

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 16, стр. 886–892 (2019)

DOI 10.33048/semi.2019.16.058

УДК 519.72

MSC 94B60

ON TRANSITIVE UNIFORM PARTITIONS OF F^n INTO
BINARY HAMMING CODES

F.I. SOLOV'EVA

ABSTRACT. We investigate transitive uniform partitions of the vector space F^n of dimension n over the Galois field $GF(2)$ into cosets of Hamming codes. A partition $P^n = \{H_0, H_1 + e_1, \dots, H_n + e_n\}$ of F^n into cosets of Hamming codes H_0, H_1, \dots, H_n of length n is said to be uniform if the intersection of any two codes H_i and H_j , $i, j \in \{0, 1, \dots, n\}$ is constant, here e_i is a binary vector in F^n of weight 1 with one in the i th coordinate position. For any $n = 2^m - 1$, $m > 4$ we found a class of nonequivalent 2-transitive uniform partitions of F^n into cosets of Hamming codes.

Keywords: Hamming code, partition, uniform partition into Hamming codes, transitive partition, 2-transitive partition, Reed – Muller code, dual code.

1. INTRODUCTION

By F^n we denote the vector space of dimension n over the Galois field $GF(2)$ with respect to the Hamming metric. In this short correspondence using a recursive construction we prove the existence of nonequivalent 2-transitive uniform partitions of F^n into cosets of Hamming codes for any length $n = 2^m - 1$, $m > 4$.

The *Hamming distance* $d(x, y)$ between any two vectors $x, y \in F^n$ is defined as the number of coordinates in which x and y differ. The *Hamming weight* $w(x)$ of a vector x is $d(x, \mathbf{0}^n)$, where $\mathbf{0}^n$ is the all-zero vector of length n . A *code* of length n is a subset of F^n , its elements are called *codewords*. The *code distance* of a code is the minimum value of the Hamming distance between two different codewords from the code. A code C is called *perfect binary single-error-correcting* (briefly *perfect*)

SOLOV'EVA, F.I., ON TRANSITIVE UNIFORM PARTITIONS OF F^n INTO BINARY HAMMING CODES.

© 2019 SOLOV'EVA F.I.

The work is supported by RFBR (grant 19-01-00682).

Received September, 3, 2018, published 21, June, 2019.

if for any vector x from the set F^n there exists exactly one vector $y \in C$ at the Hamming distance not more than 1 from the vector x . A perfect linear code is called the *Hamming code*.

The automorphism group of a partition $P^n = \{C_0, C_1, \dots, C_n\}$ of length n of F^n into perfect codes C_0, C_1, \dots, C_n of length n , $\bigcup_{i=0}^n C_i = F^n$ is defined as the group of all isometries of F^n preserving the partition P^n . A partition P^n is called *transitive*, if for any two codes C_i and C_j , i, j from $I = \{0, 1, \dots, n\}$, there is an automorphism σ from $\text{Aut}(P^n)$ such that $\sigma(C_i) = C_j$. A partition P^n of F^n is defined to be *2-transitive*, if for any two subsets $\{i_1, i_2\}$ and $\{j_1, j_2\}$ of I there exists an automorphism σ from $\text{Aut}(P^n)$ such that $\sigma(C_{i_t}) = C_{j_t}$, $t = 1, 2$. By definition any 2-transitive partition is transitive. Let e_i be a binary vector in F^n of weight 1 with one in the i th coordinate position. A partition $P^n = \{H_0, H_1 + e_1, \dots, H_n + e_n\}$ of F^n into cosets of Hamming codes H_0, H_1, \dots, H_n of length n is said to be *uniform* if the intersection of any two codes H_i and H_j , $i, j \in I$ is constant, the number $\log \eta_n$ is called the *uniformity number*. Here and below \log stands for the binary logarithm. Note that transitive partitions are not necessarily uniform.

In [4] several classes of partitions of the space F^n into mutually nonparallel cosets of Hamming codes were presented and the lower bound on the number of nonequivalent partitions was given. The partitions in these constructions were not generally transitive. In [12] constructions of transitive, vertex-transitive and 2-transitive partitions of F^n into perfect codes and lower bounds on the number of nonequivalent such partitions were given. Note that the partitions presented in [4, 12] were not uniform. Uniform partitions of F^n into cosets of Hamming codes and into extended Hamming codes with the smallest possible η_n were constructed for length $n = 7$ by Phelps in [9] and for any $n = 2^m$ for odd $m > 3$, using the Gold function by Krotov in [5]. In [11] partitions into pairwise nonparallel Hamming codes were constructed and among them there were uniform partitions. An overview of results till 1998 on utilizing partitions to construct q -ary perfect codes can be found in [1]. See [10] for a survey concerning some new results on partitions and all other necessary definitions and notions.

The investigation of the partitions of F^n into perfect codes is important due to the connection of the classification problem of all partitions with the analogous problem for perfect binary codes. It is known that the limit for the relation of double logarithms of the numbers of different perfect binary codes and different partitions equals 1, although the number of nonequivalent partitions significantly exceeds the number of nonequivalent codes. Since an extended Hamming code of length n is the Reed – Muller code of the same length and order $n - 2$, see [6], it is natural to investigate the problem of constructing partitions of F^n into Reed – Muller codes of any admissible order. The intersection of two Hamming codes often gives a good cyclic code, see, for example, [6]. Moreover partitions are connected with the perfect colorings called also regular codes, partition designs or equitable partitions [3]. In some cases partitions of F^n into codes induce colorings associated with fibre optic nets [8].

2. CONSTRUCTION

In order to give a recursive construction of the class of uniform partitions into Hamming codes we exploit the construction B from [11] based on the classical

Mollard construction for perfect codes [7] and the results given in [5, 9, 12]. For the sake of completeness we recall the definition of the Mollard construction and the construction B.

Let C^l and C^t be two binary codes of lengths l and t respectively with the code distance not less than 3 containing the all-zero vectors. Let

$$x = (x_{11}, x_{12}, \dots, x_{1t}, x_{21}, \dots, x_{2t}, \dots, x_{l1}, \dots, x_{lt}) \in F^{lt}.$$

We use a matrix notation of vector x : the i th row of the matrix is equal to $x_{i1} x_{i2} \dots x_{it}$, where $i = 1, \dots, l$. Functions $p_1(x)$ and $p_2(x)$ are defined as

$$p_1(x) = \left(\sum_{j=1}^t x_{1j}, \dots, \sum_{j=1}^t x_{lj} \right) \in F^l,$$

$$p_2(x) = \left(\sum_{i=1}^l x_{i1}, \dots, \sum_{i=1}^l x_{it} \right) \in F^t.$$

Let f be an arbitrary function from C^l to F^t . The set

$$C^n = \{(x, y + p_1(x), z + p_2(x) + f(y)) \mid x \in F^{lt}, y \in C^l, z \in C^t\}$$

is a binary *Mollard code* of length $n = lt + l + t$ with the code distance 3, see [7].

Let C^l and C^t be the Hamming codes of lengths l and t denoted as H^l and H^t respectively and the function f be the constant function from H^l to $\mathbf{0}^t$. Then we obtain the Hamming code of length $n = lt + l + t$ by the Mollard construction

$$\{(x, y + p_1(x), z + p_2(x)) \mid x \in F^{lt}, y \in H^l, z \in H^t\}.$$

Below we use a partial case of the *construction B* from [11]. It should be noted that in [11] the construction B was described for any binary single-errors-correcting codes, not necessarily perfect. Let $P^l = \{H_0^l, H_1^l + e_1, \dots, H_l^l + e_l\}$ and $P^t = \{H_0^t, H_1^t + e_1, \dots, H_t^t + e_t\}$ be arbitrary partitions of F^l and F^t into cosets of Hamming codes $H_0^l, H_1^l, \dots, H_l^l$ and $H_0^t, H_1^t, \dots, H_t^t$ respectively. Then the set of the codes

$$(1) \quad \{(x, y + p_1(x), z + p_2(x)) \mid x \in F^{lt}, y \in H_i^l + e_i, z \in H_j^t + e_j\},$$

$0 \leq i \leq l, 0 \leq j \leq t$ defines a *partition P^n of the space F^n into cosets of Hamming codes* of length $n = lt + l + t$, see [11]. The Hamming code corresponding to (1) we denote by $M^n(H_i^l, H_j^t)$ to emphasize that this Hamming code of length n is obtained from the Hamming codes H_i^l and H_j^t of lengths l and t respectively.

Lemma 1. (See [12].) *The construction B applied for 2-transitive partitions into perfect codes of length l and t gives a 2-transitive partition into perfect codes of length $l + t + lt$.*

The partition of F^n into cosets of any Hamming code of length n is called *trivial*, so in this case the uniformity number $\log \eta_n$ is equal to the dimension of the code, i.e. $n - \log(n + 1)$.

Lemma 2. *The construction B applied for uniform partitions into Hamming codes of length l and t gives a uniform partition into Hamming codes of length $l + t + lt$.*

Proof. Let $\log \eta_l$ and $\log \eta_t$ be the uniformity numbers of two initial uniform partitions into Hamming codes of lengths l and t respectively. Then it is easy to see that applying the construction B to both of these uniform partitions we obtain the uniform partition of length $n = l + t + lt$ with the uniformity number $\log \eta_n = \log \eta_l + \log \eta_t + lt$. □

In Lemma 3, see [2], one can find the simple fact that two perfect codes of length n intersected by s codewords exist if and only if there exist two extended perfect codes of length $n + 1$ intersected by s codewords. Therefore the partitions of F^n into Hamming codes obtained from the uniform partitions given in [5] by puncturing any coordinate position are uniform. Recall that in [5] the uniform partitions of the set of all odd weight vectors of F^{n+1} into extended Hamming codes are given. We will further call such uniform partitions into cosets of Hamming codes *punctured*.

Lemma 3. *There are uniform partitions of F^n into cosets of Hamming codes of length $n \in \{31, 127, 1023\}$ with the uniformity numbers $\log \eta_{31} = 24$, $\log \eta_{127} = 116$ and $\log \eta_{1023} = 1007$ respectively.*

Proof. The case $\log \eta_{31} = 24$ is covered applying Lemma 2 to the trivial partition for $l = 3$ with $\log \eta_3 = 1$ and the uniform partition from [9] for $t = 7$ having $\log \eta_7 = 2$. So we have the uniform partition of length 31 with the uniformity number $\log \eta_{31} = 1 + 2 + 21 = 24$.

The case $\log \eta_{127} = 116$ is achieved by Lemma 2 utilizing the trivial partition for $l = 3$ with $\log \eta_3 = 1$ and the punctured uniform partition for $t = 31$ given in [5] with the uniformity number $\log \eta_{31} = 22$: $\log \eta_{127} = 1 + 22 + 93 = 116$.

For the last case we need the uniform partition of length 255 with $\log \eta_{255} = 241$ that can be obtained by Lemma 2 applying to the partition of length $l = 7$ from [9] with the uniformity number $\log \eta_7 = 2$ and the punctured uniform partition for $t = 31$ taken from [5] with $\log \eta_{31} = 22$: $\log \eta_{255} = 2 + 22 + 217 = 241$. Then in order to obtain $\log \eta_{1023} = 1007$ for $n = 1023$ we apply again Lemma 2 to the trivial partition with $l = 3$, $\log \eta_3 = 1$ and the obtained uniform partition for $t = 255$ having the uniformity number $\log \eta_t = 241$: $\log \eta_{1023} = 1 + 241 + 765 = 1007$. □

The 2-transitivity of the uniform partition of length 7 with the uniformity number $\eta_7 = 2$ from [9] was proved in [12], see Lemma 1. The 2-transitivity of the uniform partition of F^{31} considered in Lemma 3 follows from this 2-transitive uniform partition of length 7 and $\eta_7 = 2$ and Lemma 1, so the following holds

Corollary 1. *There is a 2-transitive uniform partition of F^{31} into cosets of Hamming codes of length 31 with the uniformity number $\log \eta_{31} = 24$.*

Theorem 1. *For any $n = 2^m - 1$, $m > 2$ and $e = 1, 2, \dots, [(m + 1)/2]$, with the exception of $m = 4$, $e = 1$, there exists a uniform partition of F^n into cosets of Hamming codes of length n with η_n satisfying*

$$(2) \qquad \log \eta_n = n - 2m + 2e - \delta(m),$$

where $\delta(m) = \begin{cases} 1 & \text{for } m \equiv 1 \pmod{2}; \\ 0 & \text{for } m \equiv 0 \pmod{2}. \end{cases}$

Proof. The proof will be done by induction on $m, m \geq 3$ exploiting the construction B, see Lemma 2, and Lemma 3. In the construction B we fix the first uniform partition P^l with $l = 3$ and $\log \eta_7 = 2$ from [9] and vary the second partition P^t , $t = 2^{m-3} - 1$ to be any uniform partition of length t including the trivial one. So the corresponding Hamming codes are $M^n(H_i^7, H_j^{2^{m-3}-1})$, $i = 0, 1, 2, \dots, 7$, $j = 0, 1, 2, \dots, 2^{m-3} - 1$. The approach is valid with the exception of three special cases: $\eta_{31} = 24$, $\eta_{127} = 116$, $\eta_{1023} = 1007$, where $l = 3$ and $\log \eta_7 = 1$ that were covered by Lemma 3.

By the specification of the construction B we need for induction base the following three initial cases: $m = 3, 4$ and 5 . For $m = 3$, i.e. $n = 7$ there exist only two uniform partitions described by Phelps in [9]. The first one has the uniformity number $\log \eta_7 = 2$, the second one is the trivial partition into the cosets of any Hamming code of length 7 with the uniformity number $\log \eta_7 = 4$. For $m = 4$ we know only the trivial partition. For $m = 5$ there are known three nonequivalent uniform partitions of length 31: the punctured partition from [5] with the uniformity number 22, the partition with the uniformity number 24 from Lemma 3 and the trivial one.

Let the statement of the theorem hold for any number not greater than $m - 1$, so for partitions of length $2^{m-1} - 1$. We prove that it is valid for m , i.e. for $n = 2^m - 1$. In the induction steps of the proof of the theorem we will also use trivial partitions into the cosets of any Hamming code of length t by the reason that if one partition in the construction B is nontrivial uniform partition and another one is trivial then the resulting partition is uniform nontrivial.

Let m be even. So $m - 3$ is odd and by (2) and induction hypothesis we have $\delta(m - 3) = 1$ and the following admissible values for uniformity number $\log \eta_t$ for partitions of length $t = 2^{m-3} - 1$:

$$\log \eta_t = t - 2(m - 3) + 2e' - 1, \quad e' = 1, 2, \dots, [(m - 2)/2].$$

Applying the construction B and taking into account that $n = 7 + t + 7t$ we calculate the uniformity number for the partition (1):

$$\begin{aligned} \log \eta_n &= \log(|M^n(H_i^7, H_j^t) \cap M^n(H_r^7, H_s^t)|) \\ &= 2 + (t - 2(m - 3) + 2e' - 1) + 7t \\ &= 2 + 8(2^{m-3} - 1) - 2(m - 3) + 2e' - 1 = n - 2m + 2e, \end{aligned}$$

where $e = e' = 1, 2, \dots, (m - 2)/2$, $i, r = 0, 1, 2, \dots, 7$, $j, s = 0, 1, 2, \dots, 2^{m-3} - 1$. We obtain $|M^n(H_i^7, H_j^t) \cap M^n(H_r^7, H_s^t)|$ to be constant regardless of the choice of i, j, r, s . So we construct all the required uniform partitions of length n , i.e. (2), since there is the trivial partition of length n with the case $\eta_n = n - m$, i.e. $e = m/2 = [(m + 1)/2]$.

Let m be odd. This case is analogous to the previous one taking into account that $m - 3$ is even and so $\delta(m - 3) = 0$ in (2). Therefore we have

$$\log \eta_t = t - 2(m - 3) + 2e', \quad \text{where } e' = 1, 2, \dots, [(m - 2)/2].$$

Then for any $i, r = 0, 1, 2, \dots, 7$, $j, s = 0, 1, 2, \dots, 2^{m-3} - 1$ we have

$$\begin{aligned} \log \eta_n &= \log(|M^n(H_i^7, H_j^t) \cap M^n(H_r^7, H_s^t)|) \\ &= 2 + t - 2(m - 3) + 2e' + 7t \\ &= 2 + 8(2^{m-3} - 1) - 2(m - 3) + 2e' = n - 2m + 2e - 1, \end{aligned}$$

where $e = e' + 1 = 2, 3, 4, \dots, (m - 1)/2$. We obtain (2) adding the first minimal uniformity number given by the Krotov punctured construction [5], i.e. the case when $\log \eta_m = n - 2m + 1$ with $e = 1$, and the trivial uniform partition into cosets of a Hamming code of length n with $\eta_n = n - m$, i.e. $e = (m + 1)/2$. \square

Extended codes or partitions are often objects with larger automorphism groups, see, for example, [10]. It is easy to see that extending by parity check any uniform partition of length n gives the uniform partition of the set of all even weight vectors in F^{n+1} . It is clear that any two partitions of F^n into cosets of Hamming codes of length n from Theorem 1 are nonequivalent since by the construction they have different uniformity numbers. So the following holds

Corollary 2. *For any $n = 2^m - 1$, $m > 2$, with the exception $m = 4$, there exist at least $\lfloor (m + 1)/2 \rfloor$ nonequivalent uniform partitions of F^n into cosets of Hamming codes of length n and of the set of all even weight vectors in F^{n+1} into cosets of extended Hamming codes of length $n + 1$.*

Note that some of the partitions constructed in Theorem 1 are 2-transitive. The 2-transitivity may not hold for the construction B applied to punctured partitions from [5] as we do not know whether the latter are 2-transitive or not.

Theorem 2. *For any $n = 2^m - 1$, $m \geq 6$ there exist at least $\lfloor m/3 \rfloor$ nonequivalent 2-transitive uniform partitions of F^n into cosets of Hamming codes of length n . For $m = 3$ there exist two and for $m = 5$ there exist at least two nonequivalent such partitions.*

The proof of this theorem is the same as that for Theorem 1. The 2-transitivity of all obtained uniform partitions follows by Lemma 1, Corollary 1 and the 2-transitivity of the initial partition of length 7 with the uniformity number $\eta_7 = 2$ given by [9]. The 2-transitivity of the latter one was proved in [12]. Then for $n \geq 6$ there exist at least $\lfloor m/3 \rfloor$ nonequivalent 2-transitive partitions among $\lfloor (m + 1)/2 \rfloor$ nonequivalent uniform partitions from Theorem 1 having different and the largest uniformity numbers. For $m = 3$ there exist two and for $m = 5$ there exist at least two nonequivalent such partitions.

Extending by parity check a 2-transitive uniform partition of length n leads to the 2-transitive uniform partition of the set of all even weight vectors in F^{n+1} . In the next corollary we take into account that partitions in [5] are 2-transitive.

Corollary 3. *For any $n + 1 = 2^m$, $m \geq 6$ there exist at least $\lfloor m/3 \rfloor + 1$ nonequivalent 2-transitive uniform partitions of the set of all even weight vectors in F^{n+1} into cosets of extended Hamming codes of length $n + 1$. For $m = 3$ there exist two and for $m = 5$ there exist at least three nonequivalent such partitions.*

Remarks. It should be noted that Theorem 1 covers a half of possible values of the numbers η_n . Another part is still open, so the problem of finding of all numbers η_n is still open as far as the problem of the description of all nonequivalent uniform partitions of F^n into cosets of Hamming codes of length n . This part of values of the numbers η_n could be covered using the technique above if we found, for example, the uniform partition in F^{15} with the uniformity number 9 (if such partition exists). It should also be noted that in the proof of the theorems it is possible to choose other variations of the initial partitions into the construction

B. Speaking more precisely one can consider $l > 3$, that could give different (or perhaps nonequivalent) partitions of length n with the same uniformity number.

The author is very grateful to the anonymous referee for the useful remarks.

REFERENCES

- [1] G. Cohen, I. Honkala, A. Lobstein, S. Litsyn, *Covering codes*, Elsevier, 1998.
- [2] T. Etzion, A. Vardy, *On perfect codes and tilings: problems and solutions*, SIAM J. Discrete Math., **11**:2 (1998), 205–223. Zbl 0908.94035
- [3] D. G. Fon-Der-Flaass, *Perfect 2-colorings of a hypercube*, Siberian Math. J., **22**:4 (2007), 740–745. Zbl 1164.05348
- [4] O. Heden, F. I. Solov'eva, *Partitions of F^n into non-parallel Hamming codes*, Adv. Math. Commun., **3**:4 (2009), 385–397. Zbl 1191.94126
- [5] D. S. Krotov, *A partition of the hypercube into maximally nonparallel Hamming codes*, Journal of Combinatorial Designs, **22**:4 (2014), 179–187. Zbl 1286.05114
- [6] F. J. MacWilliams, N. J. A. Sloane, *The theory of error-correcting codes*, North-Holland Publishing Company, 1977. Zbl 0369.94008
- [7] M. Mollard, *A generalized parity function and its use in the construction of perfect codes*, SIAM J. Alg. Discrete Math., **7**:1 (1986), 113–115. Zbl 0585.94015
- [8] P. R. J. Östergård, *On a hypercube coloring problem*, J. Combin. Theory, Ser. A, **108** (2004), 199–204. Zbl 1061.05035
- [9] K. T. Phelps, *An enumeration of 1-perfect binary codes*, Australas. J. Comb., **21** (2000), 287–298. Zbl 0972.94049
- [10] F. I. Solov'eva, *Survey on perfect codes*, Mathematical Problems of Cybernetics, **18** (2013), 5–34 (in Russian).
- [11] F. I. Solov'eva, *On transitive partitions of an n -cube into codes*, Probl. of Inform. Transm., **45**:1 (2009) 23–31. Zbl 1173.94462
- [12] F. I. Solov'eva, G. K. Gus'kov, *On construction of vertex-transitive partitions of n -cube into perfect codes*, Journal of Applied and Industrial Math., **5**:2 (2011), 84–100.

FAINA IVANOVNA SOLOV'eva
 SOBOLEV INSTITUTE OF MATHEMATICS,
 PR. AC. KOPTYUGA 4,
 630090, NOVOSIBIRSK, RUSSIA
E-mail address: sol@math.nsc.ru