAINS

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ABSTRACT. This paper is the first part of a cycle of three articles, which is a survey devoted to the discussion of problems of the unique determination of conformal type for domains in Euclidean spaces. The main goal of the survey is to present a new apparently, very interesting and yet very difficult trend in the classical geometric topic of the unique determination of convex surfaces by their intrinsic metrics. This (the first) article of the cycle relies upon the author's talk "Unique Determination of Polyhedral Domains and p -Moduli of Path Families" given at the International Conference "Metric Geometry of Surfaces and Polyhedra" (Moscow, August 2010) dedicated to the 100th anniversary of Prof. N. V. Efimov, and the article itself is an extended version of this talk. Note that the author developed problems of the unique determination of conformal type for domains in Euclidean spaces in earlier papers. In the present article, we expose new results on the problem of the unique determination of conformal type for domains in \mathbb{R}^n . In particular, we show that a (generally speaking) nonconvex bounded polyhedral domain in \mathbb{R}^n ($n \geq 4$) whose boundary is an $(n - 1)$ -dimensional connected manifold of class C^0 without boundary and is representable as a finite union of pairwise nonoverlapping $(n - 1)$ -dimensional cells is uniquely determined by the relative conformal moduli of its boundary condensers.

Results on the unique determination (of polyhedral domains) of isometric type are also obtained. In contrast to the classical case, these results present a new approach in which the notion of the p -modulus of path families is used.

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1. INTRODUCTION

In development of the classical topic of the unique determination of closed convex surfaces by their intrinsic metrics [1], in [2]-[4], the author started a search for a complete description of the boundary values of conformal mappings of domains in the space $\bar{\mathbb{R}}^n$ (by $\bar{\mathbb{R}}^n$ we denote the one-point compactification $\mathbb{R}^n \cup \{\infty\}$ of the real Euclidean space \mathbb{R}^n). This search is based on the notion of the n -modulus of a family of curves first introduced in [5], which plays a very important role in various domains of mathematics. In particular, using the notion of the n -modulus of a family of curves, one can obtain the following characterization of conformal mappings [6] (see also [7], [8]): A homeomorphism $f : U \rightarrow V$ of domains U and V in \mathbb{R}^n ($n \geq 2$) is conformal if and only if every family of curves Γ in U satisfies the condition

$$(1.1) \quad M_n(f(\Gamma)) = M_n(\Gamma),$$

where $f(\Gamma) = \{f(\gamma) : \gamma \in \Gamma\}$ (in other words, a mapping f is conformal if and only if it is 1-quasiconformal).

Further, suppose that the boundaries $\text{fr } U$ and $\text{fr } V$ of the domains U and V are sufficiently regular (e.g., they are bounded and are Lipschitz manifolds of dimension $n - 1$ without boundary). Then any quasiconformal mapping of these domains can be extended to a homeomorphism $H : \text{cl } U \rightarrow \text{cl } V$ of their closures $\text{cl } U$ and $\text{cl } V$ [6]; moreover, in the case of conformal mappings, the extension H satisfies (1.1) as well. In particular, (1.1) holds for the n -moduli of the families Γ of paths joining in U the components F_1 and F_2 of the boundary condensers $F = \{F_1, F_2\}$ of the domain U (in this case, $f = H|_{\text{fr } U}$ in (1.1)). This gives rise to the natural question: Are domains U and V conformally equivalent if the boundary of one of them can be mapped onto the boundary of the other by means of a homeomorphism $f : \text{fr } U \rightarrow \text{fr } V$ preserving the n -moduli of the families of paths joining the components of boundary condensers in the domain U ?

In [2]-[4], we gave a positive answer to this question in the case of convex domains. Namely, therein, we proved the following

Theorem 1.1. *If $n \geq 4$ then any bounded convex polyhedral domain $U \subset \mathbb{R}^n$ (i.e., a nonempty bounded intersection of finitely many open n -dimensional half-spaces) is uniquely determined by the relative conformal moduli of its boundary condensers in the class $\mathcal{P} = \mathcal{P}(n)$ of all bounded convex polyhedral domains $V \subset \mathbb{R}^n$.*

This paper continues the study of problems of the unique determination of conformal type initiated in [2]-[4] by Theorem 1.1. First, we briefly recall the notions from [2]-[4] used in Theorem 1.1 that we will need also in the sequel.

Let $U \subset \mathbb{R}^n$ ($U \neq \mathbb{R}^n$) be a domain in \mathbb{R}^n for which $\text{fr } U$ is a Lipschitz $(n - 1)$ -manifold without boundary. A boundary condenser $F = \{F_1, F_2\}$ of U is a pair of disjoint closed subsets F_1 and F_2 of the boundary $\text{fr } U$ of this domain (at least one of which is bounded). A relative conformal modulus $M^U(F)$ of a boundary condenser F of the domain U is by definition the n -modulus

$$(1.2) \quad M_n(\Gamma_{F_1, F_2, U}) = \inf_{\rho \in \mathcal{R}(\Gamma_{F_1, F_2, U})} \int_{\mathbb{R}^n} [\rho(x)]^n dx$$

of the family $\Gamma_{F_1, F_2, U}$ of all continuous paths $\gamma : [0, 1] \rightarrow \text{cl}U$, where $\text{cl}U$ denotes the closure of the domain U , such that $\gamma(0) \in F_1$, $\gamma(1) \in F_2$, and $\gamma(t) \in U$ for $0 < t < 1$ (in (1.2), $\mathcal{R}(\Gamma_{F_1, F_2, U})$ is the set of all nonnegative Borel measurable functions $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$, where $\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}$ is the two-point compactification of the real line $\mathbb{R} = \mathbb{R}^1$, satisfying the condition $\int_\gamma \rho ds \geq 1$ for every rectifiable path $\gamma \in \Gamma_{F_1, F_2, U}$).

Let $\mathcal{L}_0 = \mathcal{L}_0(n)$ be a subclass in the class $\mathcal{L} = \mathcal{L}(n)$ of all domains U in \mathbb{R}^n with $n \geq 3$ different from \mathbb{R}^n and such that the boundary of each of these domains is a Lipschitz $(n-1)$ -manifold without boundary. Following [2]-[4], we say that a domain $U \in \mathcal{L}_0$ is uniquely determined by the relative conformal moduli of its boundary condensers in the class \mathcal{L}_0 if the following conditions hold. Suppose that $V \in \mathcal{L}_0$ and there exists a homeomorphism $f : \text{fr} V \rightarrow \text{fr} U$ of the boundary $\text{fr} V$ of the domain V to the boundary $\text{fr} U$ of U preserving the relative conformal moduli of the boundary condensers, i.e., such that $M^V(F) = M^U(f(F))$ (where $f(F) = \{f(F_1), f(F_2)\}$) for each boundary condenser F of V . Then V can be mapped conformally onto U .

In connection with Theorem 1.1, there arises the question of whether the convexity condition in its statement is substantial. Among the main results of this paper, there is Theorem 2.1 below, which make it possible to waive the convexity condition in Theorem 1.1.

The second part of the article is devoted to a complete description of the boundary values of isometric mappings of n -dimensional domains in terms of the p -moduli of path families. In this connection, we now briefly recall some facts of the theory of quasi-isometric mappings that we will need below.

Definition 1.1. Let $K \in [1, \infty[$. A homeomorphism $f : U_1 \rightarrow U_2$ of domains U_1 and U_2 in \mathbb{R}^n is called K -quasi-isometric if

$$K^{-1} \leq \liminf_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|} \leq \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|} \leq K$$

for any $x \in U_1$. A homeomorphism $f : U_1 \rightarrow U_2$ is called quasi-isometric if it is K -quasi-isometric for some $K \in [1, \infty[$.

We have (see [9])

Theorem 1.2. Suppose that $f : U_1 \rightarrow U_2$ is a K -quasi-isometric homeomorphism of bounded domains U_1 and U_2 in \mathbb{R}^n , where $n \geq 2$ ($1 \leq K < \infty$). Then

$$(1.3) \quad K^{2-p-n} M_p(\Gamma) \leq M_p(f(\Gamma)) \leq K^{p+n-2} M_p(\Gamma)$$

for every $p \in]1, \infty[$ and any family Γ of paths γ such that $\text{Im} \gamma \subset \text{cl}U_1$.

Remark 1.1. The quantity $M_p(\Gamma)$, where $1 \leq p < \infty$, is called the p -modulus of the path family Γ and defined by analogy with the conformal modulus $M_n(\Gamma)$ as

$$M_p(\Gamma) = \inf_{\rho \in \mathcal{R}(\Gamma)} \int_{\mathbb{R}^n} [\rho(x)]^p dx,$$

where $\mathcal{R}(\Gamma)$ is the set of all nonnegative Borel measurable functions $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\int_\gamma \rho ds \geq 1$ for every rectifiable path $\gamma \in \Gamma$.

It is also well known that if the boundaries of domains U_1 and U_2 are sufficiently regular (e.g., these domains belong to the class \mathcal{L}), then any K -quasi-isometric homeomorphism of these domains admits a natural extension to a K -quasi-isometric homeomorphism H of their closures $\text{cl}U_1$ and $\text{cl}U_2$ satisfying condition (1.3). In

particular, (1.3) holds for the p -moduli $M_p^{U_1}(F) = M_p^{U_1}(\{F_1, F_2\}) = M_p(\Gamma_{F_1, F_2, U_1})$ of the boundary condensers $F = \{F_1, F_2\}$ of the domain U_1 , and if $K = 1$ then inequalities (1.3) turn into the equality $M_p^{U_2}(f(F)) = M_p^{U_1}(F)$ (in this case, the mapping f in (1.3) coincides with the restriction $H|_{\text{fr}U}$ of H to the boundary of U).

These facts and Theorem 2.1 lead to the following question: Do there exist analogs of Theorem 2.1 characterizing the boundary values of isometric mappings in terms of the p -moduli of path families? In Sec. 3, we answer this question in the positive.

In what follows, for $x \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$, $\text{dist}(x, E) = \inf_{y \in E} |x - y|$, all paths $\gamma : [\alpha, \beta] \rightarrow \mathbb{R}^n$, where $\alpha, \beta \in \mathbb{R}$, are assumed continuous and non-constant, and $l(\gamma)$ means the length of a path γ .

2. UNIQUE DETERMINATION OF NONCONVEX POLYHEDRAL DOMAINS
BY THE RELATIVE CONFORMAL MODULI OF THEIR BOUNDARY CONDENSERS

Let $\mathcal{P}_1 = \mathcal{P}_1(n)$ be the class of all bounded domains U in \mathbb{R}^n satisfying the following conditions:

- (i) $\text{fr}U$ is an $(n - 1)$ -manifold of class C^0 without boundary;
- (ii) $\text{fr}U$ is representable as a finite union of pairwise nonoverlapping $(n - 1)$ -dimensional cells.

Remark 2.1. Recall (see, e.g., [10]) that a cell σ in \mathbb{R}^n is a nonempty closed bounded subset in \mathbb{R}^n which can be represented as a finite intersection of closed half-spaces. The plane $\Omega(\sigma)$ of a cell σ is the minimal affine subspace containing σ ; the dimension $\text{dim}(\sigma)$ of a cell σ coincides with that of the plane $\Omega(\sigma)$, and if this dimension equals r then σ is called an r -cell. If the dimensions of two cells σ_1 and σ_2 coincide and $\{\text{Int}(\sigma_1)\} \cap \{\text{Int}(\sigma_2)\} = \emptyset$ then we say that the cells σ_1 and σ_2 do not overlap. Here $\text{Int}(\sigma_j) = \sigma_j \setminus \partial\sigma_j$ is the open kernel of the cell σ_j ($j = 1, 2$), and $\partial\sigma_j$ denotes the boundary of the cell σ_j treated as a subset of the plane $\Omega(\sigma_j)$.

The first main result of the article is the following

Theorem 2.1. Suppose that $n \geq 4$. Then every domain U in \mathbb{R}^n belonging to the class \mathcal{P}_1 and having connected boundary is uniquely determined in this class by the relative conformal moduli of its boundary condensers. Moreover, U can be determined in the class \mathcal{P}_1 up to an additional affine conformal transformation (i.e., a similarity transformation) $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Remark 2.2. It should be mentioned in relation to Theorem 2.1 that the boundary of any domain of class \mathcal{P}_1 is a Lipschitz $(n - 1)$ -manifold without boundary (this follows directly from the definition of the class \mathcal{P}_1).

To prove Theorem 2.1, we first need to introduce a number of notions and remind some assertions of auxiliary nature from author's article [3].

We will begin with the notion of an $(n - 1)$ -face of the boundary of a domain U from the class \mathcal{P}_1 .

Let Ξ be a collection of pairwise nonoverlapping cells of dimension $n - 1$ whose union coincides with the boundary of the domain U , and suppose that a hyperplane τ contains at least one cell from Ξ . We say that δ is an $(n - 1)$ -face of the boundary

$\text{fr } U$ of U contained in τ if it is a maximal union

$$(2.1) \quad \delta = \bigcup_{s=1}^k \xi_s$$

of those cells in Ξ that (i) are contained in Ξ , moreover, (ii) the interior $\text{Int}_{\text{fr } U} \delta$ of this union (calculated in the interior metric of the boundary $\text{fr } U$ of U) is a connected set. The maximality of the union (2.1) means that it is impossible to add (at least) one more cell from Ξ to this union with the preservation of properties (i) and (ii). Clearly, the just-introduced definition of $(n - 1)$ -face of the boundary $\text{fr } U$ is correct and $\text{fr } U$ itself is a (uniquely defined) union of pairwise nonoverlapping $(n - 1)$ -faces. Moreover, since the boundary of U is an $(n - 1)$ -dimensional manifold of class C^0 without boundary, we have the following assertions:

Lemma 2.1. *If δ is an $(n - 1)$ -face of the boundary of U contained in a hyperplane τ then the set $\text{Int } \delta$ ($= \text{Int}_{\text{fr } U} \delta$) has an open neighborhood D such that $D \cup \mathbb{R}_1^n \subset U$ and $D \cup \mathbb{R}_2^n \subset \mathbb{R}^n \setminus U$, where \mathbb{R}_1^n and \mathbb{R}_2^n are open half-spaces satisfying the conditions $\mathbb{R}_1^n \cap \mathbb{R}_2^n = \emptyset$ and $\mathbb{R}_1^n \cup \mathbb{R}_2^n = \mathbb{R}^n \setminus \tau$.*

Lemma 2.2. *Let $\{\delta_j\}_{j=1, \dots, k}$ be a proper subset of the set of all $(n - 1)$ -faces of the boundary $\text{fr } U$. Then the boundary $\varkappa = \text{fr}_{\text{fr } U} (\bigcup_{j=1}^l \delta_j)$ of the union $\bigcup_{j=1}^l \delta_j$ with respect to $\text{fr } U$ is a nonempty subset of the union of boundaries of the cells $\xi \in \Xi$ and so \varkappa is the union of a finite set $\Theta = \Theta(\varkappa)$ of pairwise nonoverlapping $(n - 2)$ -dimensional cells v .*

Furthermore, if $x \in \text{Int } v$ ($v \in \Theta$) then the contingency $\text{contg}_U x$ of U at x is the set $\tilde{V}_\alpha = \text{cl}(P(V_\alpha))$, where P is a similarity transformation and

$$(2.2) \quad V_\alpha = \{x = (x_1, x_2, \dots, x_{n-2}, x_{n-1}, x_n) \in \mathbb{R}^n : x_j \in \mathbb{R}, j = 1, 2, \dots, n - 2, \\ x_{n-1} = r \cos \theta, x_n = r \sin \theta, 0 < r < \infty, 0 < \theta < \alpha\},$$

$0 < \alpha < 2\pi$, $\alpha \neq \pi$, moreover, there exists a number $r = r_x > 0$ such that $B(x, r) \cap U = B(x, r) \cap \tilde{V}_\alpha$.

Remark 2.3. *The proofs of Lemmas 2.1 and 2.2 are rather simple. For this reason, we omit them.*

Lemma 2.3. *Let p_1, p_2 be points of the hyperplane $\tau_{n-1} = \{x \in \mathbb{R}^n : x_n = 0\}$ with $|p_j| = 1$ ($j = 1, 2$) and let F_1, F_2 be disjoint continua on τ_{n-1} such that F_1 is bounded and contains the points 0 and p_1 , whereas F_2 is unbounded and contains p_2 . Then*

$$(2.3) \quad M^{\mathbb{R}_+^n}(\{F_1, F_2\}) \geq \lambda_n = \frac{(n - 1)v_{n-1} \log 3}{16} \left(\frac{[\Gamma(\frac{1}{2(n-1)})]^2}{\Gamma(\frac{1}{n-1})} \right)^{1-n},$$

where $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$, v_n is the volume of the n -dimensional unit ball, and Γ is the Euler gamma-function.

Remark 2.4. *Lemma 2.3 goes back to Theorem 3.10 in [11] and the results of [12] and [13]. By Theorem 3.10 in [11], for example,*

$$(2.4) \quad M^{\mathbb{R}^3 \setminus \{F_1 \cup F_2\}}(\{F_1, F_2\}) \geq \lambda_3,$$

where F_1 and F_2 are disjoint continua in \mathbb{R}^3 such that F_1 is bounded and contains the points 0 and p_1 , and F_2 is unbounded and contains p_2 , moreover, $|p_j| = 1$,

$j = 1, 2$. In [11], a way was chosen that made it possible to obtain estimate (2.4) by rather rough but straightforward calculations. Using the same calculations, the author proved Lemma 2.3 in [3] (see Lemma 8.1 in [3]).

Lemma 2.4. *The relative conformal modulus $M(A_t) = M_{\mathbb{R}_+^n}(A_t)$ of the boundary condenser A_t ($0 < t < \infty$) of the half-space \mathbb{R}_+^n whose components are the segments*

$$(2.5) \quad F_1 = F_1(t) = \{x \in \mathbb{R}^n : |x_1| \leq \frac{t}{2}, x_2 = -\frac{1}{2}, x_j = 0, j = 3, 4, \dots, n\}$$

and

$$(2.6) \quad F_2 = F_2(t) = \{x \in \mathbb{R}^n : |x_1| \leq \frac{t}{2}, x_2 = \frac{1}{2}, x_j = 0, j = 3, 4, \dots, n\}$$

has the following properties: (i) $0 < M(A_t) < \infty$; (ii) $M(A_t) \rightarrow 0$ as $t \rightarrow 0$ and $M(A_t) \rightarrow \infty$ as $t \rightarrow \infty$; (iii) $M(A_t)$ is an increasing function (in the wide sense) of the parameter $t : M(A_{t_1}) \leq M(A_{t_2})$ if $t_1 < t_2$; (iv) if $0 < t_1 < t_2 < \infty$ then $t_2^{-1}M(A_{t_2}) \leq t_1^{-1}M(A_{t_1})$.

Remark 2.5. *The condenser A_t was first considered (in the case $n = 3$) in [14] in connection with study of the boundary values of quasiconformal mappings of domains in \mathbb{R}^3 : by Lemma 3.6 in [14], the function*

$$\zeta_3(t) = t^{-1}M^{\mathbb{R}^3 \setminus \{F_1(t) \cup F_2(t)\}}(\{F_1(t), F_2(t)\}), \quad 0 < t < \infty,$$

decreases (in the wide sense: $\zeta_3(t_2) \leq \zeta_3(t_1)$ if $0 < t_1 < t_2 < \infty$), moreover, $\zeta_3(t) \rightarrow \alpha \in]0, 1[$ as $t \rightarrow \infty$. In contrast to [14], here (as well as in [3]), we consider A_t as a boundary condenser of the half-space \mathbb{R}_+^n and use its properties from Lemma 2.4, whose proof can be found in [3] (see Lemma 8.2 in [3]). Below, we use the notion of a growth point t_n of the function $M^{\mathbb{R}_+^n}(A_t)$ which is introduced in [3] as follows: t_n is a point in $]0, \infty[$ such that

$$(2.7) \quad M^{\mathbb{R}_+^n}(A_t) > M^{\mathbb{R}_+^n}(A_{t_n})$$

for every $t > t_n$ (the existence of growth points for $M^{\mathbb{R}_+^n}(A_t)$ is immediate from Lemma 2.4. For definiteness, we will further assume that t_n is the least number $t_n^* \geq 1$ satisfying (2.7) if inserted instead of t_n . Clearly, this number is a growth point of $M^{\mathbb{R}_+^n}(A_t)$ too).

Lemma 2.5. *Assume that $0 < t < \infty$, $A_t = \{F_1, F_2\} = \{F_1(t), F_2(t)\}$ is the boundary condenser of the half-space \mathbb{R}_+^n whose components are the segments $F_1 = F_1(t)$ and $F_2 = F_2(t)$ defined by (2.5) and (2.6) and*

$$F_j^\tau = \{x \in \mathbb{R}^n : \text{dist}(x, F_j) \leq \tau, x_n \geq 0\}, \quad 0 < \tau < 1/2, \quad j = 1, 2.$$

Then

$$M(A_t) = \lim_{\tau \rightarrow 0} M_n(\Gamma(\tau)),$$

where $\Gamma(\tau) = \Gamma_{F_1^\tau, F_2^\tau, \{F_1^\tau \cup F_2^\tau\}}$ is the family of paths joining F_1^τ and F_2^τ in $\mathbb{R}_+^n \setminus \{F_1^\tau \cup F_2^\tau\}$.

Remark 2.6. *Lemma 2.5 goes back to Lemma 3.4 in [14] on the continuity of moduli, by which*

$$M(\Gamma) = \lim_{\tau \rightarrow 0} M_n(\Gamma(\tau)),$$

where Γ and $\Gamma(\tau)$ are the families of curves joining disjoint bounded continua E_1 and E_2 and $E_1(\tau)$ and $E_2(\tau)$, respectively in an open set $U \subset \mathbb{R}^3$ (here $E_j(\tau) =$

$\{x \in U : \text{dist}(x, E_j) \leq \tau\}$, $j = 1, 2$, moreover, τ is sufficiently small). The proof of Lemma 2.5 can be found in [3] (see Lemma 8.4 in [3]).

Lemma 2.6. *Assume that $n \geq 4$ and $0 < \alpha \leq 2\pi$. Put $F_1 = \{x \in \mathbb{R}^n : -1 \leq x_1 \leq 0, x_j = 0, j = 2, 3, \dots, n\}$ and $F_2 = \{x \in \mathbb{R}^n : 1 \leq x_1 \leq \infty, x_j = 0, j = 2, 3, \dots, n\}$. If Γ_α is the family of all paths connecting F_1 and F_2 in V_α then $M^{V_\alpha}(A) = M_n(\Gamma_\alpha) = \frac{\alpha}{\pi} M_n(\Gamma_\pi) = \frac{\alpha}{\pi} M^{V_\pi}(A)$, where $A = \{F_1, F_2\}$ is the boundary condenser of V_α with components F_1 and F_2 ; moreover, $V_\alpha = \{(x_1, x_2, \dots, x_{n-2}, x_{n-1}, x_n) \in \mathbb{R}^n : x_j \in \mathbb{R}, j = 1, 2, \dots, n-2, x_{n-1} = r \cos \theta, x_n = r \sin \theta, 0 < r < \infty, 0 < \theta < \alpha\}$, $0 < \alpha \leq 2\pi$.*

Remark 2.7. *Lemma 2.6 is a generalization of Lemma 7.1 in [14] (which deals with the case of $n = 3$) to $n \geq 4$ (see Lemma 8.7 in [3]).*

Proof of Theorem 2.1. Suppose that U_1 is as in the hypothesis of the theorem, U_2 is a domain of class \mathcal{P}_1 , and $f : \text{fr } U_1 \rightarrow \text{fr } U_2$ is a homeomorphism of the boundary $\text{fr } U_1$ of U_1 onto the boundary $\text{fr } U_2$ of U_2 preserving the relative conformal moduli of boundary condensers. It suffices to show that there exists a similarity transformation $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying the condition $U_2 = F(U_1)$ (and $f = F|_{\text{fr } U_1}$).

To this end, note first that since the boundary $\text{fr } U_1$ of U_1 is connected and f is a homeomorphism, the boundary $\text{fr } U_2$ of U_2 is also connected, and then consider an $(n - 1)$ -dimensional face s of $\text{fr } U_1$. Clearly, there exists an $(n - 1)$ -dimensional face \tilde{s} of $\text{fr } U_2$ such that

$$B(x, r) \cap f(s) = B(x, r) \cap \text{Int } \tilde{s} \neq \emptyset$$

for some $x \in \text{Int } \tilde{s}$ and $r > 0$.

Let $\tilde{\sigma}$ be a connected component of the set $\text{Int } \tilde{s} \cap \text{Int}_{\text{fr } U_2} f(s)$. Assume that $\sigma = f^{-1}(\tilde{\sigma})$. We assert that the restriction $f|_\sigma$ of f to σ is an $(n - 1)$ -dimensional conformal mapping.

Indeed, taking into account that the relative conformal modulus $M^U(F)$ of a boundary condenser F of U is a conformal invariant and using Lemma 2.1, applying additional conformal mappings if necessary, we can come to the following situation: s and \tilde{s} are subsets of the hyperplane $\tau_{n-1} = \{x \in \mathbb{R}^n : x_n = 0\}$, and the sets $\text{Int } s$ and $\text{Int } \tilde{s}$ have open neighborhoods D_1 and D_2 (respectively) such that $D_j \cap \mathbb{R}_+^n \subset U_j$ and $D_j \cap \mathbb{R}_-^n \subset \mathbb{R}^n \setminus U_j$, where $j = 1, 2$ and $\mathbb{R}_\pm^n = \mathbb{R}^n \setminus (\text{cl } \mathbb{R}_\mp^n)$. Suppose that $x_0 \in \sigma$, $k \in \{2, 3, \dots\}$, and r is a sufficiently small positive number and consider an $(n - 1)$ -dimensional ball $B_{n-1}(x_0, r) = \{x \in \mathbb{R}_n : |x - x_0| < r, x_n = 0\}$ in τ_{n-1} such that $\text{cl } B_{n-1}(x_0, r) \in \sigma$, $\text{cl } B_{n-1}(f(x_0), L) \in \tilde{\sigma}$ ($L = L(x_0, f, r) = \max_{|x-x_0|=r, x \in \tau_{n-1}} |f(x) - f(x_0)|$), $B_n^+(f(x_0), L) = \{y \in \mathbb{R}_+^n : |y - f(x_0)| < L, y_n > 0\} \subset U_2$, and

$$(2.8) \quad B_n^+(x_0, kL^*) = \{x \in \mathbb{R}^n : |x - x_0| < kL^*, x_n > 0\} \subset \{D_1 \cap \mathbb{R}_+^n\} \quad (\subset U_1),$$

where

$$(2.9) \quad \{B_n(x_0, kL^*) \cap \tau_{n-1}\} \subset \sigma$$

and $L^* = \max\{|x - x_0| : x \in E_1\}$, moreover, $E_1 = \{y \in \mathbb{R}^n : |y - f(x_0)| = L, y_n = 0\}$ (note that we can obtain (2.8) and (2.9) by using the continuity of f and the smallness of the values of r). The following estimate holds for the relative conformal modulus $M^{U_2}(\{E_1, E_2\})$ of the boundary condenser $\{E_1, E_2\}$ of U_2 , where $E_2 =$

$$\{y \in \mathbb{R}^n : |y - f(x_0)| = l, y_n = 0\} \quad (l = l(x_0, f, r) = \min_{|x-x_0|=r, x \in \tau_{n-1}} |f(x) - f(x_0)|):$$

$$(2.10) \quad nv_n \left(\log \frac{L}{l} \right)^{1-n} \geq M^{U_2}(\{E_1, E_2\}).$$

This stems from the fact that the family $\Gamma_{S_L, S_l, A}$ of all paths connecting the spheres $S_L = \{y \in \mathbb{R}^n : |y - f(x_0)| = L\}$ and $S_l = \{y \in \mathbb{R}^n : |y - f(x_0)| = l\}$ in the spherical ring $A = \{y \in \mathbb{R}^n : l < |y - f(x_0)| < L\}$ minorizes the family Γ_{E_1, E_2, U_2} (i.e., for each path $\gamma : [\alpha, \beta] \rightarrow \mathbb{R}^n, \gamma \in \Gamma_{E_1, E_2, U_2}$, there exists a segment $[\varkappa, \delta] \subset [\alpha, \beta]$) such that $\gamma|_{[\alpha, \beta]} \in \Gamma_{S_L, S_l, A}$, and from assertions 6.4 and 7.5 in [6].

Now, estimate $M^{U_1}(\{f^{-1}(E_1), f^{-1}(E_2)\})$ (which is equal to $M^{U_2}(\{E_1, E_2\})$ since f preserves the relative conformal moduli of boundary condensers) from below. To this end, note first that

$$M^{U_1}(\{f^{-1}(E_1), f^{-1}(E_2)\}) \geq M_n(\Gamma_k),$$

where Γ_k is the subfamily of the family $\Gamma_{f^{-1}(E_1), f^{-1}(E_2), U_1}$ of all paths connecting the components of the boundary condenser $\{f^{-1}(E_1), f^{-1}(E_2)\}$ of the domain U_1 in this domain, which consists of the paths $\gamma \in \Gamma_{f^{-1}(E_1), f^{-1}(E_2), U_1}$ such that $\text{Im } \gamma \subset \text{cl } B_n^+(x_0, kL^*)$. Furthermore, consider the family $\Gamma_{f^{-1}(E_1), f^{-1}(E_2), \mathbb{R}_+^n}$ of paths connecting the components of the condenser $\{f^{-1}(E_1), f^{-1}(E_2)\}$ in \mathbb{R}_+^n , which is also a boundary condenser for the half-space \mathbb{R}_+^n . It is clear that

$$(2.11) \quad \Gamma_{f^{-1}(E_1), f^{-1}(E_2), \mathbb{R}_+^n} \subset \{\Gamma_k \cup \Gamma^k\},$$

where Γ^k is the subfamily of paths γ in $\Gamma_{f^{-1}(E_1), f^{-1}(E_2), \mathbb{R}_+^n}$ satisfying the condition $\{\text{Im } \gamma \cap (\mathbb{R}^n \setminus B_n^+(x_0, kL^*))\} \neq \emptyset$. Moreover, since Γ^k is minorized by the family $\Gamma_{S(x_0, kL^*), S(x_0, L^*), A^*}$ of all paths connecting the boundary spheres $S(x_0, kL^*)$ and $S(x_0, L^*)$ of the spherical ring $A^* = \{x \in \mathbb{R}^n : L^* < |x - x_0| < kL^*\}$ in this ring, by the above-mentioned assertions 6.4 and 7.5 in [6], we have

$$(2.12) \quad M^n(\Gamma^k) \leq nv_n \left(\log \frac{kL^*}{L^*} \right)^{1-n} = nv_n (\log k)^{1-n} = \mu_k \rightarrow 0$$

as $k \rightarrow \infty$. On the other hand, (2.11) and (2.12) imply

$$M^{\mathbb{R}_+^n}(\{f^{-1}(E_1), f^{-1}(E_2)\}) \leq M_n(\Gamma_k) + M_n(\Gamma^k) \leq M_n(\Gamma_k) + \mu_k.$$

From these relations and Lemma 2.3 it follows that

$$(2.13) \quad M_n(\Gamma_k) \geq M^{\mathbb{R}_+^n}(\{f^{-1}(E_1), f^{-1}(E_2)\}) - \mu_k \geq \lambda_n - \mu_k,$$

where λ_n is from (2.3). Involving also the fact that $\lambda_n - \mu_k > 0$ when k is sufficiently large, and reckoning with (2.10) and (2.13), we easily obtain the relation

$$(2.14) \quad \frac{L}{l} = \frac{L(x_0, f, r)}{l(x_0, f, r)} \leq \exp \left\{ \left(\frac{nv_n}{\lambda_n - \mu_k} \right)^{\frac{1}{n-1}} \right\}.$$

Passing to the limit first as $r \rightarrow 0$ and then as $k \rightarrow \infty$ in (2.14), we get

$$(2.15) \quad H(x, f) = \limsup_{r \rightarrow 0} \frac{L(x, f, r)}{l(x, f, r)} \leq \Lambda_n = \exp \left\{ \left(\frac{8nv_n}{(n-1)v_{n-1} \log 3} \right)^{\frac{1}{n-1}} \frac{[\Gamma(\frac{1}{2(n-1)})]^2}{\Gamma(\frac{1}{n-1})} \right\}$$

for $x \in \sigma$. From (2.15) we see that $f|_\sigma$ is an $(n - 1)$ -dimensional quasiconformal mapping (moreover, for the same reasons, the inverse mapping $(f|_\sigma)^{-1}$ is also quasiconformal). Following the proof of Theorem 8.1 in [3] and making necessary corrections to it concerned with the specific nature of the general case discussed in Theorem 2.1, we will now show that $f|_\sigma$ is a conformal mapping.

Indeed, assume that $x_0 \in \sigma$ is a nondegenerate differentiability point of $f|_\sigma$, i.e., x_0 is a point at which f is differentiable, moreover, the value $J(x_0, f)$ of its Jacobian at x_0 is nonzero (by the just-proven quasiconformality of $f|_\sigma$, $\text{mes}_{(n-1)}$ -almost all points $x \in \sigma$ have this property; here $\text{mes}_{(n-1)}$ is the $(n - 1)$ -dimensional Lebesgue measure), and suppose that the differential $f'(x_0)$ is not a conformal mapping. Consider points $e_1, e_2 \in \tau_{n-1}$ such that $|e_j| = 1$ ($j = 1, 2$) and $|f'(x_0)e_1| = \max_e |f'(x_0)e| > |f'(x_0)e_2| = \min_e |f'(x_0)e|$, where the maximum and minimum are calculated over the set of all vectors $e \in \tau_{n-1}$ with $|e| = 1$. By the conformal invariance of the relative conformal moduli of boundary condensers, we can assume that e_1, e_2, \dots, e_n is the canonical basis in \mathbb{R}^n ; $s, \tilde{s} \subset \tau_{n-1}$ (as above, s and \tilde{s} are $(n - 1)$ -dimensional faces of the boundaries $\text{fr} U_1$ and $\text{fr} U_2$ of the polyhedrons $\text{cl} U_1$ and $\text{cl} U_2$ containing σ and $\tilde{\sigma} = f(\sigma)$ respectively); $x_0 = f(x_0) = 0$; $B_n^+(0, \sqrt{1 + t_n^2}) (= B_n(0, \sqrt{1 + t_n^2}) \cap \mathbb{R}_+^n) \subset D_1 \cap \mathbb{R}_+^n \subset U_1, \{B_n(0, \sqrt{1 + t_n^2}) \cap \tau_{n-1}\} \subset \sigma$; $B_n^+(0, \sqrt{1 + (\Lambda_n t_n)^2}) \subset D_2 \cap \mathbb{R}_+^n \subset U_2, \{B_n(0, \sqrt{1 + (\Lambda_n t_n)^2}) \cap \tau_{n-1}\} \subset \tilde{\sigma}, f'(0)e_2 = e_2$, and $f'(0)e_1 = ue_1$ ($1 < u \leq \Lambda_n$). Here Λ_n is defined by (2.15) and t_n is a growth point (see Remark 2.5) of the function $t \mapsto M^{\mathbb{R}_+^n}(A_t), 0 < t < \infty$, where A_t is the boundary condenser with components (2.5) and (2.6).

Starting from this situation, consider the parameter $\mu = 2, 3, \dots$ and the boundary condenser

$$\mu^{-1}A_{t_n} = \{F_1^\mu, F_2^\mu\} = \{\mu^{-1}F_1, \mu^{-1}F_2\} = \{\{x \in \mathbb{R}^n : \mu x \in F_1\}, \{x \in \mathbb{R}^n : \mu x \in F_2\}\}$$

of the half-space \mathbb{R}_+^n , where F_j ($j = 1, 2$) are the components of the boundary condenser A_{t_n} of \mathbb{R}_+^n defined by (2.5) and (2.6) for $t = t_n$, and then construct the mapping $f_\mu : \text{fr} U_1 \rightarrow \text{fr} U_2$ by setting $f_\mu(x) = \mu f(\mu^{-1}x)$ where $x \in \text{fr}(\mu U_1)$. By the nondegenerate differentiability of $f|_\sigma$ (and hence that of the inverse mapping $(f|_\sigma)^{-1}$) at 0, we have

$$(2.16) \quad f_\mu(x) = Lx + |x|\alpha(\mu^{-1}x), \quad x \in \text{fr}(\mu U_1),$$

and

$$(2.17) \quad f_\mu^{-1}(x) (= \mu f^{-1}(\mu^{-1}x)) = L^{-1}x + |x|\beta(\mu^{-1}x), \quad x \in \text{fr}(\mu U_2).$$

Note that, in (2.16) and (2.17), $L = f'(0)$ is the derivative (differential) of the mapping f at the point 0,

$$(2.18) \quad L\left(\frac{\tau e_1 \pm e_2}{2}\right) = \frac{u\tau e_1 \pm e_2}{2}, \quad \tau \in \mathbb{R},$$

and

$$(2.19) \quad \lim_{x \rightarrow 0, x \in \tau_{n-1}} \alpha(x) = 0, \quad \lim_{x \rightarrow 0, x \in \tau_{n-1}} \beta(x) = 0;$$

moreover, the mappings α and β are independent of μ .

Consider the boundary condenser $f_\mu^{-1}(A_{ut_n})$ of \mathbb{R}_+^n with the components $f_\mu^{-1}(F_j(ut_n))$ ($j = 1, 2$), where F_j are defined by (2.5) and (2.6). For this condenser, we have

$$(2.20) \quad \Gamma_{f_\mu^{-1}(F_1(ut_n)), f_\mu^{-1}(F_2(ut_n)), \mu U_1} \subset \{\Gamma_{f_\mu^{-1}(F_1(ut_n)), f_\mu^{-1}(F_2(ut_n)), \mu B_n^+(0, \sqrt{1+t_n^2})} \cup \Gamma_\mu^*\},$$

where Γ_μ^* is the subfamily of paths in $\Gamma_{f_\mu^{-1}(F_1(ut_n)), f_\mu^{-1}(F_2(ut_n)), \mu U_1}$ such that $\text{Im } \gamma \cap (\mathbb{R}^n \setminus \mu B_n(0, \sqrt{1+t_n^2})) \neq \emptyset$; moreover, from the fact that Γ_μ^* is minorized by the family of all paths connecting the boundary spheres of the spherical ring $\{x \in \mathbb{R}^n : \sqrt{1+t_n^2} < |x| < \mu\sqrt{1+t_n^2}\}$ in this ring we (by the same arguments as those used to deduce (2.12)) obtain the inequality

$$(2.21) \quad M_n(\Gamma_\mu^*) \leq \frac{nv_n}{(\log \mu)^{n-1}}.$$

On the other hand, we can easily verify that

$$(2.22) \quad \Gamma_{f_\mu^{-1}(F_1(ut_n)), f_\mu^{-1}(F_2(ut_n)), \mu B_n^+(0, \sqrt{1+t_n^2})} \subset \Gamma_{f_\mu^{-1}(F_1(ut_n)), f_\mu^{-1}(F_2(ut_n)), \mathbb{R}_+^n}.$$

Therefore (by (2.20)-(2.22)),

$$(2.23) \quad M^{\mu U_1}(f^{-1}(A_{ut_n})) \leq M^{\mathbb{R}_+^n}(f^{-1}(A_{ut_n})) + \frac{nv_n}{(\log \mu)^{n-1}}.$$

Now,

$$(2.24) \quad \Gamma_{F_1(ut_n), F_2(ut_n), \mathbb{R}_+^n} \subset \{\Gamma_{F_1(ut_n), F_2(ut_n), \mu B_n^+(0, \sqrt{1+(\Lambda_n t_n)^2})} \cup \bar{\Gamma}_\mu\},$$

where $\bar{\Gamma}_\mu$ is the subset of paths $\Gamma_{F_1(ut_n), F_2(ut_n), \mathbb{R}_+^n}$ satisfying the condition $\text{Im } \gamma \subset \{\mathbb{R}^n \setminus (B_n(0, \sqrt{1+(\Lambda_n t_n)^2}))\} \neq \emptyset$. Since $\bar{\Gamma}_\mu$ is minorized by the family of all paths connecting the boundary spheres of the spherical ring $\{x \in \mathbb{R}^n : \sqrt{1+(\Lambda_n t_n)^2} < |x| < \mu\sqrt{1+(\Lambda_n t_n)^2}\}$ in this ring, it follows that, by repeating the arguments used in deriving (2.12) and (2.21), we get the estimate

$$(2.25) \quad M_n(\bar{\Gamma}_\mu) \leq \frac{nv_n}{(\log \mu)^{n-1}}.$$

Using the obvious relation

$$\Gamma_{F_1(ut_n), F_2(ut_n), \mu\sqrt{1+(\Lambda_n t_n)^2}} \subset \Gamma_{F_1(ut_n), F_2(ut_n), \mu U_2},$$

and (2.24) and (2.25), we have

$$M^{\mathbb{R}_+^n}(A_{ut_n}) \leq M^{U_2}(A_{ut_n}) + \frac{nv_n}{(\log \mu)^{n-1}}.$$

Involving also (2.23) and the circumstance that the mapping f_μ (together with f) preserves the relative conformal moduli of boundary condensers, we have

$$(2.26) \quad M^{\mathbb{R}_+^n}(A_{ut_n}) \leq M^{U_1}(f^{-1}(A_{ut_n})) + \frac{nv_n}{(\log \mu)^{n-1}} \\ \leq M^{\mathbb{R}_+^n}(f^{-1}(A_{ut_n})) + 2\frac{nv_n}{(\log \mu)^{n-1}}.$$

The proof of the conformality of $f|_\sigma$ is finished by analogy with that of Theorem 8.1 in [3]. We will only confine ourselves to its brief exposition. First, starting from (2.16)-(2.19), we arrive at the estimate

$$(2.27) \quad M^{\mathbb{R}_+^n}(f^{-1}(A_{ut_n})) \leq M_n(\Gamma(\beta_\mu^*)),$$

where

$$(2.28) \quad \beta_\mu^* = \frac{\sqrt{1 + (\Lambda_n t_n)^2}}{2} \left\{ \sup_{|y| \leq \frac{\sqrt{1 + (\Lambda_n t_n)^2}}{2\mu}} |\beta(y)| \right\} \rightarrow 0$$

as $\mu \rightarrow \infty$ (in what follows, μ is so large that $\beta_\mu^* < 1/2$) and $\Gamma(\tau) = \Gamma_{F_1^\tau, F_2^\tau, \mathbb{R}_+^n \setminus \{F_1^\tau \cup F_2^\tau\}}$ is the family of all paths connecting F_1^τ and F_2^τ in $\mathbb{R}_+^n \setminus \{F_1^\tau \cup F_2^\tau\}$. Here

$$F_j^\tau = \{x \in \mathbb{R}^n : \text{dist}(x, F_j) \leq \tau, \quad x_n \geq 0\},$$

$0 < \tau < 1/2$ ($j = 1, 2$), moreover, the sets F_j are the components of the boundary condenser $A(t)$ of \mathbb{R}_+^n defined by (2.5) and (2.6).

Finally, combining (2.26) and (2.27), we arrive at the inequalities

$$(2.29) \quad M^{\mathbb{R}_+^n}(A_{ut_n}) - \frac{2nv_n}{(\log \mu)^{n-1}} \leq M_n(\Gamma(\beta_\mu^*)),$$

and then, letting μ tend to ∞ , apply Lemma 2.5 to the right-hand side in (2.29). In particular, this lemma implies the equality

$$M^{\mathbb{R}_+^n}(A_{ut_n}) = \lim_{\tau \rightarrow 0} M_n(\Gamma(\tau)).$$

Now, involving (2.28), we obtain the inequality

$$M^{\mathbb{R}_+^n}(A_{ut_n}) \leq M^{\mathbb{R}_+^n}(A_{t_n})$$

which contradicts the fact that t_n is a growth point of the function $t \mapsto M^{\mathbb{R}_+^n}(A_t)$, $0 < t < \infty$. Therefore, $f|_\sigma$ is a conformal mapping.

Note also that since σ is a connected component of the set $(\text{Int}_{\text{fr } U_1} f^{-1}(\tilde{s})) \cap \text{Int } s$, $\tilde{\sigma} = f(\sigma)$ and f^{-1} preserves the relative conformal moduli of boundary condensers, $f^{-1}|_\sigma$ is also conformal.

The next step in proving the theorem is the proof of the equality $\text{cl } \sigma = s$.

To this end, assume that $s \setminus \text{cl } \sigma \neq \emptyset$ and then consider a point $x_0 \in \{(\text{fr}_s \sigma) \cap (\text{Int } s)\}$. The image $y_0 = f(x_0)$ of this point belongs to the set $\partial \tilde{s}$, moreover, by the continuity of f^{-1} , there exists an n -dimensional ball $B(y_0, r)$ such that $f^{-1}(B(y_0, r) \cap \partial \tilde{s}) \subset \{(\text{fr}_s \sigma) \cap (\text{Int } s)\}$. In the set $B(y_0, r) \cap \partial \tilde{s}$, there is a point y_0^* belonging to the interior $\text{Int } v$ of a certain $(n - 2)$ -dimensional cell $v \in \Theta_2$, where Θ_2 is a finite set (chosen and fixed a priori) of pairwise nonoverlapping $(n - 2)$ -dimensional cells whose union is the boundary $\varkappa = \text{fr}_{\text{fr } U_2} \tilde{s}$ of the face \tilde{s} (see Lemma 2.2). Let $x_0^* = f^{-1}(y_0^*)$. Basing on the conformal invariance of the relative conformal moduli of boundary condensers, assume that x_0^* is just the initially-chosen point x_0 and, what is more, $x_0 = f(x_0) = 0$; $s, \tilde{s} \subset \tau_{n-1}$; the set $\text{Int } \tilde{s}$ has

an open neighborhood D_2 such that $(D_2 \cap \mathbb{R}_+^n) \subset U_2$ and $(D_2 \cap \mathbb{R}_-^n) \subset \mathbb{R}^n \setminus U_2$, and $v \subset \{x \in \mathbb{R}^n : x_{n-1} = x_n = 0\}$. Moreover, since the condition $n \geq 4$ and the well-known properties of space conformal mappings imply that the mapping $f|_\sigma$ is a restriction to σ of a certain Möbius mapping $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we can also assume that $\tilde{\sigma} = \sigma$ and $f|_\sigma = \text{Id} \sigma$.

Further, suppose that a number $r_0 > 0$ satisfies the condition $\{B(0, r_0) \cap V_\pi\} \subset \{D_1 \cap \mathbb{R}_+^n\} \subset U_1$ and condition (2.10), where now $x_0 = 0$ and $r = r_0$, and let $\text{cont}_{U_2} 0 = V_{\alpha_2}$ ($\alpha_2 \in]0, \pi[\cup]\pi, 2\pi[$); moreover, V_α is the domain defined for $\alpha \in]0, 2\pi[$ by (2.2). Setting $r_0 = 2$ (which is possible because of the conformal invariance of the relative conformal moduli of boundary condensers), construct the sequence $\{f_\mu\}_{\mu=2,3,\dots}$ of the mappings $f_\mu : \text{fr}(\mu U_1) \rightarrow \text{fr}(\mu U_2)$, where (as above) $f_\mu(x) = \mu f(\mu^{-1}x)$, $x \in \text{fr}(\mu U_1)$. The mapping f_μ has the following properties:

$$(2.30) \quad B(0, 2\mu) \cap v \subset \mu v,$$

$$(2.31) \quad \{B(0, 2\mu) \cap V_\pi\} \subset \mu U_1,$$

$$(2.32) \quad \{B(0, 2\mu) \cap V_{\alpha_2}\} \subset \mu U_2$$

and

$$(2.33) \quad f_\mu|_{\mu\sigma} = \text{Id}(\mu\sigma).$$

Starting from the mapping f_μ and taking into account (2.30)-(2.33), for the boundary condenser A_μ of the domains μU_j ($j = 1, 2$) whose components are the sets

$$F_1^\mu = \{x \in \mathbb{R}^n : -1 \leq x_1 \leq 0, x_\nu = 0, \nu = 2, 3, \dots, n\}$$

and

$$F_2^\mu = \{x \in \mathbb{R}^n : 1 \leq x_1 \leq \mu, x_\nu = 0, \nu = 2, 3, \dots, n\},$$

we now obtain the relations

$$(2.34) \quad \begin{aligned} M^{V_\pi}(A) &\leq M^{\mu U_1}(A_\mu) + M_n(\Gamma_\mu) + M_n(A_\mu^*, \mathbb{R}^n) \\ (M^{V_{\alpha_2}}(A) &\leq M^{\mu U_2}(A_\mu) + M_n(\Gamma_\mu) + M_n(A_\mu^*, \mathbb{R}^n)), \end{aligned}$$

where A is the boundary condenser of the domains V_π and V_{α_2} (defined above by (2.2)) with the components

$$(2.35) \quad F_1 = \{x \in \mathbb{R}^n : -1 \leq x_1 \leq 0, x_\nu = 0, \nu = 2, 3, \dots, n\}$$

and

$$(2.36) \quad F_2 = \{x \in \mathbb{R}^n : 1 \leq x_1 \leq \infty, x_\nu = 0, \nu = 2, 3, \dots, n\},$$

Γ_μ is the family of paths connecting of F_1 and F_2 in $\mathbb{R}^n \setminus \{F_1 \cup F_2\}$ and such that $\text{Im} \gamma \cap \{\mathbb{R}^n \setminus B(0, 2\mu)\} \neq \emptyset$; finally, A_μ^* is the condenser in \mathbb{R}^n whose components are the sets F_1^μ and

$$F_2^{\mu*} = \{x \in \mathbb{R}^n : \mu \leq x_1 < \infty, x_\nu = 0, \nu = 2, 3, \dots, n\},$$

moreover,

$$M_n(A_\mu^*, \mathbb{R}^n) = M_n(\Gamma_{F_1^\mu, F_2^{\mu*}, \mathbb{R}^n \setminus \{F_1^\mu \cup F_2^{\mu*}\}}).$$

Indeed, (2.30)-(2.33) imply the relations

$$\begin{aligned} \Gamma_{F_1, F_2, V_\pi} &\subset \Gamma_{F_1^\mu, F_2^\mu, V_\pi} \cup \Gamma_{F_1^\mu, F_2^{\mu*}, \mathbb{R}^n \setminus \{F_1^\mu \cup F_2^{\mu*}\}} \\ (\Gamma_{F_1, F_2, V_{\alpha_2}} &\subset \Gamma_{F_1^\mu, F_2^\mu, V_{\alpha_2}} \cup \Gamma_{F_1^\mu, F_2^{\mu*}, \mathbb{R}^n \setminus \{F_1^\mu \cup F_2^{\mu*}\}}) \end{aligned}$$

and

$$\begin{aligned} \Gamma_{F_1^\mu, F_2^\mu, V_\pi} &\subset \Gamma_{F_1^\mu, F_2^\mu, \mu U_1} \cup \Gamma_\mu \\ (\Gamma_{F_1^\mu, F_2^\mu, V_{\alpha_2}} &\subset \Gamma_{F_1^\mu, F_2^\mu, \mu U_2} \cup \Gamma_\mu). \end{aligned}$$

Thus, by Theorem 6.2 in [6], we have

$$\begin{aligned} M^{V_\pi}(A) &\leq M^{V_\pi}(A_\mu) + M_n(A_\mu^*, \mathbb{R}^n) \leq M^{\mu U_1}(A_\mu) + M_n(\Gamma_\mu) + M_n(A_\mu^*, \mathbb{R}^n) \\ (M^{V_{\alpha_2}}(A) &\leq M^{V_{\alpha_2}}(A_\mu) + M_n(A_\mu^*, \mathbb{R}^n) \leq M^{\mu U_2}(A_\mu) + M_n(\Gamma_\mu) + M_n(A_\mu^*, \mathbb{R}^n). \end{aligned}$$

Taking into account that the families $\Gamma_{F_1^\mu, F_2^\mu, \mathbb{R}^n \setminus \{F_1^\mu \cup F_2^\mu\}}$ and Γ_μ are minorized by the families $\Gamma_{S_1, S_2, A'_\mu}$ and $\Gamma_{S_1, S'_2, A''_\mu}$ of paths connecting the boundary spheres $S_1 = S_{n-1}(0, 1)$ and $S'_2 = S_{n-1}(0, \mu)$ in the spherical ring $A'_\mu = \{x \in \mathbb{R}^n : 1 < |x| < \mu\}$ and the boundary spheres S_1 and $S''_2 = S_{n-1}(0, 2\mu)$ in the spherical ring $A''_\mu = \{x \in \mathbb{R}^n : 1 < |x| < 2\mu\}$, respectively, by Theorems 6.2, 6.4 and 7.5 in [6], we obtain

$$(2.37) \quad M_n(A_\mu^*, \mathbb{R}^n) \leq nv_n(\log \mu)^{1-n}$$

and

$$(2.38) \quad M_n(\Gamma_\mu) \leq nv_n\{\log(2\mu)\}^{1-n} < nv_n(\log \mu)^{1-n}.$$

Inequalities (2.34), (2.37) and (2.38) imply the relations

$$(2.39) \quad \begin{aligned} M^{V_\pi}(A) &\leq M^{\mu U_1}(A_\mu) + 2nv_n(\log \mu)^{1-n} \\ (M^{V_{\alpha_2}}(A) &\leq M^{\mu U_2}(A_\mu) + 2nv_n(\log \mu)^{1-n}). \end{aligned}$$

On the other hand,

$$\begin{aligned} \Gamma_{F_1^\mu, F_2^\mu, \mu U_1} &\subset \Gamma_{F_1^\mu, F_2^\mu, B(0, 2\mu) \cap V_\pi} \cup \Gamma^\mu \subset \Gamma_{F_1^\mu, F_2^\mu, V_\pi} \cup \Gamma^\mu \\ (\Gamma_{F_1^\mu, F_2^\mu, \mu U_2} &\subset \Gamma_{F_1^\mu, F_2^\mu, B(0, 2\mu) \cap V_{\alpha_2}} \cup \Gamma^\mu \subset \Gamma_{F_1^\mu, F_2^\mu, V_{\alpha_2}} \cup \Gamma^\mu), \end{aligned}$$

where $\Gamma_{F_1^\mu, F_2^\mu, B(0, 2\mu) \cap V_\pi}$ ($\Gamma_{F_1^\mu, F_2^\mu, B(0, 2\mu) \cap V_{\alpha_2}}$) is the subfamily of paths in $\Gamma_{F_1^\mu, F_2^\mu, \mu U_1}$ ($\Gamma_{F_1^\mu, F_2^\mu, \mu U_2}$) whose images are in the ball $B(0, 2\mu)$ (note that here we have reckoned with (2.31) ((2.32))), and Γ^μ is the subfamily of all paths γ in the same family such that $\text{Im } \gamma \cap \{\mathbb{R}^n \setminus B(0, 2\mu)\} \neq \emptyset$; moreover, just as Γ_μ , the family Γ^μ is minorized by $\Gamma_{S_1, S'_2, A'_\mu}$. Hence,

$$(2.40) \quad M^{\mu U_1}(A_\mu) \leq M^{V_\pi}(A_\mu) + nv_n\{\log(2\mu)\}^{1-n} \leq M^{V_\pi}(A) + nv_n(\log \mu)^{1-n}.$$

Combining (2.39) and (2.40), we finally prove that

$$M^{V_\pi}(A) - 2nv_n(\log \mu)^{1-n} \leq M^{\mu U_1}(A_\mu) \leq M^{V_\pi}(A) + nv_n(\log \mu)^{1-n},$$

which in turn implies the relation

$$(2.41) \quad \lim_{\mu \rightarrow \infty} M^{\mu U_1}(A_\mu) = M^{V_\pi}(A).$$

Similar arguments also enable us to obtain the inequalities

$$M^{V_{\alpha_2}}(A) - 2nv_n(\log \mu)^{1-n} \leq M^{\mu U_2}(A_\mu) \leq M^{V_{\alpha_2}}(A) + nv_n(\log \mu)^{1-n},$$

which imply the equality

$$(2.42) \quad \lim_{\mu \rightarrow \infty} M^{\mu U_2}(A_\mu) = M^{V_{\alpha_2}}(A).$$

Next, the fact that f_μ (together with f) preserves the relative conformal moduli of boundary condensers imply the equality

$$(2.43) \quad M^{\mu U_1}(A_\mu) = M^{\mu U_2}(A_\mu).$$

Thus, by (2.41)-(2.43),

$$M^{V_{\alpha_2}}(A) = M^{V_{\pi}}(A).$$

At the same time, Lemma 2.6 and the condition $\alpha_2 \in (]0, \pi[\cup]\pi, 2\pi[)$ imply the inequality

$$(0 <) M^{V_{\alpha_2}}(A) \neq M^{V_{\pi}}(A).$$

The so-obtained contradiction completes the proof of the equality $\text{cl } \sigma = s$. It should be noted that, taking f^{-1} instead of f in the above-mentioned arguments, we also establish the equality $\text{cl } \tilde{\sigma} = \tilde{s}$. Hence, f generates a bijection between the sets of all $(n - 1)$ -dimensional faces of the boundaries $\text{fr } U_1$ and $\text{fr } U_2$ of U_1 and U_2 .

Turning to the final step in the proof of the theorem, choose an arbitrary $(n - 1)$ -dimensional face s_1 of the boundary $\text{fr } U_1$ of the polyhedron $\text{cl } U_1$. As above, we may assume that $s_1 \in \tau_{n-1}$, $f|_{s_1} = \text{Id } s_1$, $(D_j \cap \mathbb{R}_+^n) \subset U_j$ and $(D_j \cap \mathbb{R}_+^n) \subset \mathbb{R}^n \setminus U_j$ ($j = 1, 2$) (D_j are the open neighborhoods of the faces s_1 and $\tilde{s}_1 = f(s_1)$ defined for the domains U_j by Lemma 2.1). Let s_2 be an $(n - 1)$ -dimensional face of the boundary $\text{fr } U_1$ such that the intersection $s_1 \cap s_2$ of s_1 and s_2 contains an $(n - 2)$ -dimensional cell v_0 (from an a priori fixed finite set Θ of pairwise nonoverlapping $(n - 2)$ -dimensional cells whose union is $\text{fr}_{\text{fr } U_1} s_1$ (see Lemma 2.2)). We assert that $f(s_2) = s_2$. Indeed, since $f|_{\text{Int } s_2}$ is a conformal mapping of the $(n - 1)$ -dimensional domain $\text{Int } s_2$ onto the $(n - 1)$ -dimensional domain $f(\text{Int } s_2)$, the condition $n \geq 4$ and the properties of space conformal mappings imply that $f|_{s_2} = h|_{s_2}$, where $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Möbius transformation. Taking into account the relation $f|_{v_0} = \text{Id } v_0$, we conclude that h is an isometric mapping of \mathbb{R}^n . Let $x_0 \in \text{Int } v_0$. Repeating the arguments used above for proving the equality $\text{cl } \sigma = s$ almost verbatim and applying Lemmas 2.2 and 2.6, we obtain the equality $\text{contg}_{U_1} x_0 = \text{contg}_{U_2} x_0$ ($= \text{contg}_{U_2} f(x_0)$), from which (and what was said above) we have the desired equality $f(s_2) = s_2$.

Continuing these arguments by induction, say, at l th step, we will either establish the conformal equivalence of the domains U_1 and U_2 or obtain the following situation: there exists a proper subset $\{s_{\nu} : \nu = 1, 2, \dots, l\}$ of the set of all $(n - 1)$ -dimensional faces of the boundary $\text{fr } U_1$ such that

$$f|_{\cup_{\nu=1}^l s_{\nu}} = (P \circ L)|_{\cup_{\nu=1}^l s_{\nu}},$$

where $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry and L is a Möbius transformation. Show that, in the so-obtained situation, we can make at least one more step.

Indeed, consider the set $\{s_{\nu} : \nu = l + 1, l + 2, \dots, m\}$ of all remaining $(n - 1)$ -dimensional faces of the boundary $\text{fr } U_1$. Applying to this set Lemma 2.2 and comparing it with the set $\{s_{\nu} : \nu = 1, 2, \dots, l\}$, it is easy to conclude that there are faces $s_{\nu_1} \in \{s_{\nu} : \nu = 1, 2, \dots, l\}$ and $s_{\nu_2} \in \{s_{\nu} : \nu = l + 1, l + 2, \dots, m\}$ such that their intersection contains an $(n - 2)$ -dimensional cell v_0 . As a result, for the pair of $(n - 1)$ -dimensional faces s_{ν_1} and s_{ν_2} , we find ourselves in the situation described above (at the first step) for s_1 and s_2 . Therefore, it is not difficult to conclude that

$$f|_{\cup_{\nu=1}^{l+1} s_{\nu}} = (P \circ L)|_{\cup_{\nu=1}^{l+1} s_{\nu}},$$

where now $s_{l+1} = s_{\nu_2}$. Continuing our arguments by induction and taking into account the finiteness of the set of all $(n - 1)$ -dimensional faces of the boundary $\text{fr } U_1$ of the domain U_1 , we finally obtain the conformal equivalence of U_1 and U_2 .

The existence of a similarity transformation $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying the condition $U_2 = P(U_1)$ can be established in the same way as in the proof of Theorem 8.1 in [3].

Namely, if $H : U_1 \rightarrow \mathbb{R}^n$ is a conformal mapping from U_1 into \mathbb{R}^n that is not the restriction to U_1 of a similarity transformation then the image $\tilde{H}(s)$ of at least one of the $(n - 1)$ -dimensional faces s of the boundary $\text{fr } U_1$ under a conformal mapping $\tilde{H} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $H = \tilde{H}|_{U_1}$ is a subset of a certain sphere $S_{n-1}(x, r)$, $x \in \mathbb{R}^n$, $r \in \mathbb{R}_+$. But this is impossible because $H(U_1) \in \mathcal{P}_1$. Thus, the theorem is completely proved. \square

Remark 2.8. *If the components of a boundary condenser F of a domain U are connected then this condenser is a ring (in the sense of [6]). It is well known that, in this case, the relative conformal modulus of the condenser F is equal to its relative conformal capacity $\text{cap}^U(F) = \text{cap}_n^U(F)$, i.e., its n -capacity with respect to the domain U (see, e.g., [16] and [15] for the definition of the p -capacity of a ring and its very close relationship with the theory of quasiconformal and quasi-isometric (bi-Lipschitz) mappings). The proof of Theorems 2.1 use only ring-shaped boundary condensers. This allows us to reformulate Theorem 2.1 as follows:*

Theorem 2.2. *If $n \geq 4$ then any domain U in \mathbb{R}^n belonging to the class $\mathcal{P}_1(n)$ and having connected boundary is uniquely determined in this class by the relative conformal capacities of its ring-shaped boundary condensers.*

3. BOUNDARY VALUES OF ISOMETRIC MAPPINGS AND THE p -MODULI OF PATH FAMILIES

The facts of the theory of quasi-isometric mappings stated in the Sec. 1 and Theorem 2.1 lead to the following question: Do there exist analogs of Theorem 2.1 characterizing the boundary values of isometric mappings in terms of p -moduli of path families? At present, we can give the following answer to this question (corollary of Theorems 2.1):

Corollary 3.1. *Let $n \geq 4$. Suppose that U_1 and U_2 are bounded domains of class \mathcal{P}_1 having connected boundaries for which there exist a homeomorphism $f : \text{fr } U_1 \rightarrow \text{fr } U_2$ of the boundaries of these domains and a number $p \in \{1, n[\cup]n, \infty\}$ such that the following conditions hold: f preserves both the relative n -moduli and the relative p -moduli of boundary condensers. Then there exists an isometry $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying the condition $H(U_1) = U_2$.*

Proof. The proof of the corollary is based on Theorem 2.1 and Lemma 3.1 which will be formulated immediately after the proof of Corollary 3.1.

The hypothesis of the corollary and Theorem 2.1 imply the existence of an affine conformal mapping $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $U_2 = H(U_1)$ and $H|_{\text{fr } U_1} = f$. Nevertheless, if H is not an isometry then it has the form $H(x) = \varkappa\Omega x + \nu$ ($x \in \mathbb{R}^n$), where $0 < \varkappa \neq 1$, ν is a fixed point in \mathbb{R}^n , and Ω is an orthogonal mapping. By Lemma 3.1, there exists a ring-shaped boundary condenser F of U_1 satisfying the condition $0 < M_p^{U_1}(F) < \infty$. Furthermore, Theorem 8.2 in [6] immediately implies the following assertion: *If $\varkappa > 0$ and $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an affine conformal mapping, i.e., a mapping defined by the relation $G(x) = \varkappa\Omega x + \nu$ (Ω is as above an orthogonal mapping) then $M_p(G(\Gamma)) = \varkappa^{n-p} M_p(\Gamma)$ for any path family Γ .* Using this assertion and considerations from Sec. 1, we get the relation

$$(3.1) \quad M_p^{U_2}(f(F)) = M_p^{U_2}(H(F)) = \varkappa^{n-p} M_p^{U_1}(F)$$

which is a contradiction to the facts that f preserves the relative p -moduli of boundary ring-shaped condensers.

The corollary is proved. □

Lemma 3.1. *Assume that a domain U is bounded and belongs to the class \mathcal{P}_1 . Then there exists a ring-shaped boundary condenser F of U such that $0 < M_p^U(F) < \infty$ for every $p \in [1, \infty[$.*

Proof. Consider a ball $B = B(x_0, r)$ satisfying the condition $\text{cl } B(x_0, r) \subset U$ and then the set $A = T \cap U$, where $T = \{x \in \mathbb{R}^n : \sum_{\nu=1}^{n-1} (x_\nu - x_{0\nu})^2 < r^2\}$. Let Γ be the family of all paths $\gamma : [0, 1] \rightarrow \text{cl } A$ such that

$$\gamma(t) = \sum_{\nu=1}^{n-1} \alpha_\nu(\gamma) e_\nu + \{b_n(\gamma)t + a_n(\gamma)(1-t)\} e_n,$$

where $a_n(\gamma) < b_n(\gamma)$; $\gamma(0), \gamma(1) \in \text{fr } U$; $\gamma(t) \in U$ if $t \in]0, 1[$; finally, $B \cap \text{Im } \gamma \neq \emptyset$. Starting from Γ , construct the boundary condenser $F = \{F_1, F_2\}$ by setting $F_1 = \text{cl}\{\sum_{\nu=1}^{n-1} \alpha_\nu(\gamma) e_\nu + a_n(\gamma) e_n : \gamma \in \Gamma\}$ and $F_2 = \text{cl}\{\sum_{\nu=1}^{n-1} \alpha_\nu(\gamma) e_\nu + b_n(\gamma) e_n : \gamma \in \Gamma\}$. Clearly,

$$(3.2) \quad \inf_{\gamma \in \Gamma_{F_1, F_2, U}} l(\gamma) = \lambda > 0.$$

Recalling also that $U \in \mathcal{P}_1$, it is easy to verify the existence of a cylinder $T^* = \{x \in \mathbb{R}^n : \sum_{\nu=1}^{n-1} (x_\nu - x_{*\nu})^2 < r_*^2\}$ satisfying the following conditions: (i) $\text{cl } T^* \subset T$ and (ii) $F_j^* = F_j \cap \text{cl } T^*$ is a subset of a certain hyperplane τ_j ($j = 1, 2$). We assert that the ring-shaped boundary condenser $F^* = \{F_1^*, F_2^*\}$ is a desired one.

Indeed, by (3.2) and Theorem 7.1 in [6],

$$M_p^U(F^*) \leq \frac{\text{mes } U}{\lambda^p} < \infty.$$

On the other hand, the boundedness of U implies the existence of numbers a_n^*, b_n^* ($0 < a_n^* < b_n^* < \infty$), having the following properties: for every path $\gamma^* : [0, 1] \rightarrow \mathbb{R}^n$ of the form

$$(3.3) \quad \gamma^*(t) = \sum_{\nu=1}^{n-1} \alpha_\nu e_\nu + \{b_n^* t + a_n^*(1-t)\} e_n, \quad t \in [0, 1],$$

where $\sum_{\nu=1}^{n-1} (\alpha_\nu - x_{*\nu})^2 < r_*^2$, there exist numbers t_1, t_2 ($0 < t_1 < t_2 < 1$) such that $\gamma^*|_{[t_1, t_2]} \in \Gamma_{F_1^*, F_2^*, U}$.

The family $\Gamma_{F_1^*, F_2^*, U}$ minorizes the family Γ^* of all paths having the form (3.3). Hence, assertions 6.4 and 7.2 in [6] imply the relations

$$M_p^U(F^*) \geq M_p(\Gamma^*) = \frac{r_*^{n-1} v_{n-1}}{(b_n^* - a_n^*)^{p-1}} > 0.$$

The lemma is proved. □

To conclude the paper, we note that the main results were announced in [9], [17], [18], [19].

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