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SOME CHARACTERIZATION OF CURVES IN $\widetilde{\mathbf{SL}_2\mathbb{R}}$ SPACE

B. SENOUSSI, M. BEKKAR

Abstract. In 1997 Emil Molnár introduced [15] the hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space. In this paper, we obtained characterizations of a curve with respect to the Frenet frame of $\widetilde{\mathbf{SL}_2\mathbb{R}}$. Rectifying curves are introduced in [3] as space curves whose position vector always lies in its rectifying plane. We characterize rectifying curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$.

Keywords: $\widetilde{\mathbf{SL}_2\mathbb{R}}$ geometry, biharmonic curves, general helix, rectifying curve.

1. Introduction

First we should recall some notions and results related to the harmonic and the biharmonic maps between Riemannian manifolds.

Harmonic maps $\psi : (M, g) \to (N, \tilde{g})$ between Riemannian manifolds are the critical points of the energy functional

$$E_1 : C^\infty(M, N) \to \mathbb{R}, \quad E_1(\psi) = \frac{1}{2}\int_M |d\psi|^2 \, v_g,$$

and is characterized by the vanishing of the first tension field

$$\tau_1(\psi) = trace \nabla d\psi.$$

We remind that the bienergy of ψ is given by

$$E_2 : C^\infty(M, N) \to \mathbb{R}, \quad E_2(\psi) = \frac{1}{2}\int_M |\tau(\psi)|^2 \, v_g,$$

and the bitension field $\tau_2(\psi)$ has the expression

$$\tau_2(\psi) = -\Delta^\psi \tau(\psi) - trace_g R^N(d\psi, \tau(\psi))d\psi,$$

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where $\Delta^{\psi} = -trace(\nabla^{\psi})^2 = -trace(\nabla^{\psi}\nabla^{\psi} - \nabla^{\psi}_{\nabla})$.

A smooth map ψ is *biharmonic* if it satisfies the following biharmonic equation

$$\tau_2(\psi) = 0.$$

Biharmonic maps are the critical points of the bienergy functional E_2. We call proper biharmonic the non-harmonic biharmonic maps. Biharmonic curves ψ of a Riemannian manifold are the solutions of the fourth order differential equation

$$\nabla^3_{\phi'}\phi' - R(\phi', \nabla_{\phi'}\phi')\phi' = 0. \tag{1}$$

In 1997 Emil Molnár introduced [15] the hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space. In the theory of curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space, one of the important and interesting problems is the characterizations of a regular curve. In [12] T. Ikawa obtained the following equation

$$D^3_X X - (k^2 - \tau^2)D_X X = 0,$$

for the circular helices which corresponds to the case that the curvatures k and τ of a timelike curve γ on the Lorentzian manifold M are constant.

In [9] N. Ekmekçi and H. H. Hacisalihoğlu generalized T. Ikawa's result to the case of general helices and gave the following characterization

$$D^3_X X - \left(\frac{3k'}{k}\right) D^2_X X - \left(\frac{k''}{k} - \frac{3k'}{k^2} + k^2 - \tau^2\right) D_X X = 0$$

for timelike curve with its tangent vector fields on any point. The authors in [7] determined the equation of the geodesic curve in the hyperboloid model. Recently, in [19] Y. Nakanishi prove the following lemma.

LEMMA 1.1. *A unit speed curve c in M_c is a helix if and only if there exist a constant λ such that*

$$D^3_X X = \lambda D_X X.$$

2. Preliminaries

2.1. Hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space.

In [15] Emil Molnár proposed a projective spherical model as unified geometrical model of homogeneous geometries. The hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbb{R}}$ geometry is described in details in [7, 11, 17, 18].

The real 2×2 matrices $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ with unit determinant $ad - bc = 1$ constitute a Lie transformation group by the usual product operation, taken to act on row matrices as on point coordinates on the right as follows [5]

$$(z^0, z^1)\begin{pmatrix} d & b \\ c & a \end{pmatrix} = (z^0 d + z^1 c, z^0 b + z^1 a) = (\omega^0, \omega^1) \tag{2}$$

with

$$\omega = \frac{\omega^1}{\omega^0} = \frac{b + \frac{z^1}{z^0}a}{d + \frac{z^1}{z^0}c} = \frac{b + za}{d + zc}$$

as right action on the complex projective line C^{∞}. This group is a $3-$ dimensional manifold, because of its 3 independent real coordinates and with its usual neighbourhood topology. In order to model the above structure in the projective sphere $\mathcal{PS}^3$ and in the projective space $\mathcal{P}^3$, we introduce the new projective coordinates (x^0, x^1, x^2, x^3) where

$$a := x^0 + x^3,\ b := x^1 + x^2,\ c := -x^1 + x^2,\ d := x^0 - x^3,$$

with positive, then the non-zero multiplicative equivalence as a projective freedom in $\mathcal{PS}^3$ and in $\mathcal{P}^3$, respectively. Meanwhile we turn to the proportionality

$$\mathbf{SL}_2\mathbb{R} < \mathbf{PSL}_2\mathbb{R},$$

natural in this context. Then it follows that

$$0 > bc - ad = -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 \tag{3}$$

describes the interior of the above one-sheeted hyperboloid solid $\mathcal{H}$ in the usual Euclidean coordinate simplex, with the origin $E_0(1;0;0;0)$ and the ideal points of the axes

$E_1^\infty(0;1;0;0)$, $E_2^\infty(0;0;1;0)$, $E_3^\infty(0;0;0;1)$. We consider the collineation group $\mathbf{G}_*$ that acts on the projective sphere $\mathcal{PS}^3$ and preserves a polarity, i.e. a scalar product of signature $(--++)$, this group leaves the one sheeted hyperboloid solid $\mathcal{H}$ invariant. We have to choose an appropriate subgroup $\mathbf{G}$ of $\mathbf{G}_*$ as isometry group, then the universal covering group and space $\tilde{\mathcal{H}}$ of $\mathcal{H}$ will be the hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbb{R}}$.

The specific isometries $\mathbf{S}(\phi)$ ($\phi \in \mathbb{R}$) constitute a one parameter group given by the matrices

$$\mathbf{S}(\phi) : (s_i^j(\phi)) = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}. \tag{4}$$

The elements of $\mathbf{S}(\phi)$ are the so-called *fibre translations*. We obtain a unique fibre line to each $X(x^0;\ x^1;\ x^2;\ x^3) \in \tilde{\mathcal{H}}$ as the orbit by right action of $\mathbf{S}(\phi)$ on X. The coordinates of points lying on the fibre line through X can be expressed as the images of X by $\mathbf{S}(\phi)$:

$$\begin{aligned}(x^0;x^1;x^2;x^3) \xrightarrow{\mathbf{S}(\phi)} (&x^0\cos\phi - x^1\sin\phi; x^0\sin\phi + x^1\cos\phi;\\ &x^2\cos\phi + x^3\sin\phi; -x^2\sin\phi + x^3\cos\phi)\end{aligned} \tag{5}$$

for the Euclidean coordinates $x := \frac{x^1}{x^0}$, $y := \frac{x^2}{x^0}$, $z := \frac{x^3}{x^0}$, $x^0 \neq 0$ as well.

The π periodicity for the above coordinates in the above maps can be seen from the formula (5).

In (4) and (5) we can see the 2π periodicity of ϕ. Moreover, we see the (logical) extension to $\phi \in \mathbb{R}$, as real parameter, to have the universal covers $\tilde{\mathcal{H}}$ and $\widetilde{\mathbf{SL}_2\mathbb{R}}$, respectively, through the projective sphere $\mathcal{PS}^3$. The elements of the isometry group of $\mathbf{SL}_2\mathbb{R}$ (and so by the above extension the isometries of $\widetilde{\mathbf{SL}_2\mathbb{R}}$) can be described by the matrix (a_i^j).

$$(a_i^j) = \begin{pmatrix} a_0^0 & a_0^1 & a_0^2 & a_0^3 \\ \mp a_0^1 & \pm a_0^0 & \pm a_0^3 & \mp a_0^2 \\ a_2^0 & a_2^1 & a_2^2 & a_2^3 \\ \pm a_2^1 & \mp a_2^0 & \mp a_2^3 & \pm a_2^2 \end{pmatrix},$$

where

$$(6)\qquad \begin{array}{rcl} -(a_0^0)^2-(a_0^1)^2+(a_0^2)^2+(a_0^3)^2 &=& -1, \\ -(a_2^0)^2-(a_2^1)^2+(a_2^2)^2+(a_2^3)^2 &=& 1, \\ -a_0^0a_2^0-a_0^1a_2^1+a_0^2a_2^2+a_0^3a_2^3 &=& 0, \\ -a_0^0a_2^1+a_0^1a_2^0-a_0^2a_2^3+a_0^3a_2^2 &=& 0, \end{array}$$

and we allow positive proportionality, of course, as projective freedom. We define the translation group $\mathbf{G}_T$, as a subgroup of the isometry group of $\mathbf{SL}_2\mathbb{R}$, those isometries acting transitively on the points of $\mathcal{H}$ and by the above extension on the points of $\tilde{\mathcal{H}}$. $\mathbf{G}_T$ maps the origin $E_0(1;0;0;0)$ onto $X(x^0;\ x^1;\ x^2;\ x^3)$. These isometries and their inverses (up to a positive determinant factor) can be given by

$$(7)\qquad \mathbf{T}\ :\ (t_i^j)=\begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^2 & x^3 & x^0 & x^1 \\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix}$$

$$\mathbf{T}^{-1}\ :\ (T_j^k)=\begin{pmatrix} x^0 & -x^1 & -x^2 & -x^3 \\ x^1 & x^0 & -x^3 & x^2 \\ -x^2 & -x^3 & x^0 & -x^1 \\ -x^3 & x^2 & x^1 & x^0 \end{pmatrix}.$$

Horizontal intersection of the hyperboloid solid $\mathcal{H}$ with the plane $E_0E_2^\infty E_3^\infty$ provides the *base plane* of the model $\tilde{\mathcal{H}}=\widetilde{\mathbf{SL}_2\mathbb{R}}$. The fibre through X intersects the hyperbolic ($\mathbf{H}^2$) base plane $z^1=x=0$ in the foot point

$$(8)\qquad Z(z^0=x^0x^0+x^1x^1;\ z^1=0;\ z^2=x^0x^2-x^1x^3;\ z^3=x^0x^3+x^1x^2).$$

We generally introduce a so-called hyperboloid parametrization as follows

$$(9)\qquad \begin{array}{rcl} x^0 &=& \cosh r\cos\phi, \\ x^1 &=& \cosh r\sin\phi, \\ x^2 &=& \sinh r\cos(\vartheta-\phi), \\ x^3 &=& \sinh r\sin(\vartheta-\phi), \end{array}$$

where (r,ϑ) are the polar coordinates of the $\mathbf{H}^2$ base plane, and ϕ is the fibre coordinate. We note that

$$-x^0x^0-x^1x^1+x^2x^2+x^3x^3=-\cosh^2 r+\sinh^2 r=-1<0.$$

The inhomogeneous coordinates in (10), which will play an important role in the later $\mathbb{E}^3-$ visualization of the prism tilings in $\widetilde{\mathbf{SL}_2\mathbb{R}}$, are given by

$$(10)\qquad \begin{array}{rcl} x &=& \dfrac{x^1}{x^0}=\tan\phi, \\ y &=& \dfrac{x^2}{x^0}=\tanh r\dfrac{\cos(\vartheta-\phi)}{\cos\phi}, \\ z &=& \dfrac{x^3}{x^0}=\tanh r\dfrac{\sin(\vartheta-\phi)}{\cos\phi}. \end{array}$$

An invariant infinitesimal arc length square in $\widetilde{\mathbf{SL}_2\mathbb{R}}$, is given by [15]

$$ds^2=\lambda(\omega_1^2+\omega_2^2+\omega_3^2),$$

where $\lambda = [-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2]^{-2}$ and

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \left(x^0, x^1, x^2, x^3\right) \begin{pmatrix} -dx^1 & -dx^2 & -dx^3 \\ dx^0 & -dx^3 & dx^2 \\ dx^3 & -dx^0 & dx^1 \\ -dx^2 & -dx^1 & -dx^0 \end{pmatrix}.$$

By using (9) we obtain the following result

$$ds^2 = dr^2 + (\cosh r \sinh r)^2 d\vartheta^2 + (d\phi + \sinh^2 r d\vartheta)^2.$$

If

$$\{\theta^1 = dr, \theta^2 = \cosh r \sinh r d\vartheta, \theta^3 = d\phi + \sinh^2 r d\vartheta\}$$

is an orthonormal coframe then the dual orthonormal frame fields are given by

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sinh 2r} & -\tanh r \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\vartheta \\ \partial_\phi \end{pmatrix},$$

where $\partial_r = \frac{\partial}{\partial r}$, $\partial_\vartheta = \frac{\partial}{\partial \vartheta}$, $\partial_\phi = \frac{\partial}{\partial \phi}$.

One can easily check that $g(e_i, e_j) = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta.

The symmetric metric tensor field g is given by

$$G = (g_{ij})_{1 \le i,j \le 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\sinh r)^2((\cosh r)^2 + (\sinh r)^2) & (\sinh r)^2 \\ 0 & (\sinh r)^2 & 1 \end{pmatrix}.$$

The inverse matrix $(g^{ij})_{1 \le i,j \le 3}$ of $(g_{ij})_{1 \le i,j \le 3}$ is given by

$$G^{-1} = \frac{1}{(\sinh r \cosh r)^2} \begin{pmatrix} (\sinh r \cosh r)^2 & 0 & 0 \\ 0 & 1 & -(\sinh r)^2 \\ 0 & -(\sinh r)^2 & (\sinh r)^2((\cosh r)^2 + (\sinh r)^2) \end{pmatrix}.$$

The Levi-Civita connection ∇ is defined by $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$, where the Cristoffel symbols Γ^k_{ij} are given by

$$\begin{aligned}
\Gamma^1_{ij} &= -\frac{1}{2} \sinh 2r \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 + 4(\sinh r)^2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\Gamma^2_{ij} &= \frac{2}{\sinh 2r} \begin{pmatrix} 0 & 1 + 3(\sinh r)^2 & 1 \\ 1 + 3(\sinh r)^2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\Gamma^3_{ij} &= -\tanh r \begin{pmatrix} 0 & 2(\sinh r)^2 & 1 \\ 2(\sinh r)^2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

The Levi-Civita connection ∇ is given by

$$\begin{pmatrix} \nabla_{e_1}e_1 \\ \nabla_{e_1}e_2 \\ \nabla_{e_1}e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$$\begin{pmatrix} \nabla_{e_2}e_1 \\ \nabla_{e_2}e_2 \\ \nabla_{e_2}e_3 \end{pmatrix} = \begin{pmatrix} 0 & 2\coth 2r & 1 \\ -2\coth 2r & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$$\begin{pmatrix} \nabla_{e_3}e_1 \\ \nabla_{e_3}e_2 \\ \nabla_{e_3}e_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

The characterising properties of this algebra are the following commutation relations:

$$[e_1, e_2] = -2\coth(2r)e_2 - 2e_3, \quad [e_1, e_3] = [e_2, e_3] = 0.$$

The components of the Riemannian curvature tensor de ned by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$

are [10]

$$\begin{aligned} R(e_1, e_2)e_1 &= 7e_2 \quad R(e_1, e_2)e_2 = -7e_1 \quad R(e_1, e_3)e_1 = -e_3 \\ R(e_1, e_3)e_3 &= e_1 \quad R(e_3, e_2)e_2 = e_3 \quad R(e_3, e_2)e_3 = -e_2. \end{aligned} \tag{11}$$

The Riemannian curvature tensor field R is described by

$$R_{1212} = -7, \quad R_{1313} = R_{2323} = 1.$$

The product

$$g_\lambda (X \wedge Y, Z) = [X, Y, Z] \tag{12}$$

is called the mixed product.

2.1.1. *Translation curves.* Let us consider for a given vector $(q : u : v : w)$ a curve $C(t) = (x^0(t) : x^1(t) : x^2(t) : x^3(t))$, $t \geq 0$, in $\mathcal{H}$ starting at the origin: $C(0) = E_0(1; 0; 0; 0)$ and such that

$$\dot{C}(0) = (\dot{x}^0(0) : \dot{x}^1(0) : \dot{x}^2(0) : \dot{x}^3(0)) = (q : u : v : w),$$

where $\dot{C}(0)$ is the tangent vector at any point of the curve.

For $t \geq 0$ there exists a matrix

$$\mathbf{T}(t) = \begin{pmatrix} x^0(t) & x^1(t) & x^2(t) & x^3(t) \\ -x^1(t) & x^0(t) & x^3(t) & -x^2(t) \\ x^2(t) & x^3(t) & x^0(t) & x^1(t) \\ x^3(t) & -x^2(t) & -x^1(t) & x^0(t) \end{pmatrix}$$

which defines the translation from $C(0)$ to $\dot{C}(0)$.

Definition 2.1 ([5]). The curve $C(t)$, $t \geq 0$, is said to be a translation curve if

$$\dot{C}(0).\mathbf{T}(t) = \dot{C}(t), \quad t \geq 0.$$

The solution, depending on $(q : u : v : w)$ had been determined in [16].

2.1.2. *Geodesic curves.* The second order differential equation system of the $\widetilde{\mathbf{SL}_2\mathbb{R}}$ geodesic curve is the following: [5]

$$\begin{cases} \ddot{r} = \dot{\vartheta}\dot{\phi}\sinh(2r) + \frac{1}{2}(\sinh(4r) - \sinh(2r))\dot{\vartheta}\dot{\vartheta} \\ \ddot{\vartheta} = -\frac{2\dot{r}}{\sinh(2r)}(\dot{\vartheta}(3\cosh(2r) - 1) + 2\dot{\phi}) \\ \ddot{\phi} = 2\dot{r}(\dot{\phi} + 2\dot{\vartheta}\sinh^2 r)\tanh(r). \end{cases}$$

The parametrization of a geodesic curve in the hyperboloid model with the geographical sphere coordinates (λ, α), as longitude and altitude, $(-\pi < \lambda \leq \pi$, $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2})$, and the arc-length parameter $s \geq 0$, has been determined in [7].

3. Biharmonic curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space

In this section we study biharmonic curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space. Z. Erjavec and D. Horvat in [8] introduced the notion of biharmonic maps as a natural generalization of the well-known harmonic maps. Chen and Ishikawa in [4] classified biharmonic curves in semi-Euclidean 3-space. The biharmonic curves in the Heisenberg group $\mathbb{H}_3$ are investigated in [1].

Let $\phi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a non-geodesic timelike biharmonic curve in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ parametrized by arclength and let $\{T, N, B\}$ be the orthonormal moving Frenet frame along the curve ϕ in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ such that $T = \phi^{'}$ is the unit vector field tangent to ϕ, N is the unit vector field in the direction $\nabla_T T$ normal to ϕ (principal normal) and $B = T \wedge N$ (binormal vector). Then we have the following Frenet equations

$$\begin{pmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{13}$$

where

$$k^2 = g(\nabla_T T, \nabla_T T),$$

is the curvature of ϕ and τ is its torsion.

The planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are respectively known as the osculating, the rectifying and the normal plane.

From (13) we have

$$\nabla_T^3 T = (-3kk')T + (k'' - k^3 - k\tau^2)N + (2k'\tau + k\tau')B, \tag{14}$$

where $k' = \frac{dk}{ds}$, $k'' = \frac{d^2k}{ds^2}$, $\tau' = \frac{d\tau}{ds}$.

Using (11) one obtains [10]

$$R(T, N, T, N) = 1 - 8B_3^2, \quad R(T, N, T, B) = 8B_3N_3, \tag{15}$$

where

$$\begin{cases} T = T_1e_1 + T_2e_2 + T_3e_3 \\ N = N_1e_1 + N_2e_2 + N_3e_3 \\ B = T \wedge N = B_1e_1 + B_2e_2 + B_3e_3. \end{cases}$$

Proposition 3.1 ([10]). *Let $\phi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parametrized by arc length. Then ϕ is a proper non-geodesic biharmonic curve if and only if*

$$\begin{cases} k = constant \neq 0 \\ k^2 + \tau^2 = 1 - 8B_3^2 \\ \tau' = 8N_3B_3. \end{cases} \tag{16}$$

COROLLARY 3.2. *If $\tau = 0$ and $k = constant \neq 0$ for a timelike curve ϕ. ϕ is a non-geodesic biharmonic curve if and only if*

$$\begin{cases} k^2 = 1 - 8B_3^2 \\ B_3N_3 = 0. \end{cases}$$

4. General helix in $\widetilde{\mathbf{SL}_2\mathbb{R}}$

In 1845, de Saint Venant first proved that a space curve is a general helix if and only if the ratio of curvature to torsion be constant (see [20] for details).

DEFINITION 4.1. Let ϕ be a curve in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ and $\{T, N, B\}$ be the Frenet frame on $\widetilde{\mathbf{SL}_2\mathbb{R}}$ along ϕ.

1) If both k and τ are constant along ϕ, then is called circular helix with respect to Frenet frame.

2) A curve ϕ such that

$$\frac{k}{\tau} = c, \ c \in \mathbb{R},$$

is called a general helix with respect to Frenet frame.

If k =constant $\neq 0$ and $\tau = 0$, then the curve ϕ is a circle.

THEOREM 4.2 ([3]). *A twisted curve $\phi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ is a rectifying curve if and only if the ratio $\frac{\tau}{k}$ is a nonconstant linear function in arclength function s.*

THEOREM 4.3. *Let $\phi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parameterized by arc length. Then ϕ is a general helix with respect to the Frenet frame $\{T, N, B\}$ if and only if*

1)

$$\nabla_T^3 T - \sigma_1(s)\nabla_T T = 3k'\nabla_T N, \tag{17}$$

where $\sigma_1(s) = \frac{k''}{k} - (k^2 + \tau^2)$.

2)

$$\nabla_T^3 B - \sigma_2(s)\nabla_T B + 3\tau'\nabla_T N = 0, \tag{18}$$

where $\sigma_2(s) = \tau^2 - k^2 - \frac{\tau''}{\tau}$.

Proof. Suppose that ϕ is general helix with respect to the Frenet frame $\{T$, N, $B\}$.

Then from (13), we have

$$N = \left(\frac{1}{k}\right)\nabla_T T, \quad B = \left(\frac{1}{\tau}\right)\nabla_T N + \left(\frac{k}{\tau}\right)T. \tag{19}$$

Since ϕ is general helix, we have

$$k'\tau = k\tau'. \tag{20}$$

If we substitute the equations (19) and (20) in (14), we obtain (17).

Conversely let us assume that the equation (17) holds. Differentiating covariantly (19) we obtain

$$\nabla_T N = -\left(\frac{k'}{k^2}\right)\nabla_T T + \left(\frac{1}{k}\right)\nabla_T^2 T$$

and so

$$\nabla_T^2 N = \left(-\frac{k'}{k^2}\right)' \nabla_T T - 2\left(\frac{k'}{k^2}\right)\nabla_T^2 T + \left(\frac{1}{k}\right)\nabla_T^3 T$$

$$= -k'T - (k^2+\tau^2)N + \left(\frac{k'\tau}{k}\right)B. \tag{21}$$

Also we obtain

$$\nabla_T^2 N = -k'T - (k^2+\tau^2)N + \tau' B. \tag{22}$$

Since (21) and (22) are equal, then

$$\tau' = \frac{k'\tau}{k}. \tag{23}$$

From (23), we obtain

$$\frac{k}{\tau} = \text{constant}.$$

This means that ϕ is a general helix.

2) Suppose that ϕ is general helix with respect to the Frenet frame. Then, from (13), we have

$$\nabla_T^3 B = (k'\tau + 2\tau' k)T + (\tau^3 + k^2\tau - \tau'')N - 3\tau\tau' B \tag{24}$$

$$N = -\left(\frac{1}{\tau}\right)\nabla_T B,\ B = \frac{1}{\tau}\nabla_T N + \frac{k}{\tau}T. \tag{25}$$

Now we replace (25) in the above expression of $\nabla_T^3 B$, and we obtain (18).

Conversely (18) holds.

Differentiating covariantly of

$$N = -\left(\frac{1}{\tau}\right)\nabla_T B,$$

we obtain

$$\nabla_T^2 N = -\left(\frac{1}{\tau}\right)'' \nabla_T B - 2\left(\frac{1}{\tau}\right)' \nabla_T^2 B - \left(\frac{1}{\tau}\right)\nabla_T^3 B. \tag{26}$$

If we use (13) and (18) we get,

$$\nabla_T^2 N = -\left(\frac{\tau' k}{\tau}\right)T - (k^2+\tau^2)N + \tau' B. \tag{27}$$

Also we obtain

$$\nabla_T^2 N = -k'T - (k^2+\tau^2)N + \tau' B. \tag{28}$$

By comparing (27) and (28), we obtain

$$\left(\frac{k}{\tau}\right)' = 0.$$

From this, we have

$$\frac{k}{\tau} = \text{constant}.$$

Then ϕ is a general helix.

COROLLARY 4.4. *Let ϕ be a curve in $\widetilde{\mathbf{SL}_2\mathbb{R}}$. ϕ is a circular helix with respect to the Frenet frame $\{T,\ N,\ B\}$ if and only if*

1)
$$\nabla_T^3 B = \tau(\tau^2 + k^2)N,$$

2)
$$\nabla_T^3 T = -k(\tau^2 + k^2)N.$$

THEOREM 4.5. *Let ϕ be a curve in $\widetilde{\mathbf{SL}_2\mathbb{R}}$. ϕ is a general helix with respect to the Frenet frame $\{T, N, B\}$, then*

$$\nabla_T^3 T + \sigma_1(s)\nabla_T B = 3k'\nabla_T N, \tag{29}$$

where $\sigma_1(s) = \frac{1}{\tau}\big(k'' - k^3 - k\tau^2\big)$.

Proof. Suppose that ϕ is general helix with respect to the Frenet frame $\{T,\ N,\ B\}$. If we substitute $\tau' k = k'\tau$ in (14), we obtain (29).

THEOREM 4.6. *Let $\phi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parameterized by arc length. Then ϕ is a general helix in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ if and only if*

1)
$$[\nabla_T T, \nabla_T^2 T, \nabla_T^3 T] = 0, \tag{30}$$

2)
$$[\nabla_T B, \nabla_T^2 B, \nabla_T^3 B] = 0, \tag{31}$$

3)
$$[\nabla_T N, \nabla_T^2 N, \nabla_T^3 N] = 0. \tag{32}$$

Proof. 1) From (12) and

$$\begin{pmatrix} \nabla_T T \\ \nabla_T^2 T \\ \nabla_T^3 T \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k^2 & k' & \tau k \\ -(3kk') & k'' - k^3 - k\tau^2 & 2\tau k' + k\tau' \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

we have

$$[\nabla_T T, \nabla_T^2 T, \nabla_T^3 T] = g(\nabla_T T \wedge \nabla_T^2 T, \nabla_T^3 T) = k^5 \left(\frac{\tau}{k}\right)'. \tag{33}$$

Since ϕ is general helix and (33), it follows

$$[\nabla_T T, \nabla_T^2 T, \nabla_T^3 T] = 0.$$

The proof is completed.

2) From (12) and

$$\begin{pmatrix} \nabla_T B \\ \nabla_T^2 B \\ \nabla_T^3 B \end{pmatrix} = \begin{pmatrix} 0 & -\tau & 0 \\ k\tau & -\tau' & -\tau^2 \\ k'\tau + 2k\tau' & -\tau'' + \tau^3 + \tau k^2 & -3\tau\tau' \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

we have

$$[\nabla_T B, \nabla_T^2 B, \nabla_T^3 B] = \tau^5 \left(\frac{\tau}{k}\right)' = 0.$$

The proof is completed.

3) From (12) and

$$\begin{pmatrix} \nabla_T N \\ \nabla_T^2 N \\ \nabla_T^3 N \end{pmatrix} = \begin{pmatrix} -k & 0 & \tau \\ -k' & -(k^2+\tau^2) & \tau' \\ -k''+k^3+k\tau^2 & -3(kk'+\tau\tau') & \tau''-\tau^3-\tau k^2 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

we have

$$[\nabla_T N, \nabla_T^2 N, \nabla_T^3 N] = (\tau^2+k^2)(k\tau''-\tau k'') - 3(kk'+\tau\tau')(\tau' k - k'\tau). \tag{34}$$

Since ϕ is general helix, we have

$$k\tau''-\tau k''=0, \quad k'\tau-\tau' k=0.$$

The proof is completed.

THEOREM 4.7. *Let $\phi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parameterized by arc length. If ϕ is a general helix in $\widetilde{\mathbf{SL}_2\mathbb{R}}$, then*

$$\det(\nabla_T^2 T, \nabla_T^3 T, \nabla_T^4 T) = 0. \tag{35}$$

Proof. From (13)

$$\begin{pmatrix} \nabla_T^2 T \\ \nabla_T^3 T \\ \nabla_T^4 T \end{pmatrix} = \begin{pmatrix} -k^2 & k' & \tau k \\ -(3kk') & k''-k^3-k\tau^2 & 2\tau k'+k\tau' \\ a(s) & b(s) & c(s) \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where

$$\begin{aligned} a(s) &= k^4+k^2\tau^2-4kk''-3k'^2 \\ b(s) &= k'''-3\tau(k'\tau+k\tau')-6k'k^2 \\ c(s) &= 3k'\tau'+3k''\tau-k\tau(k^2+\tau^2)+k\tau'', \end{aligned}$$

we have

$$\begin{aligned} \det(\nabla_T^2 T, \nabla_T^3 T, \nabla_T^4 T) &= (k'''k^2-2k^4k'+6k'^3-4kk'k''-3k^3\tau\tau'+k^2k'\tau^2)(k\tau'-k'\tau)+ \\ (36) \qquad & \quad (k''k^2-k^3\tau^2-k^5)(k''\tau-\tau''k)+3k^2k'(k'\tau''-k''\tau'). \end{aligned}$$

Since ϕ is a general helix in $\widetilde{\mathbf{SL}_2\mathbb{R}}$, we have

$$k\tau' = k'\tau, \quad k\tau'' = k''\tau. \tag{37}$$

Putting (37) in (36), we get (35).

5. RECTIFYING CURVE IN $\widetilde{\mathbf{SL}_2\mathbb{R}}$

Rectifying curves in the three-dimensional sphere, Euclidean space and Minkowski space are studied in [2, 14, 3, 6, 13]. If the position vector ϕ lies on its rectifying plane then $\phi(s)$ is called rectifying curve. Therefore, the position vector ϕ of a rectifying curve satisfies by definition [3] the equation

$$\phi(s) = \rho(s)T(s) + \sigma(s)B(s), \tag{38}$$

for some differentiable functions $\rho(s)$ and $\sigma(s)$.

THEOREM 5.1. *Let $\phi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parameterized by arc length. If ϕ is a rectifying curve in $\widetilde{\mathbf{SL}_2\mathbb{R}}$, then*

$$[\nabla_T N, \nabla_T^2 N, \nabla_T^3 N] = -ak^3(k'(1+(as+b)^2)+3ak(as+b)). \tag{39}$$

Proof. ϕ is a rectifying curve if and only if the ratio $\frac{\tau}{k}$ satisfies

$$\frac{\tau}{k} = as + b, \tag{40}$$

for some constants a, b with a $a \neq 0$.

By differentiating (40) we find

$$\begin{cases} \tau' k - k'\tau = ak^2 \\ \frac{\tau''}{\tau} - \frac{k''}{k} = 2a\frac{k'}{\tau}. \end{cases} \tag{41}$$

Moreover, using (34), (40) and (41), we get (39).

Corollary 5.2. *Let $\phi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ is a rectifying curve. Then*

$$[\nabla_T N, \nabla_T^2 N, \nabla_T^3 N] = 0$$

if and only if

$$k = \frac{\lambda}{(1 + (as+b)^2)^{\frac{3}{2}}}, \quad a\lambda \neq 0. \tag{42}$$

Proof. From (39) we get

$$k'(1 + (as+b)^2) + 3ak(as+b) = 0. \tag{43}$$

After solving (43) we find (42).

Corollary 5.3. *From (42) and (40) we get*

$$\tau = \frac{\lambda(as+b)}{(1 + (as+b)^2)^{\frac{3}{2}}}, \quad a\lambda \neq 0.$$

Theorem 5.4. *Let $\phi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ is a rectifying curve. Then*

$$g(\phi, \nabla_T^3 T) = k^2, \quad g(\phi, \nabla_T^3 B) = -\tau k, \quad g(\phi, \nabla_T^3 N) = 2k'.$$

Proof. Let us suppose that $\phi = \phi(s)$ is a unit speed rectifying curve. Then the position vector ϕ of the curve satisfies the equation (38).

Differentiating (38) with respect to s and applying (13) gives

$$\begin{cases} \rho' = 1 \\ \rho k - \sigma\tau = 0 \\ \sigma' = 0. \end{cases}$$

Therefore, it follows that

$$\begin{cases} \rho(s) = s + c_1 \\ \sigma(s) = c_2 \\ \rho(s)k - \sigma(s)\tau = 0. \end{cases} \tag{44}$$

Then

$$\phi(s) = (s + c_1)T(s) + c_2 B(s). \tag{45}$$

From the first and third equations of (44) we get

$$\frac{\tau}{k} = \frac{1}{c_2}s + \frac{c_1}{c_2}, \quad c_1, c_2 \in \mathbb{R},\ c_2 \neq 0. \tag{46}$$

Then from (14) and (46) we have $g(\phi, \nabla_T^3 T) = k^2$.

From (45), (46) and (24) we have $g(\phi, \nabla_T^3 B) = -\tau k$.

From (13) we obtain

$$\nabla_T^3 N = (-k'' + k^3 + k\tau^2)T - 3(kk' + \tau\tau')N + (\tau'' - \tau^3 - \tau k^2)B. \tag{47}$$

From the equations (45) and (47) we easily find $g(\phi, \nabla_T^3 N) = 2k'$.

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Bendehiba Senoussi
Department of Mathematics,
Ecole Normale Supérieure,
Mostaganem, Algeria
E-mail address: se2014bendhiba@gmail.com

Mohammed Bekkar
Department of Mathematics,
Faculty of Sciences,
University of Oran, Algeria
E-mail address: bekkar_99@yahoo.fr