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ON THE ASCENDING AND DESCENDING CHAIN
CONDITIONS IN THE LATTICE OF MONOID VARIETIES

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ABSTRACT. In this work we consider monoids as algebras with an associative binary operation and the nullary operation that fixes the identity element. We found an example of two varieties of monoids with finite subvariety lattices such that their join covers one of them and has a continuum cardinality subvariety lattice that violates the ascending chain condition and the descending chain condition.

Keywords: monoid, variety, lattice of varieties, ascending chain condition, descending chain condition.

1. INTRODUCTION AND SUMMARY

This paper is devoted to the examination of the lattice MON of all monoid varieties (referring to monoid varieties, we consider monoids as algebras with an associative binary operation and the nullary operation that fixes the identity element). There are a lot of articles about the monoid varieties. However, these articles are devoted mainly to the examination of identities of monoids. At the same time, although the first results about the lattice MON was found a long time ago (see [5, 15, 19]), only a little information was recently known about the lattice MON .

The situation has recently changed. There are papers, devoted to examination of identities of monoids, that contain also some non-trivial results about the lattice MON (see [6, 10, 11, 12], for instance). Several works devoted to the examination of the lattice MON was published in 2018 [2, 3, 4]. In these articles, several restrictions

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on the lattices of monoid varieties formulated in terms that are somehow connected with lattice identities were studied. At the same time, when studying lattices of varieties of algebras of various types, much attention has been also paid to the *finiteness conditions*, i.e., conditions that hold in every finite lattice (see [18, Section 10], for instance).

The subvariety lattice of a variety \mathbf{V} is denoted by $L(\mathbf{V})$. A variety is called *finitely generated* if it is generated by a finite algebra. In [13, Theorems 2.3, 2.4 and 2.5] I.V.L'vov proved that for an associative ring variety \mathbf{V} , the following are equivalent:

- a) the lattice $L(\mathbf{V})$ is finite;
- b) the lattice $L(\mathbf{V})$ satisfies the ascending chain condition;
- c) \mathbf{V} is finitely generated.

A similar result does not hold for group varieties. Indeed, there are only countably many finitely generated group varieties. At the same time, there are uncountably many periodic non-locally finite varieties of groups with subvariety lattice isomorphic to the 3-element chain [9]. As far as we know, in the group case the question about the equivalence of the claims a) and b) still remains open. In the semigroup case the claim c) is not equivalent to the claims a) and b). This follows from the folklore fact that the 2-element semigroup with zero multiplication with a new identity element adjoined generates a semigroup variety with countably infinitely many subvarieties [1, Fig. 5(b)]. In the semigroup case the claims a) and b) are not equivalent too. This fact follows from the results of the work [16] where an example of semigroup varieties whose subvariety lattice satisfies the descending chain condition but violates the ascending chain condition is given. Also, this example shows that the classes of semigroup varieties whose subvariety lattices are finite or satisfy the descending chain condition are not closed with respect to the join of the varieties and to coverings. The similar questions about the class of semigroup varieties whose subvariety lattices satisfy the ascending chain condition remain open.

In [7, Subsection 3.2] two monoid varieties \mathbf{U} and \mathbf{W} are exhibited such that the subvariety lattices of both varieties are finite, while the lattice $L(\mathbf{U} \vee \mathbf{W})$ is uncountably infinite and does not satisfy the ascending chain condition. Moreover, it follows from the proof of Theorem 3.4 in [7] that $L(\mathbf{U} \vee \mathbf{W})$ violates the descending chain condition. So, the classes of monoid varieties whose subvariety lattices are finite, satisfy the descending chain condition or satisfy the ascending chain condition are not closed with respect to the join of the varieties. At the same time, the results of [7] leave open the question about stability of these classes of varieties with respect to coverings.

In this work we find two monoid varieties with finite subvariety lattices such that their join covers one of them and has a continuum cardinality subvariety lattice that violates the ascending chain condition and the descending chain condition. Thus, we give a negative answer to the question noted in the previous paragraph.

In order to formulate the main result of the article, we need some definitions and notation. We denote by F the free semigroup over a countably infinite alphabet. As usual, elements of F and the alphabet are called *words* and *letters* respectively. The words and the letters are denoted by small Latin letters. However, the words unlike the letters are written in bold. The symbol F^1 stands for the semigroup F with a new identity element adjoined. We treat this identity element as the empty

word and denote it by λ . Expressions like $\mathbf{u} \approx \mathbf{v}$ are used for identities, whereas $\mathbf{u} = \mathbf{v}$ means that the words \mathbf{u} and \mathbf{v} coincide. One can introduce notation for the following three identities:

$$\sigma_1 : xyzxty \approx yxzaty, \quad \sigma_2 : xtyzxy \approx xtyzyx, \quad \sigma_3 : xzxyty \approx xzyxty.$$

Note that the identities σ_1 and σ_2 are dual to each other. A letter is called *simple* [*multiple*] *in a word* \mathbf{w} if it occurs in \mathbf{w} once [at least twice]. Note also that the identity σ_1 [respectively, σ_2] allows us to swap the adjacent non-latest [respectively, non-first] occurrences of two multiple letters, while the identity σ_3 allows us to swap a non-latest occurrence and a non-first occurrence of two multiple letters whenever these occurrences are adjacent to each other. Put

$$\Phi = \{x^2y \approx yx^2, x^2yz \approx xyxzx, \sigma_3\}.$$

The trivial variety of monoids is denoted by \mathbf{T} , while \mathbf{SL} denotes the variety of all semilattice monoids. For an identity system Σ , we denote by $\text{var } \Sigma$ the variety of monoids given by Σ . Let us fix notation for the following varieties:

$$\begin{aligned} \mathbf{C} &= \text{var}\{x^2 \approx x^3, xy \approx yx\}, \quad \mathbf{D}_1 = \text{var}\{x^2 \approx x^3, x^2y \approx xyx \approx yx^2\}, \\ \mathbf{D}_2 &= \text{var}\{\Phi, \sigma_1, \sigma_2\}, \quad \mathbf{M} = \text{var}\{\Phi, xyzxy \approx yxzxy, \sigma_2\}, \quad \mathbf{N} = \text{var}\{\Phi, \sigma_2\}. \end{aligned}$$

If \mathbf{V} is a monoid variety then we denote by $\overleftarrow{\mathbf{V}}$ the variety *dual to* \mathbf{V} , i.e., the variety consisting of monoids antiisomorphic to monoids from \mathbf{V} .

The main result of the paper is the following

- Theorem 1.1.** (i) *The variety $\mathbf{N} \vee \overleftarrow{\mathbf{M}}$ covers the variety \mathbf{N} .*
 (ii) *The interval $[\mathbf{M} \vee \overleftarrow{\mathbf{M}}, \mathbf{N} \vee \overleftarrow{\mathbf{M}}]$ of the lattice $L(\mathbf{N} \vee \overleftarrow{\mathbf{M}})$ contains continuum many subvarieties and does not satisfy the ascending chain condition and the descending chain condition.*

The proof of Theorem 1.1 implies that the lattice $L(\mathbf{N} \vee \overleftarrow{\mathbf{M}})$ "modulo" the interval $[\mathbf{M} \vee \overleftarrow{\mathbf{M}}, \mathbf{N} \vee \overleftarrow{\mathbf{M}}]$ has the form shown in Fig. 1.

We note also that one of the main goals of the work [7] is to construct several examples of finitely generated monoid varieties with continuum many subvarieties. It is verified in Erratum to [6] that $\overleftarrow{\mathbf{M}}$ and $\overleftarrow{\mathbf{N}}$ are finitely generated. Therefore, the variety $\mathbf{N} \vee \overleftarrow{\mathbf{M}}$ is finitely generated too. So, Theorem 1.1 gives a new example of finitely generated variety of monoids with continuum many subvarieties. Besides that, this theorem provides some more new examples of finitely generated varieties of monoids with continuum many subvarieties (see Corollary 3.1 below). Since the variety \mathbf{N} is finitely generated, this variety is locally finite. Moreover, \mathbf{N} is finitely based and has only finite many subvarieties, i.e., it is a Cross variety. It follows that the covering of a Cross monoid variety can have a continuum cardinality subvariety lattice that violates the ascending chain condition and the descending chain condition. Since the cover $\mathbf{M} \vee \overleftarrow{\mathbf{M}}$ of the Cross varieties \mathbf{M} and $\overleftarrow{\mathbf{M}}$ is non-finitely based [6], the class of Cross monoid varieties is not closed with respect to the formation of joins and of covers.

The article consists of three sections. Section 2 contains definitions, notation and auxiliary results, while Section 3 is devoted to the proof of Theorem 1.1.

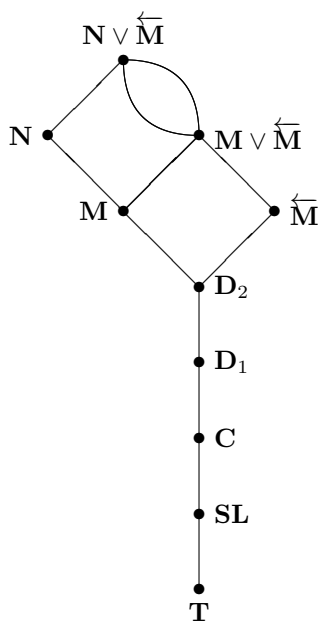


FIG. 1. The lattice $L(N \vee \overleftarrow{M})$

2. PRELIMINARIES

2.1. A useful construction. The following notion was introduced by Perkins [14] and often appeared in the literature (see [4, 6, 7, 8], for instance). Let W be a set of possibly empty words. We denote by \overline{W} the set of all subwords of words from W and by $I(\overline{W})$ the set $F^1 \setminus \overline{W}$. It is clear that $I(\overline{W})$ is an ideal of F^1 . Then $S(W)$ denotes the Rees quotient monoid $F^1/I(\overline{W})$. If $W = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$ then we will write $S(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k)$ rather than $S(\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\})$. A word \mathbf{w} is called an *isoterm* for a class of semigroups if no semigroup in the class satisfies any non-trivial identity of the form $\mathbf{w} \approx \mathbf{w}'$.

Lemma 2.1. *Let \mathbf{V} be a monoid variety and W a set of possibly empty words. Then $S(W)$ lies in \mathbf{V} if and only if each word in W is an isoterm for \mathbf{V} .*

Proof. It is easy to verify that it suffices to consider the case when W consists of one word (see the paragraph after Lemma 3.3 in [6]). Then necessity is obvious, while sufficiency is proved in [8, Lemma 5.3]. □

The following statement is dual to Proposition 1 in Erratum to [6].

Lemma 2.2. *The variety \mathbf{M} is generated by monoid $S(xysxty)$.* □

2.2. Word problems for the varieties \mathbf{M} and \mathbf{N} . We introduce a series of new notions and notation. The set of all simple [multiple] letters in a word \mathbf{w} is denoted by $\text{sim}(\mathbf{w})$ [respectively $\text{mul}(\mathbf{w})$]. The *content* of a word \mathbf{w} , i.e., the set of all letters occurring in \mathbf{w} , is denoted by $\text{con}(\mathbf{w})$. For a word \mathbf{w} and letters $x_1, x_2, \dots, x_k \in \text{con}(\mathbf{w})$, let $\mathbf{w}(x_1, x_2, \dots, x_k)$ denotes the word obtained from \mathbf{w} by retaining the letters x_1, x_2, \dots, x_k .

Let \mathbf{w} be a word and $\text{sim}(\mathbf{w}) = \{t_1, t_2, \dots, t_m\}$. We can assume without loss of generality that $\mathbf{w}(t_1, t_2, \dots, t_m) = t_1 t_2 \cdots t_m$. Then $\mathbf{w} = \mathbf{w}_0 t_1 \mathbf{w}_1 \cdots t_m \mathbf{w}_m$ where $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m$ are possibly empty words and $t_0 = \lambda$. The words $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m$ are called *blocks* of \mathbf{w} , while t_0, t_1, \dots, t_m are said to be *dividers* of \mathbf{w} . The representation of the word \mathbf{w} as a product of alternating dividers and blocks, starting with the divider t_0 and ending with the block \mathbf{w}_m is called a *decomposition* of the word \mathbf{w} . A block of the word \mathbf{w} is called a *k-block* if this block consists of k th occurrences of letters in \mathbf{w} . If every block of the word \mathbf{w} is either 1-block or 2-block then we say that the word \mathbf{w} is *reduced*. Recall that a word \mathbf{w} is called *linear* if every letter from $\text{con}(\mathbf{w})$ is simple in \mathbf{w} . We note that if \mathbf{w} is a reduced word and x is a multiple letter in \mathbf{w} then x cannot occur twice in the same block of \mathbf{w} . In other words, the following is true.

Remark 2.3. *Every block of a reduced word is a linear word.*

Further, let \mathbf{w} be a reduced word. Let us consider an arbitrary 1-block \mathbf{w}_i of \mathbf{w} . The maximal subwords of this block consisting of the letters whose second occurrences in \mathbf{w} lie in the same 2-block of \mathbf{w} are called *subblocks of 1-block* \mathbf{w}_i of the word \mathbf{w} . The representation of 1-block as a product of subblocks is called a *decomposition* of this 1-block. The notions of 2-blocks and 2-decompositions of the word \mathbf{w} are defined dually. The representation of the reduced word \mathbf{w} as a product of alternating dividers and decompositions of blocks is called *full decomposition* of \mathbf{w} . Below we underline dividers to distinguish them from blocks and we divide subblocks of the same block by the symbol ”|”. We illustrate the introduced notions by the following

Example 2.4. Put $\mathbf{w} = abcdxbyezaed$. Clearly, $\text{sim}(\mathbf{w}) = \{x, y, z\}$. Therefore, the letters x, y and z are the dividers of \mathbf{w} , while the words $abcd, cb, e$ and aed are the blocks of \mathbf{w} . Evidently, the blocks $abcd$ and e consist of the first occurrences of letters in \mathbf{w} , while the blocks cb and aed consist of the second occurrences of letters in \mathbf{w} . Consequently, $abcd$ and e are the 1-blocks of the word \mathbf{w} , while cb and aed are the 2-blocks of this word. So, the word \mathbf{w} is reduced. The second occurrences of the letters a, b and the letters c, d lie in the different 2-blocks, while the second occurrences of the letters b and c lie in the same 2-blocks. Therefore, the decomposition of 1-block $abcd$ has the form $a|bc|d$. The decomposition of 2-block cb consist of one subblock because the first occurrences of letters b and c lie in the same 1-block. Finally, the decomposition of 2-block aed equals $a|e|d$ because the first occurrences of the letters a, d and the letter e lie in different 1-blocks. Thus, the full decomposition of the word \mathbf{w} has the form $a|bc|dx|cb|ye|za|e|d$.

An identity $\mathbf{u} \approx \mathbf{v}$ is called *reduced* if the words \mathbf{u} and \mathbf{v} are reduced. The words \mathbf{u} and \mathbf{v} are said to be *equivalent* if their decompositions are

$$(2.1) \quad \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_m \mathbf{u}_m,$$

$$(2.2) \quad \mathbf{v}_0 t_1 \mathbf{v}_1 \cdots t_m \mathbf{v}_m$$

respectively and $\text{con}(\mathbf{u}_i) = \text{con}(\mathbf{v}_i)$ for every $0 \leq i \leq m$. Let \mathbf{u} and \mathbf{v} be equivalent reduced words. Suppose that (2.1) and (2.2) are the decompositions of these words respectively. In this case, the blocks \mathbf{u}_i and \mathbf{v}_i are said to be *corresponding* to each other. Let's say that the corresponding blocks \mathbf{u}_i and \mathbf{v}_i are *equivalent* if their

decompositions are

$$(2.3) \quad \mathbf{u}_{i1}\mathbf{u}_{i2} \dots \mathbf{u}_{ik_i},$$

$$(2.4) \quad \mathbf{v}_{i1}\mathbf{v}_{i2} \dots \mathbf{v}_{ik_i}$$

respectively and $\text{con}(\mathbf{u}_{ij}) = \text{con}(\mathbf{v}_{ij})$ for every $1 \leq j \leq k_i$. In this case, the subblocks \mathbf{u}_{ij} and \mathbf{v}_{ij} are said to be *corresponding* to each other. The equivalent words \mathbf{u} and \mathbf{v} are said to be *1-equivalent* if every two corresponding 1-blocks of these words are equivalent each other.

Lemma 2.5. *A reduced identity $\mathbf{u} \approx \mathbf{v}$ holds in the variety \mathbf{M} if and only if the words \mathbf{u} and \mathbf{v} are 1-equivalent.*

Proof. Necessity. Suppose the reduced identity $\mathbf{u} \approx \mathbf{v}$ holds in the variety \mathbf{M} .

First, we will prove that \mathbf{u} and \mathbf{v} are equivalent. It is proved in [4, Proposition 2.13] that an identity $\mathbf{u} \approx \mathbf{v}$ holds in the variety \mathbf{D}_1 if and only if $\text{sim}(\mathbf{u}) = \text{sim}(\mathbf{v})$, $\text{mul}(\mathbf{u}) = \text{mul}(\mathbf{v})$ and the simple letters appear in the words \mathbf{u} and \mathbf{v} in the same order. This fact and inclusion $\mathbf{D}_1 \subset \mathbf{M}$ imply that if (2.1) is the decomposition of the word \mathbf{u} then the decomposition of the word \mathbf{v} has the form (2.2). Let $z \in \text{mul}(\mathbf{u})$. Suppose that, the first occurrence of z in \mathbf{u} lies in the block \mathbf{u}_p , while the second occurrence of z in \mathbf{u} lies in the block \mathbf{u}_q for some $0 \leq p, q \leq m$. Remark 2.3 implies that $p < q$. The word ztz is an isoterm for the variety \mathbf{M} by Lemmas 2.1 and 2.2. Consequently, $\mathbf{u}(z, t_r) = zt_rz = \mathbf{v}(z, t_r)$ for every $p < r \leq q$. Moreover, $\mathbf{v}(z, t_s) \neq zt_sz$ and $\mathbf{v}(z, t_\ell) \neq zt_\ell z$ for all $1 \leq s \leq p$ and $q < \ell \leq m$. Therefore, the first occurrence of z in \mathbf{v} lies in the block \mathbf{v}_p , while the second occurrence of z in \mathbf{v} lies in the block \mathbf{v}_q . This implies that the words \mathbf{u} and \mathbf{v} are equivalent.

Suppose now that the words \mathbf{u} and \mathbf{v} are not 1-equivalent. Then there exist a 1-block \mathbf{u}_i of \mathbf{u} and letters $x, y \in \text{con}(\mathbf{u}_i)$ such that the second occurrences of x and y lie in different 2-blocks of \mathbf{u} , say \mathbf{u}_j and \mathbf{u}_k respectively, while the first occurrence of x precedes the first occurrence of y in \mathbf{u} , but the first occurrence of y precedes the first occurrence of x in \mathbf{v} . In view of Remark 2.3, we have that $i < j$. Besides that, we can assume without loss of generality that $j < k$. Since the words \mathbf{u} and \mathbf{v} are equivalent, $x \in \text{con}(\mathbf{v}_i) \cap \text{con}(\mathbf{v}_j)$ and $y \in \text{con}(\mathbf{v}_i) \cap \text{con}(\mathbf{v}_k)$. Then \mathbf{M} satisfies the identity $\mathbf{u}(x, y, t_{i+1}, t_{j+1}) \approx \mathbf{v}(x, y, t_{i+1}, t_{j+1})$, where

$$\mathbf{u}(x, y, t_{i+1}, t_{j+1}) = xyt_{i+1}xt_{j+1}y \text{ and } \mathbf{v}(x, y, t_{i+1}, t_{j+1}) = yxt_{i+1}xt_{j+1}y,$$

so that $xysxty$ is not an isoterm for \mathbf{M} . But this is impossible by Lemmas 2.1 and 2.2.

Sufficiency. Let (2.1) be the decomposition of \mathbf{u} . Since the words \mathbf{u} and \mathbf{v} are 1-equivalent, the decomposition of \mathbf{v} has the form (2.2). We are going to verify that the word $xysxty$ is an isoterm for the variety $\text{var}\{\mathbf{u} \approx \mathbf{v}\}$. Arguing by contradiction, we suppose that this variety satisfies a non-trivial identity $xysxty \approx \mathbf{w}$. We can assume without any loss that $xysxty = \mathbf{a}\xi(\mathbf{u})\mathbf{b}$ and $\mathbf{w} = \mathbf{a}\xi(\mathbf{v})\mathbf{b}$ for some words \mathbf{a} , \mathbf{b} and some endomorphism ξ of F^1 . Since the identity $xysxty \approx \mathbf{w}$ is non-trivial and the words \mathbf{u} and \mathbf{v} are equivalent, there are letters $c, d \in \text{mul}(\mathbf{u})$ such that $x \in \text{con}(\xi(c))$ and $y \in \text{con}(\xi(d))$. Clearly, $c \neq d$, whence $\xi(c) = x$ and $\xi(d) = y$. This implies that $\mathbf{a} = \mathbf{b} = \lambda$ and $s = \xi(t_\ell)$, $t = \xi(t_r)$ for some $1 \leq \ell < r \leq m$. Then, since the word \mathbf{u} is reduced, $c \in \text{con}(\mathbf{u}_i) \cap \text{con}(\mathbf{u}_j)$ and $d \in \text{con}(\mathbf{u}_p) \cap \text{con}(\mathbf{u}_q)$ for some i, j, p and q such that $i \leq p < \ell \leq j < r \leq q$. Further, we have that $c \in \text{con}(\mathbf{v}_i) \cap \text{con}(\mathbf{v}_j)$ and $d \in \text{con}(\mathbf{v}_p) \cap \text{con}(\mathbf{v}_q)$ because the words \mathbf{u} and \mathbf{v} are equivalent. If $i < p$ then the first occurrence of c precedes the first occurrence of d

in \mathbf{v} . We obtain a contradiction with the fact that $\mathbf{w} \neq xysxty$. If $i = p$ then the first occurrence of c precedes the first occurrence of d in \mathbf{v} again because the \mathbf{u} and \mathbf{v} are 1-equivalent and the second occurrences of the letters c and d lie in different blocks of the words \mathbf{u} and \mathbf{v} . So, the word $xysxty$ is an isoterms for $\text{var}\{\mathbf{u} \approx \mathbf{v}\}$. In view of Lemmas 2.1 and 2.2, the identity $\mathbf{u} \approx \mathbf{v}$ holds in \mathbf{M} . \square

If \mathbf{u} and \mathbf{v} are words and ε is an identity then we will write $\mathbf{u} \overset{\varepsilon}{\approx} \mathbf{v}$ in the case when the identity $\mathbf{u} \approx \mathbf{v}$ follows from ε .

Lemma 2.6. *A reduced identity $\mathbf{u} \approx \mathbf{v}$ holds in the variety \mathbf{N} if and only if the words \mathbf{u} and \mathbf{v} are equivalent and corresponding 1-blocks of the words \mathbf{u} and \mathbf{v} equal each other.*

Proof. Necessity. Suppose the reduced identity $\mathbf{u} \approx \mathbf{v}$ holds in the variety \mathbf{N} . Lemma 2.5 and inclusion $\mathbf{M} \subset \mathbf{N}$ imply that the words \mathbf{u} and \mathbf{v} are 1-equivalent. In particular, the words \mathbf{u} and \mathbf{v} are equivalent. Then if (2.1) is the decomposition of the word \mathbf{u} then the decomposition of the word \mathbf{v} has the form (2.2). Suppose that there exist two corresponding 1-blocks \mathbf{u}_i and \mathbf{v}_i such that $\mathbf{u}_i \neq \mathbf{v}_i$. Clearly, $i < m$. Since the 1-blocks \mathbf{u}_i and \mathbf{v}_i are 1-equivalent, there is a subblock \mathbf{u}' of the block \mathbf{u}_i which does not coincide with the corresponding subblock \mathbf{v}' of the block \mathbf{v}_i . In view of Remark 2.3, the subblocks \mathbf{u}' and \mathbf{v}' are linear words. At the same time, $\text{con}(\mathbf{u}') = \text{con}(\mathbf{v}')$ because the blocks \mathbf{u}_i and \mathbf{v}_i are 1-equivalent. Therefore, there exist letters x and y such that x precedes y in \mathbf{u}' , while y precedes x in \mathbf{v}' . Then \mathbf{N} satisfies the identity

$$xyt_{i+1}\mathbf{w} = \mathbf{u}(x, y, t_{i+1}) \approx \mathbf{v}(x, y, t_{i+1}) = yxt_{i+1}\mathbf{w}',$$

where $\mathbf{w}, \mathbf{w}' \in \{xy, yx\}$. Then \mathbf{N} satisfies the identities

$$xyt_{i+1}xy \overset{\sigma_2}{\approx} xyt_{i+1}\mathbf{w} \approx yxt_{i+1}\mathbf{w}' \overset{\sigma_2}{\approx} yxt_{i+1}xy.$$

We obtain a contradiction with the fact that the varieties \mathbf{M} and \mathbf{N} are different.

Sufficiency. Let (2.1) be the decomposition of \mathbf{u} . Since the word \mathbf{u} and \mathbf{v} are equivalent, the decomposition of \mathbf{v} has the form (2.2). Consider arbitrary corresponding 2-blocks \mathbf{u}_i and \mathbf{v}_i . The words \mathbf{u}_i and \mathbf{v}_i are linear (see Remark 2.3) and depend on the same letters (since \mathbf{u} and \mathbf{v} are equivalent). The identity σ_2 allows us to swap the second occurrences of two multiple letters whenever these occurrences are adjacent to each other. Thus, if we replace the 2-block \mathbf{u}_i by \mathbf{v}_i in \mathbf{u} then the word we obtain should be equal to \mathbf{u} in \mathbf{N} . By the hypothesis, the corresponding 1-blocks of the words \mathbf{u} and \mathbf{v} equal each other. Therefore, the identity

$$\mathbf{u} = t_0\mathbf{u}_0t_1\mathbf{u}_1 \cdots t_m\mathbf{u}_m \overset{\sigma_2}{\approx} t_0\mathbf{v}_0t_1\mathbf{v}_1 \cdots t_m\mathbf{v}_m = \mathbf{v},$$

holds in the variety \mathbf{N} . \square

2.3. Some words and their properties. We introduce some new notation. As usual, the symbol \mathbb{N} stands for the set of all natural numbers. For all $n, k \in \mathbb{N}$, we put

$$\mathcal{M}_n^k = \{1\} \times \underbrace{\mathbb{N}_n \times \mathbb{N}_n \times \cdots \times \mathbb{N}_n}_{k-1 \text{ copies}}.$$

where $\mathbb{N}_n = \{1, 2, \dots, n\}$. If $\gamma = (1, i_1, i_2, \dots, i_{k-1}) \in \mathcal{M}_n^k$ and $1 \leq j \leq n$ then we put

$$\gamma + j = (1, i_1, i_2, \dots, i_{k-1}, j) \in \mathcal{M}_n^{k+1}.$$

The usual lexicographical order is defined on the set \mathcal{M}_n^k . So, the expression

$$\prod_{\gamma \in \mathcal{M}_n^k} \mathbf{w}_\gamma$$

mean an abbreviated notation of the product of the words \mathbf{w}_γ in ascending order γ . Put

$$\mathbf{c}_n = \prod_{\gamma \in \mathcal{M}_n^n} \left(\prod_{j=1}^n s_\gamma^{(j)} x_\gamma^{(j)} \right) \text{ and } \mathbf{d}_n^{(k)} = s_k \cdot \left(\prod_{\gamma \in \mathcal{M}_n^k} \left(\prod_{j=1}^n x_\gamma^{(j)} \left(\prod_{\ell=1}^n x_{\gamma+j}^{(\ell)} \right) \right) \right),$$

$$\mathbf{e}_m = s_m x_m t_m y_m \text{ and } \mathbf{f}_m = s_m x_m x_{m+1} y_{m+1} y_m$$

for all $n, k \geq 1$ and $m \geq 0$. Further, for any natural n put

$$\mathbf{a}_n = xy \cdot \left(\prod_{i=1}^n \mathbf{d}_{2n}^{(2i-1)} \right) \cdot \mathbf{c}_{2n} \cdot \left(\prod_{i=n-1}^1 \mathbf{d}_{2n}^{(2i)} \right) \cdot sx \cdot \left(\prod_{i=1}^{2n} x_1^{(i)} \right) \cdot y,$$

$$\mathbf{a}'_n = yx \cdot \left(\prod_{i=1}^n \mathbf{d}_{2n}^{(2i-1)} \right) \cdot \mathbf{c}_{2n} \cdot \left(\prod_{i=n-1}^1 \mathbf{d}_{2n}^{(2i)} \right) \cdot sx \cdot \left(\prod_{i=1}^{2n} x_1^{(i)} \right) \cdot y,$$

$$\mathbf{b}_n = x_0 y_0 \left(\prod_{i=1}^n \mathbf{f}_{2i-1} \right) \cdot \mathbf{e}_{2n} \cdot \left(\prod_{i=n-1}^0 \mathbf{f}_{2i} \right),$$

$$\mathbf{b}'_n = y_0 x_0 \left(\prod_{i=1}^n \mathbf{f}_{2i-1} \right) \cdot \mathbf{e}_{2n} \cdot \left(\prod_{i=n-1}^0 \mathbf{f}_{2i} \right).$$

We note that the words $\mathbf{a}_n, \mathbf{a}'_n, \mathbf{b}_n$ and \mathbf{b}'_n are reduced. The following simple fact can be easily verified directly.

Remark 2.7. *The expression*

$$(2.5) \quad \chi(xy) \cdot \left(\prod_{i=1}^{n-1} s_i \left(\prod_{\gamma \in \mathcal{M}_{2n}^{2i-1}} \left(\prod_{j=1}^{2n} x_\gamma^{(j)} \left| \prod_{\ell=1}^{2n} x_{\gamma+j}^{(\ell)} \right| \right) \right) \right) \cdot \left(\frac{s_{2n-1}}{\prod_{\gamma \in \mathcal{M}_{2n}^{2n-1}} \left(\prod_{j=1}^{2n} x_\gamma^{(j)} \left(\prod_{\ell=1}^{2n} |x_{\gamma+j}^{(\ell)}| \right) \right)} \right) \left(\prod_{\gamma \in \mathcal{M}_{2n}^{2n}} \left(\prod_{j=1}^{2n} s_\gamma^{(j)} x_\gamma^{(j)} \right) \right) \cdot \left(\prod_{i=n-1}^1 s_i \left(\prod_{\gamma \in \mathcal{M}_{2n}^{2i}} \left(\prod_{j=1}^{2n} x_\gamma^{(j)} \left| \prod_{\ell=1}^{2n} x_{\gamma+j}^{(\ell)} \right| \right) \right) \right) \cdot sx \cdot \left| \prod_{i=1}^{2n} x_1^{(i)} \right| \cdot y$$

is the full decomposition of \mathbf{a}_n whenever $\chi(xy) = xy$, and the full decomposition of \mathbf{a}'_n whenever $\chi(xy) = yx$.

The following two observations play an important role below.

Remark 2.8. *Suppose that*

$$(2.6) \quad \{\zeta_\ell(x_{2\ell+1}), \zeta_\ell(y_{2\ell+1})\} = \{x_{2\ell+1}, y_{2\ell+1}\}$$

for all $0 \leq \ell \leq n - 1$. Then every subword of length > 1 of the word

$$(2.7) \quad x_0 y_0 \left(\prod_{i=1}^n \mathbf{f}_{2i-1} \right) \cdot \mathbf{e}_{2n} \cdot \left(\prod_{i=n-1}^0 s_{2i} x_{2i} \zeta_i(x_{2i+1}) \zeta_i(y_{2i+1}) y_{2i} \right)$$

has exactly one occurrence in this word.

Remark 2.9. *Suppose that*

$$(2.8) \quad \{\zeta_\alpha(x_\alpha^{(i)}) \mid 1 \leq i \leq 2n\} = \{x_\alpha^{(i)} \mid 1 \leq i \leq 2n\}$$

for all odd $1 \leq \ell \leq 2n - 1$ and for all $\alpha \in \mathcal{M}_{2n}^\ell$. Then every subword of length > 1 of the word

$$(2.9) \quad xy \cdot \left(\prod_{i=1}^{n-1} \underline{s}_i \left(\prod_{\gamma \in \mathcal{M}_{2n}^{2i-1}} \left(\prod_{j=1}^{2n} x_\gamma^{(j)} \mid \prod_{\ell=1}^{2n} x_{\gamma+j}^{(\ell)} \right) \right) \right) \\ \cdot \left(\underline{s}_{2n-1} \prod_{\gamma \in \mathcal{M}_n^{2n-1}} \left(\prod_{j=1}^{2n} x_\gamma^{(j)} \mid \prod_{\ell=1}^{2n} |x_{\gamma+j}^{(\ell)}| \right) \right) \cdot \left(\prod_{\gamma \in \mathcal{M}_{2n}^{2n}} \left(\prod_{j=1}^{2n} \underline{s}_\gamma^{(j)} x_\gamma^{(j)} \right) \right) \\ \cdot \left(\prod_{i=n-1}^1 \underline{s}_i \left(\prod_{\gamma \in \mathcal{M}_{2n}^{2i}} \left(\prod_{j=1}^{2n} x_\gamma^{(j)} \mid \prod_{\ell=1}^{2n} \zeta_{\gamma+j}(x_{\gamma+j}^{(\ell)}) \right) \right) \right) \cdot \underline{s}x \cdot \left| \prod_{i=1}^{2n} \zeta_1(x_1^{(i)}) \right| \cdot y$$

has exactly one occurrence in this word.

Remarks 2.8 and 2.9 follow from the directly verifiable fact that if \mathbf{w} is one of the words (2.7) or (2.9) and ab is a subword of the word \mathbf{w} then this subword has exactly one occurrence in this word.

Lemma 2.10. *Let n be a natural number, ξ be an endomorphism of F^1 and $\mathbf{w} \approx \mathbf{w}'$ be a non-trivial identity. Suppose that the word \mathbf{w} coincides with the word (2.7) where $\zeta_1, \zeta_2, \dots, \zeta_{n-1}$ are endomorphisms of F^1 such that the equality (2.6) is true for all $0 \leq \ell \leq n - 1$. Then if $\mathbf{w} = \mathbf{u}\xi(\mathbf{a}_k)\mathbf{v}$ and $\mathbf{w}' = \mathbf{u}\xi(\mathbf{a}'_k)\mathbf{v}$ for some words \mathbf{u} and \mathbf{v} and some $k \geq n$ then $n = k$.*

Proof. We note that $\xi(x) \neq \lambda$ and $\xi(y) \neq \lambda$ because the identity $\mathbf{w} \approx \mathbf{w}'$ is non-trivial. It follows that the length of the word $\xi(xy)$ is more than 2. In view of Remark 2.8, the words $\xi(x)$ and $\xi(y)$ are letters. Remark 2.7 implies that the words \mathbf{a}_k and \mathbf{a}'_k are 1-equivalent. According to Lemma 2.5 the identity $\mathbf{a}_k \approx \mathbf{a}'_k$ holds in \mathbf{M} . Consequently, \mathbf{M} satisfies the identity $\mathbf{w} \approx \mathbf{w}'$. Since the first occurrences of the letters $\xi(x)$ and $\xi(y)$ occur in the words \mathbf{w} and \mathbf{w}' in the opposite order, Lemma 2.5 implies that some subblock of some 1-block of the word \mathbf{w} contains the subword $\xi(xy)$. It is easy to see that the full decomposition of the word \mathbf{w} has the form

$$(2.10) \quad x_0y_0 \cdot \left(\prod_{i=1}^{n-1} \underline{s}_{2i-1} x_{2i-1} \mid x_{2i}y_{2i} \mid y_{2i-1} \right) \cdot \underline{s}_{2n-1} x_{2n-1} \mid x_{2n} \mid y_{2n} \mid y_{2n-1} \\ \cdot \underline{s}_{2n} x_{2n} t_{2n} y_{2n} \cdot \left(\prod_{i=n-1}^0 \underline{s}_{2i} x_{2i} \mid \zeta_i(x_{2i+1}) \zeta_i(y_{2i+1}) \mid y_{2i} \right).$$

Then $\xi(xy) = x_{2p}y_{2p}$ for some $0 \leq p < n$, whence $\xi(x) = x_{2p}$ and $\xi(y) = y_{2p}$. So,

$$\zeta_p(x_{2p+1})\zeta_p(y_{2p+1}) = \prod_{i=1}^{2k} \xi(x_1^{(i)}).$$

In view of Remark 2.8 and the equality (2.6), there are $c_1 < d_1$ such that $\xi(x_1^{(c_1)}) = \zeta_p(x_{2p+1})$ and $\xi(x_1^{(d_1)}) = \zeta_p(y_{2p+1})$. Since the first occurrence of $x_1^{(c_1)}$ precedes the first occurrence of $x_1^{(d_1)}$ in \mathbf{a}_k , we have that the first occurrence of $\xi(x_1^{(c_1)})$ precedes the first occurrence of $\xi(x_1^{(d_1)})$ in \mathbf{w} , whence $\xi(x_1^{(c_1)}) = x_{2p+1}$ and $\xi(x_1^{(d_1)}) = y_{2p+1}$.

Further, we will prove by induction that for all $1 \leq q \leq 2n - 2p$ the equalities

$$(2.11) \quad x_{2p+q} = \xi(x_{\alpha_q}^{c_q}) \text{ and } y_{2p+q} = \xi(x_{\beta_q}^{d_q})$$

are true for some letters $x_{\alpha_q}^{c_q}$ and $x_{\beta_q}^{d_q}$ such that $\alpha_q, \beta_q \in \mathcal{M}_{2k}^q$ and the first occurrence of $x_{\alpha_q}^{c_q}$ precedes the first occurrence of $x_{\beta_q}^{d_q}$ in \mathbf{w} , i.e.,

$$(2.12) \quad \alpha_q \leq \beta_q \text{ and if } \alpha_q = \beta_q \text{ then } c_q < d_q.$$

The induction base is considered in the previous paragraph. Suppose that for all $1 \leq r < q \leq 2n - 2p$ there exist letters $x_{\alpha_r}^{c_r}$ and $x_{\beta_r}^{d_r}$ such that $x_{2p+r} = \xi(x_{\alpha_r}^{c_r})$ and $y_{2p+r} = \xi(x_{\beta_r}^{d_r})$, $\alpha_r, \beta_r \in \mathcal{M}_{2k}^r$, $\alpha_r \leq \beta_r$ and if $\alpha_r = \beta_r$ then $c_r \leq d_r$. We need to check that there are letters $x_{\alpha_q}^{c_q}$ and $x_{\beta_q}^{d_q}$ such that $\alpha_q, \beta_q \in \mathcal{M}_{2k}^q$ and the claims (2.11) and (2.12) are true.

If $a, b \in \text{mul}(\mathbf{w})$ and $i \in \{1, 2\}$ then $\mathbf{w}_i[a, b]$ denotes the subword of the word \mathbf{w} located between i th occurrences of a and b .

Suppose that q is odd. Then

$$\zeta_{\frac{2p+q-1}{2}}(x_{2p+q})\zeta_{\frac{2p+q-1}{2}}(y_{2p+q}) = \xi(\mathbf{w}_2[x_{\alpha_{q-1}}^{c_{q-1}}, x_{\beta_{q-1}}^{d_{q-1}}]).$$

We note that if $i = 2$ and $\alpha_{q-1} = \beta_{q-1}$ then

$$(2.13) \quad \mathbf{w}_i[x_{\alpha_{q-1}}^{c_{q-1}}, x_{\beta_{q-1}}^{d_{q-1}}] = \left(\prod_{\ell=1}^{2k} x_{\alpha_{q-1}+c_{q-1}}^{(\ell)} \right) \cdot \left(\prod_{j=c_{q-1}+1}^{d_{q-1}-1} x_{\alpha_{q-1}}^{(j)} \left(\prod_{\ell=1}^{2k} x_{\alpha_{q-1}+j}^{(\ell)} \right) \right),$$

while if $i = 2$ and $\alpha_{q-1} < \beta_{q-1}$ then

$$(2.14) \quad \begin{aligned} \mathbf{w}_i[x_{\alpha_{q-1}}^{c_{q-1}}, x_{\beta_{q-1}}^{d_{q-1}}] &= \left(\prod_{\ell=1}^{2k} x_{\alpha_{q-1}+c_{q-1}}^{(\ell)} \right) \cdot \left(\prod_{j=c_{q-1}+1}^{2k} x_{\alpha_{q-1}}^{(j)} \left(\prod_{\ell=1}^{2k} x_{\alpha_{q-1}+j}^{(\ell)} \right) \right) \\ &\cdot \left(\prod_{\alpha_{q-1} < \gamma < \beta_{q-1}} \left(\prod_{j=1}^{2k} x_{\gamma}^{(j)} \left(\prod_{\ell=1}^{2k} x_{\gamma+j}^{(\ell)} \right) \right) \right) \cdot \left(\prod_{j=1}^{d_{q-1}-1} x_{\beta_{q-1}}^{(j)} \left(\prod_{\ell=1}^{2k} x_{\beta_{q-1}+j}^{(\ell)} \right) \right). \end{aligned}$$

By the induction assumption, the endomorphism ξ maps all letters located between the first occurrences of $x_{\alpha_{q-1}}^{(c_{q-1})}$ and $x_{\beta_{q-1}}^{(d_{q-1})}$ in \mathbf{w} into the empty word, i.e.,

$$\xi(\mathbf{w}_1[x_{\alpha_{q-1}}^{(c_{q-1})}, x_{\beta_{q-1}}^{(d_{q-1})}]) = \lambda.$$

Then, taking into account the equalities (2.13) and (2.14), we have that if $\alpha_{q-1} = \beta_{q-1}$ then the equality

$$(2.15) \quad \xi(\mathbf{w}_i[x_{\alpha_{q-1}}^{c_{q-1}}, x_{\beta_{q-1}}^{d_{q-1}}]) = \prod_{j=c_{q-1}}^{d_{q-1}-1} \left(\prod_{\ell=1}^{2k} \xi(x_{\alpha_{q-1}+j}^{(\ell)}) \right)$$

with $i = 2$ is true, while if $\alpha_{q-1} < \beta_{q-1}$ then the equality

$$(2.16) \quad \begin{aligned} \xi(\mathbf{w}_i[x_{\alpha_{q-1}}^{c_{q-1}}, x_{\beta_{q-1}}^{d_{q-1}}]) &= \left(\prod_{j=c_{q-1}}^{2k} \left(\prod_{\ell=1}^{2k} \xi(x_{\alpha_{q-1}+j}^{(\ell)}) \right) \right) \\ &\cdot \left(\prod_{\alpha_{q-1} < \gamma < \beta_{q-1}} \left(\prod_{j=1}^{2k} \left(\prod_{\ell=1}^{2k} \xi(x_{\gamma+j}^{(\ell)}) \right) \right) \right) \cdot \left(\prod_{j=1}^{d_{q-1}-1} \left(\prod_{\ell=1}^{2k} \xi(x_{\beta_{q-1}+j}^{(\ell)}) \right) \right) \end{aligned}$$

with $i = 2$ is true. This fact and Remark 2.8 imply that there exist $\alpha_q, \beta_q \in \mathcal{M}_{2k}^q$ and $1 \leq c_q, d_q \leq 2k$ such that the claim (2.12) is true, $\zeta_{\frac{2p+q-1}{2}}(x_{2p+q}) = \xi(x_{\alpha_q}^{c_q})$ and $\zeta_{\frac{2p+q-1}{2}}(y_{2p+q}) = \xi(x_{\beta_q}^{d_q})$. Since the first occurrence of $x_{\alpha_q}^{c_q}$ precedes the first occurrence of $x_{\beta_q}^{d_q}$ in \mathbf{a}_k , we obtain that the first occurrence of $\xi(x_{\alpha_q}^{c_q})$ precedes the first occurrence of $\xi(x_{\beta_q}^{d_q})$ in \mathbf{w} , whence $\xi(x_{\alpha_q}^{c_q}) = \zeta_{\frac{2p+q-1}{2}}(x_{2p+q}) = x_{2p+q}$ and $\xi(x_{\beta_q}^{d_q}) = \zeta_{\frac{2p+q-1}{2}}(y_{2p+q}) = y_{2p+q}$. So, we have proved the equality (2.11) for all odd q .

Suppose now that q is even. Then $x_{2p+q}y_{2p+q} = \xi(\mathbf{w}_1[x_{\alpha_{q-1}}^{c_{q-1}}, x_{\beta_{q-1}}^{d_{q-1}}])$. Note that if $\alpha_{q-1} = \beta_{q-1}$ then the equality (2.13) is true whenever $i = 1$ and if $\alpha_{q-1} < \beta_{q-1}$ then the equality (2.14) with $i = 1$ is true. By the induction assumption, the endomorphism ξ maps all letters located between the second occurrences of $x_{\alpha_{q-1}}^{(c_{q-1})}$ and $x_{\beta_{q-1}}^{(d_{q-1})}$ in \mathbf{w} into the empty word, i.e. $\xi(\mathbf{w}_2[x_{\alpha_{q-1}}^{(c_{q-1})}, x_{\beta_{q-1}}^{(d_{q-1})}]) = \lambda$. Then, taking into account the equalities (2.13) and (2.14), we have that if $\alpha_{q-1} = \beta_{q-1}$ then the equality (2.15) with $i = 1$ is true and if $\alpha_{q-1} < \beta_{q-1}$ then the equality (2.16) with $i = 1$ is true. This fact and Remark 2.8 imply that there exist $\alpha_q, \beta_q \in \mathcal{M}_{2k}^q$ and $1 \leq c_q, d_q \leq 2k$ such that the claim (2.12) is true, $x_{2p+q} = \xi(x_{\alpha_q}^{c_q})$ and $y_{2p+q} = \xi(x_{\beta_q}^{d_q})$.

Thus, we have shown that, for any $1 \leq q \leq 2n - 2p$, there are letters $x_{\alpha_q}^{c_q}$ and $x_{\beta_q}^{d_q}$ such that $\alpha_q, \beta_q \in \mathcal{M}_{2k}^q$ and the claims (2.11) and (2.12) are true. In particular, $x_{2n} = \xi(x_{\alpha_{2n-2p}}^{c_{2n-2p}})$ and $y_{2n} = \xi(x_{\beta_{2n-2p}}^{d_{2n-2p}})$. It follows that $\xi(\mathbf{w}_2[x_{\alpha_{2n-2p}}^{c_{2n-2p}}, x_{\beta_{2n-2p}}^{d_{2n-2p}}]) = t_{2n}$. If $k > n$ then $\text{con}(\mathbf{w}_2[x_{\alpha_{2n-2p}}^{c_{2n-2p}}, x_{\beta_{2n-2p}}^{d_{2n-2p}}]) \subseteq \text{mul}(\mathbf{a}_k)$. This contradicts the fact that $t_{2n} \in \text{sim}(\mathbf{w})$. \square

3. PROOF OF THE MAIN RESULT

(i) We are going to verify that the lattice $L(\mathbf{N} \vee \overleftarrow{\mathbf{M}})$ "modulo" the interval $[\mathbf{M} \vee \overleftarrow{\mathbf{M}}, \mathbf{N} \vee \overleftarrow{\mathbf{M}}]$ has the form shown in Fig. 1. In view of [4, Proposition 5.2] and [6, Fig. 1], the lattices $L(\mathbf{N})$ and $L(\mathbf{M} \vee \overleftarrow{\mathbf{M}})$ have the form shown in Fig. 1. Let \mathbf{V} be a proper subvariety of the variety $\mathbf{N} \vee \overleftarrow{\mathbf{M}}$ which is not contained in \mathbf{N} and $\mathbf{M} \vee \overleftarrow{\mathbf{M}}$. We need to check that \mathbf{V} belongs to the interval $[\mathbf{M} \vee \overleftarrow{\mathbf{M}}, \mathbf{N} \vee \overleftarrow{\mathbf{M}}]$.

A variety of monoids is said to be *completely regular* if it consists of *completely regular monoids* (i.e., unions of groups). If the variety \mathbf{V} is completely regular then it is a variety of *bands* (i.e. idempotent monoids) because it satisfies the identity

$$(3.1) \quad x^2 \approx x^3.$$

Evidently, every variety of bands which satisfies the identity

$$(3.2) \quad x^2y \approx yx^2,$$

is commutative. Therefore, \mathbf{V} is one of the varieties \mathbf{T} or \mathbf{SL} , a contradiction. So, \mathbf{V} is non-completely regular. Suppose that $\mathbf{D}_2 \not\subseteq \mathbf{V}$. Then it follows from [4, Lemma 2.15] that \mathbf{V} satisfies the identity

$$(3.3) \quad xyx \approx x^qyx^r$$

where either $q > 1$ or $r > 1$. If $q > 1$ then \mathbf{V} satisfies the identities

$$xyx \stackrel{(3.3)}{\approx} x^qyx^r \stackrel{(3.1)}{\approx} x^2yx^r \stackrel{(3.2)}{\approx} yx^{2+r} \stackrel{(3.1)}{\approx} yx^2 \stackrel{(3.2)}{\approx} x^2y,$$

whence $\mathbf{V} \subseteq \mathbf{D}_1$, a contradiction. If $r > 1$ then the identities

$$xyx \stackrel{(3.3)}{\approx} x^qyx^r \stackrel{(3.1)}{\approx} x^qyx^2 \stackrel{(3.2)}{\approx} x^{2+q}y \stackrel{(3.1)}{\approx} x^2y \stackrel{(3.2)}{\approx} yx^2$$

hold in \mathbf{V} . We obtain a contradiction again. Thus, $\mathbf{D}_2 \subseteq \mathbf{V}$.

If $\mathbf{M} \not\subseteq \mathbf{V}$ then it follows from [4, Lemma 4.9(i)] that the variety \mathbf{V} satisfies the identity σ_1 . Therefore, $\mathbf{V} \subseteq \overleftarrow{\mathbf{N}}$. The lattice $L(\overleftarrow{\mathbf{N}})$ is isomorphic to the lattice $L(\mathbf{N})$. Besides that, $\mathbf{V} \neq \overleftarrow{\mathbf{N}}$. This implies the wrong inclusion $\mathbf{V} \subseteq \overleftarrow{\mathbf{M}} \subset \mathbf{M} \vee \overleftarrow{\mathbf{M}}$. By symmetry, if $\overleftarrow{\mathbf{M}} \not\subseteq \mathbf{V}$ then $\mathbf{V} \subseteq \mathbf{N}$. This inclusion is also impossible. Thus, we have shown that $\mathbf{M} \vee \overleftarrow{\mathbf{M}} \subseteq \mathbf{V}$. Therefore, $\mathbf{V} \in [\mathbf{M} \vee \overleftarrow{\mathbf{M}}, \mathbf{N} \vee \overleftarrow{\mathbf{M}}]$.

(ii) We denote an identity basis of the variety $\mathbf{N} \vee \overleftarrow{\mathbf{M}}$ by Σ . Let K be a subset of \mathbb{N} . Put $\Sigma_K = \{\mathbf{a}_n \approx \mathbf{a}'_n \mid n \in K\}$. We are going to verify that different subsets of the form Σ_K define different subvarieties within the variety $\mathbf{N} \vee \overleftarrow{\mathbf{M}}$. Arguing by contradiction, suppose that there are n and $K \subseteq \mathbb{N}$ such that $n \notin K$ and the identity $\mathbf{a}_n \approx \mathbf{a}'_n$ follows from the identity system $\Sigma \cup \Sigma_K$. Then there exists a sequence of words $\mathbf{a}_n = \mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m = \mathbf{a}'_n$ such that, for any $i \in \{0, 1, \dots, m-1\}$ there exist words $\mathbf{u}_i, \mathbf{v}_i \in F^1$, an endomorphism ξ_i of F^1 and an identity $\mathbf{p}_i \approx \mathbf{q}_i \in \Sigma \cup \Sigma_K$ such that $\mathbf{w}_i = \mathbf{u}_i \xi_i(\mathbf{p}_i) \mathbf{v}_i$ and $\mathbf{w}_{i+1} = \mathbf{u}_i \xi_i(\mathbf{q}_i) \mathbf{v}_i$. We can assume without loss of generality that $\mathbf{w}_i \neq \mathbf{w}_{i+1}$ for all $i \in \{0, 1, \dots, m-1\}$. The words \mathbf{a}_n and \mathbf{a}'_n are 1-equivalent by Remark 2.7. But the 1-block xy of \mathbf{a}_n does not coincide with the corresponding 1-block yx of \mathbf{a}'_n . Then Lemma 2.6 implies that the variety \mathbf{N} violates the identity $\mathbf{a}_n \approx \mathbf{a}'_n$, whence there is a number $r \in \{0, 1, \dots, m-1\}$ such that $\mathbf{p}_r \approx \mathbf{q}_r$ equals one of the identities $\mathbf{a}_k \approx \mathbf{a}'_k$ or $\mathbf{a}'_k \approx \mathbf{a}_k$ for some $k \neq n$. Let r be the least number with such a property. Then the identity $\mathbf{a}_n \approx \mathbf{w}_r$ holds in the variety $\mathbf{N} \vee \overleftarrow{\mathbf{M}}$. In view of Remark 2.7, the full decomposition of the word \mathbf{a}_n has the form (2.5) with $\chi(xy) = xy$. Then Lemma 2.6 and the dual to Lemma 2.5 imply that the word \mathbf{w}_r coincides with (2.9) where the equality (2.8) is true for all odd $1 \leq \ell \leq 2n-1$ and for all $\alpha \in \mathcal{M}_{2n}^\ell$.

We note that $\xi_r(x) \neq \lambda$ and $\xi_r(y) \neq \lambda$ because the identity $\mathbf{w}_r \approx \mathbf{w}_{r+1}$ is non-trivial. This implies that the length of the word $\xi_r(xy)$ is more than 2. In view of Remark 2.9, the words $\xi_r(x)$ and $\xi_r(y)$ are letters. Since the first occurrences of the letters $\xi_r(x)$ and $\xi_r(y)$ occur in the words \mathbf{w}_r and \mathbf{w}_{r+1} in the opposite order, Lemma 2.5 implies that some subblock of some 1-block of the word \mathbf{w}_r contains the subword $\xi_r(xy)$ whenever $\mathbf{p}_r = \mathbf{a}_k$, and the subword $\xi_r(yx)$ whenever $\mathbf{p}_r = \mathbf{a}'_k$. Recall that the full decomposition of the word \mathbf{w}_r has the form (2.9). Therefore, if the identity $\mathbf{p}_r \approx \mathbf{q}_r$ equals $\mathbf{a}'_k \approx \mathbf{a}_k$ then either $\xi_r(yx) = xy$ or $\xi_r(yx) = x_\gamma^{(p)} x_\gamma^{(p+1)}$ for some $1 \leq p < 2n$, $1 \leq h < n$ and $\gamma \in \mathcal{M}_{2n}^{2h}$. Then either $\xi_r(y) = x$ and $\xi_r(x) = y$ or $\xi_r(y) = x_\gamma^{(p)}$ and $\xi_r(x) = x_\gamma^{(p+1)}$. Since the second occurrence of x precedes the second occurrence of y in \mathbf{a}'_k , we have that the second occurrence of $\xi(x)$ precedes the second occurrence of $\xi(y)$ in \mathbf{w}_r . But this is impossible, because the second occurrence of y is preceded by the second occurrence of x in \mathbf{w}_r , while the second occurrence of $x_\gamma^{(p+1)}$ is preceded by the second occurrence of $x_\gamma^{(p)}$ in \mathbf{w}_r . So, the identity $\mathbf{p}_r \approx \mathbf{q}_r$ cannot coincide with the identity $\mathbf{a}'_k \approx \mathbf{a}_k$ and, therefore, $\mathbf{p}_r \approx \mathbf{q}_r$ equals $\mathbf{a}_k \approx \mathbf{a}'_k$.

Suppose that $k < n$. Since the full decomposition of the word \mathbf{w}_r has the form (2.9), either $\xi_r(xy) = xy$ or $\xi_r(xy) = x_\gamma^{(p)} x_\gamma^{(p+1)}$ for some $1 \leq p < 2n$,

$1 \leq h < n$ and $\gamma \in \mathcal{M}_{2n}^{2h}$. If $\xi(xy) = xy$ then

$$\prod_{i=1}^{2k} \xi_r(x_1^{(i)}) = \prod_{i=1}^{2n} \zeta_1(x_1^{(i)}).$$

Since $k < n$, there is $1 \leq i \leq 2k$ such that the length of the word $\xi(x_1^{(i)})$ is more than 1. This contradicts the claim (2.8) and Remark 2.9. If $\xi_r(xy) = x_\gamma^{(p)} x_\gamma^{(p+1)}$ for some $1 \leq p < 2n$, $1 \leq h < n$ and $\gamma \in \mathcal{M}_{2n}^{2h}$ then

$$\prod_{i=1}^{2k} \xi_r(x_1^{(i)}) = \prod_{i=1}^{2n} \zeta_{\gamma+p}(x_{\gamma+p}^{(i)}).$$

Taking into account the claim (2.8), we get a contradiction with Remark 2.9.

Suppose now that $k > n$. We need some more notation. The smallest element of the set \mathcal{M}_s^t is denoted by γ_s^t . Further, we denote by η the endomorphism which is defined by the following equalities:

$$\begin{aligned} \eta(x) &= x_0, \quad \eta(y) = y_0, \quad \eta(s) = s_0, \quad \eta(s_{q'}) = s_{q'}, \\ \eta(x_{\gamma_{2n}^{(1)}}) &= x_q, \quad \eta(x_{\gamma_{2n}^{(2)}}) = y_q, \quad \eta(s_{\gamma_{2n}^{(1)}}) = s_{2n}, \quad \eta(s_{\gamma_{2n}^{(2)}}) = t_{2n}, \\ \eta(z) &= \lambda, \end{aligned}$$

where $1 \leq q \leq 2n$, $1 \leq q' \leq 2n - 1$ and z is an arbitrary letter which differs from $x, y, s, s_{q'}, x_{\gamma_{2n}^{(1)}}, x_{\gamma_{2n}^{(2)}}, s_{\gamma_{2n}^{(1)}} and $s_{\gamma_{2n}^{(2)}}$.$

We note that the word $\eta(\mathbf{w}_r)$ equals the word (2.7) for some endomorphisms $\zeta_1, \zeta_2, \dots, \zeta_{n-1}$ of F^1 such that the equality (2.6) is true for all $0 \leq \ell \leq n - 1$. Obviously, $\eta(\mathbf{w}_r) = \eta(\mathbf{u}_r)\eta(\xi_r(\mathbf{p}_r))\eta(\mathbf{v}_r)$ and $\eta(\mathbf{w}_{r+1}) = \eta(\mathbf{u}_r)\eta(\xi_r(\mathbf{q}_r))\eta(\mathbf{v}_r)$. Note that the identity $\eta(\mathbf{w}_r) \approx \eta(\mathbf{w}_{r+1})$ is non-trivial because the first occurrences of the letters $\eta(\xi_r(x))$ and $\eta(\xi_r(y))$ occur in the words $\eta(\xi_r(\mathbf{p}_r))$ and $\eta(\xi_r(\mathbf{q}_r))$ in the opposite order. Then we have a contradiction with Lemma 2.10 and inequality $n < k$. So, we have proved that different subsets of the form Σ_K define different subvarieties within the variety $\mathbf{N} \vee \overleftarrow{\mathbf{M}}$. All these subvarieties belongs to the interval $[\mathbf{M} \vee \overleftarrow{\mathbf{M}}, \mathbf{N} \vee \overleftarrow{\mathbf{M}}]$ by Lemma 2.5, the dual to Lemma 2.5 and Remark 2.7. This implies that the lattice of all subsets of \mathbb{N} order-embeds into the interval $[\mathbf{M} \vee \overleftarrow{\mathbf{M}}, \mathbf{N} \vee \overleftarrow{\mathbf{M}}]$. It is well known that the lattice of all subsets of \mathbb{N} is uncountable and violates the ascending chain condition and the descending chain condition. Theorem 1.1 is proved. \square

The following statement provide some more new examples of finitely generated varieties of monoids with continuum many subvarieties.

Corollary 3.1. *If G is a finite non-Abelian group then the monoid $S(xtx) \times G$ generates a variety with continuum many subvarieties.*

Proof. Let \mathbf{G} denotes the monoid variety generated by G . We note that the monoid $S(xtx)$ generates the variety \mathbf{D}_2 [8]. It follows that $\mathbf{D}_2 \vee \mathbf{G}$ is generated by $S(xtx) \times G$. The word xtx is an isoterm for $\mathbf{D}_2 \vee \mathbf{G}$ by Lemma 2.1. Then [17, Fact 3.1(ii)] implies that if the variety $\mathbf{D}_2 \vee \mathbf{G}$ satisfies a non-trivial identity $xyxty \approx \mathbf{w}$ then $\mathbf{w} = yxsxty$. But every group that satisfies the identity σ_1 is Abelian one. Hence $xyxty$ is an isoterm for $\mathbf{D}_2 \vee \mathbf{G}$. According to Lemma 2.1, $S(xyxty) \in \mathbf{D}_2 \vee \mathbf{G}$. Analogously, $S(xsytxy) \in \mathbf{D}_2 \vee \mathbf{G}$. The word xtx is an isoterm for the variety \mathbf{D}_2 . Therefore, if this variety satisfies a non-trivial identity $xytxy \approx \mathbf{v}$ then $\mathbf{v} \in$

$\{xytyx, yxtxy, yxtyx\}$. But every group that satisfies one of the identities $xytxy \approx xytyx$, $xytxy \approx yxtxy$ or $xytxy \approx yxtyx$ is Abelian one. Hence $xytxy$ is an isoterms for $\mathbf{D}_2 \vee \mathbf{G}$. According to Lemma 2.1, $S(xytxy) \in \mathbf{D}_2 \vee \mathbf{G}$. Analogously, $S(xytyx) \in \mathbf{D}_2 \vee \mathbf{G}$.

In view of the dual to Lemma 2.2, $\overleftarrow{\mathbf{M}}$ is generated by $S(xsytxy)$, whence $\overleftarrow{\mathbf{M}} \subseteq \mathbf{D}_2 \vee \mathbf{G}$. It follows from the dual to Example 1 in Erratum to [6] that \mathbf{N} is generated by the monoid $S(xysxty)$ and some quotient monoid of $S(xytxy, xytyx)$. It follows that $\mathbf{N} \subseteq \mathbf{D}_2 \vee \mathbf{G}$. Therefore, $\mathbf{N} \vee \overleftarrow{\mathbf{M}} \subseteq \mathbf{D}_2 \vee \mathbf{G}$. Theorem 1.1 implies that $\mathbf{D}_2 \vee \mathbf{G}$ contains continuum many subvarieties. \square

In conclusion, we note that Fig. 1 and Theorem 1.1(ii) imply that the lattice $L(\mathbf{N} \vee \overleftarrow{\mathbf{M}})$ is non-modular. The following question seems to be interesting.

Question 3.2. Does the lattice $L(\mathbf{N} \vee \overleftarrow{\mathbf{M}})$ satisfy any non-trivial identity?

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