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A NOTE ON PERFECT PACKING OF d-DIMENSIONAL CUBES

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Abstract. The d-dimensional cubes of edges of length $1, 2^{-t}, 3^{-t}, 4^{-t}, \ldots$ can be packed perfectly into a d-dimensional box, provided $1/d < t \le 2^{d-1}/(d2^{d-1}-1)$.

Keywords: packing, tiling, d-cube.

1. Introduction and notation

Let C_n^t be a d-dimensional cube (a d-cube) of edge length n^{-t} for $n = 1, 2, \ldots$ A *d-box* is the Cartesian product of the intervals: $[0, w_1] \times \ldots \times [0, w_d]$, where $0 < w_1 \le w_2 \le \cdots \le w_d$. The *$d$-volume* of B is equal to $v(B) = w_1 \cdot \ldots \cdot w_d$. The *partial surface* of B is $s(B) = w_2 \cdot \ldots \cdot w_d$. A collection of d-cubes $C_1^t, C_2^t, C_3^t, \ldots$ can be *packed* into a d-dimensional box B if it is possible to apply translations and rotations to the sets C_n^t so that the resulting translated and rotated d-cubes are contained in B and have mutually disjoint interiors. Such packing is called *perfect* if the d-volume of B is equal to the sum of d-volumes of the d-cubes, i.e., if $v(B) = \zeta(dt) = \sum_{n=1}^{\infty} \frac{1}{n^{dt}}$.

Recently, Joós [1] showed that the d-cubes $C_1^t, C_2^t, C_3^t, \ldots$ can be packed perfectly into a d-box for all t satisfying $t_0 < t \le \frac{2^{d-1}-1}{1-(d-1)t}$, where t_0 is the unique solution of the equation

$$\zeta(dt) - 1 = \frac{2^{d-1}-1}{1-(d-1)t}$$

on the interval $[1/d, 2^{d-1}/(d2^{d-1}-1)]$.

By a small modification in one lemma of [1], we extend this result to all t from the interval $(1/d, 2^{d-1}/(d2^{d-1}-1)]$.

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2. Perfect packing of d-cubes

We will use notations, numbering of formulas as well as methods presented in [1]. Therefore, the reader should read this paper. For example, we apply:

- the inequality

$$\sum_{j=a}^{b} j^{-(d-1)t} < \frac{b^{1-(d-1)t} - (a-1)^{1-(d-1)t}}{1-(d-1)t} \tag{2}$$

  for $1/d < t < 1/(d-1)$ and for positive integers $a < b$;
- Algorithm **a** and Algorithm **b**;
- Lemmas 1, 2, 3 and 4.

Moreover, we change Lemma 5 [1] a little bit by adding

$$\beta = \frac{1-(d-1)t}{(2^{d-1}-1)(dt-1)}$$

in formulas (7) and (9).

It is easy to verify that $\beta \geq 1$, provided $1/d < t \leq 2^{d-1}/(d2^{d-1}-1)$.

Lemma 1 (a modification of Lemma 5 of [1])**.** *Given an integer $n \geq 1$ and a non-empty set of boxes $\mathscr{B}$, suppose that the following conditions hold for $t \leq \frac{2^{d-1}}{d2^{d-1}-1}$:*

$$v(\mathscr{B}) \geq \sum_{j=n}^{\infty} j^{-dt}, \tag{6}$$

$$s(\mathscr{B}) \leq \frac{\beta(2^{d-1}-1)}{1-(d-1)t}(n-1)^{1-(d-1)t}. \tag{7}$$

If the inputs of Algorithm **b** *are n and $\mathscr{B}$, then the following conditions hold at step* (**b**4) *for all $i \geq 1$ for which step* (**b**4) *is executed. The conditions are*

$$v(\mathscr{B}_{\mathbf{b}i}) \geq \sum_{j=n_i}^{\infty} j^{-dt}, \tag{8}$$

$$s(\mathscr{B}_{\mathbf{b}i}) \leq s(\mathscr{B}) + \beta(2^{d-1}-1) \cdot \sum_{j=n}^{n_i-1} j^{-(d-1)t}. \tag{9}$$

Moreover, Algorithm **b** *never fail.*

Proof. The proof is similar to the proof of Lemma 5 [1]. For the convenience of the reader we will present it in details.

First, we will show that (8) and (9) ensure that Algorithm **b** will not fail. By (9), (2), (7), and $\beta = \frac{1-(d-1)t}{(2^{d-1}-1)(dt-1)}$,

$$
\begin{aligned}
s(\mathscr{B}_{\mathbf{b}i}) &\le s(\mathscr{B}) + \beta(2^{d-1}-1)\cdot \sum_{j=n}^{n_i-1} j^{-(d-1)t} \\
&< s(\mathscr{B}) + \beta\cdot\frac{2^{d-1}-1}{1-(d-1)t}\Big((n_i-1)^{1-(d-1)t} - (n-1)^{1-(d-1)t}\Big) \\
&\le \frac{\beta(2^{d-1}-1)}{1-(d-1)t}(n-1)^{1-(d-1)t} + \frac{\beta(2^{d-1}-1)}{1-(d-1)t)}\cdot(n_i-1)^{1-(d-1)t} \\
&- \frac{\beta(2^{d-1}-1)}{1-(d-1)t}\cdot(n-1)^{1-(d-1)t} \\
&= \frac{\beta(2^{d-1}-1)}{1-(d-1)t}\cdot(n_i-1)^{1-(d-1)t} \\
&= \frac{1}{dt-1}\cdot(n_i-1)^{1-(d-1)t} \\
&< \frac{1}{dt-1}\cdot n_i^{1-(d-1)t}.
\end{aligned}
$$

By Lemma 4 [1], ($\mathbf{b}4$) will not fail.

Obviously, (8) holds for all i.

Now (9) will be proved by induction on i. Clearly (9) holds for $i=1$. Let $i+1$ be the smallest integer for which (9) is not true.

If the condition in ($\mathbf{b}9$) was true for i, then $s(\mathscr{B}_{\mathbf{b}(i+1)}) = s(\mathscr{B}_{\mathbf{b}i}) - n_i^{-(d-1)t}$ and $n_{i+1} = n_i + 1$. Thus by induction,

$$
\begin{aligned}
s(\mathscr{B}_{\mathbf{b}(i+1)}) &= s(\mathscr{B}_{\mathbf{b}i}) - n_i^{-(d-1)t} \\
&\le s(\mathscr{B}) + \beta(2^{d-1}-1)\sum_{u=n}^{n_i-1} u^{-(d-1)t} - n_i^{-(d-1)t} \\
&< s(\mathscr{B}) + \beta(2^{d-1}-1)\sum_{u=n}^{n_i-1} u^{-(d-1)t} \\
&= s(\mathscr{B}) + \beta(2^{d-1}-1)\sum_{u=n}^{n_{i+1}-2} u^{-(d-1)t} \\
&< s(\mathscr{B}) + \beta(2^{d-1}-1)\sum_{u=n}^{n_{i+1}-1} u^{-(d-1)t}.
\end{aligned}
$$

If the condition in ($\mathbf{b}9$) was not true for i, then

$$s(\mathscr{B}_{\mathbf{b}(i+1)}) = s(\mathscr{B}_{\mathbf{b}i}) + s(\mathscr{H}_i) - s(B_i) + s(E_i).$$

Observe that E_i is a subset of B_i. By Remark 1 [1],

$$s(E_i) \le s(B_i).$$

Thus

$$s(\mathscr{B}_{\mathbf{b}(i+1)}) \le s(\mathscr{B}_{\mathbf{b}i}) + s(\mathscr{H}_i).$$

By induction, by $\beta \geq 1$, and Lemma 2 [1],

$$\begin{aligned} s(\mathscr{B}_{\mathbf{b}(i+1)}) &< s(\mathscr{B}_{\mathbf{b}i}) + s(\mathscr{H}_i) \\ &\leq s(\mathscr{B}) + \beta(2^{d-1}-1)\sum_{u=n}^{n_i-1} u^{-(d-1)t} + (2^{d-1}-1)\sum_{u=n_i}^{n_{i+1}-1} u^{-(d-1)t} \\ &\leq s(\mathscr{B}) + \beta(2^{d-1}-1)\sum_{u=n}^{n_i-1} u^{-(d-1)t} + \beta(2^{d-1}-1)\sum_{u=n_i}^{n_{i+1}-1} u^{-(d-1)t} \\ &= s(\mathscr{B}) + \beta(2^{d-1}-1)\sum_{u=n}^{n_{i+1}-1} u^{-(d-1)t}, \end{aligned}$$

which completes the proof.

□

Theorem 1. *The d-cubes C_n^t $(n \geq 1)$ can be packed perfectly into a d-box of dimensions* $1 \times \ldots \times 1 \times \zeta(dt)$, *provided* $1/d < t \leq 2^{d-1}/(d2^{d-1}-1)$.

Proof. Observe that

$$\zeta(dt) = \sum_{n=1}^{\infty} \frac{1}{n^{dt}} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{dt}} < 1 + \int_1^{\infty} x^{-dt}\mathrm{d}x = 1 + \frac{1}{dt-1}.$$

Let $B = [0,1] \times \ldots \times [0,1] \times [0, \zeta(dt)]$. The first d-cube C_1^t (of edges of unit length) is packed into $[0,1] \times \ldots \times [0,1] \times [0,1] \subset B$. The remaining d-cubes $C_2^t, C_3^t, \ldots$ are packed into $B^- = [0,1] \times \ldots \times [0,1] \times [1, \zeta(dt)] \subset B$. Since

$$s(B^-) = \zeta(dt) - 1 < 1 + \frac{1}{dt-1} - 1 = \frac{1}{dt-1} = \frac{\beta(2^{d-1}-1)}{1-(d-1)t} \cdot (2-1)^{1-(d-1)t},$$

by Lemma 1, the Algorithm $\mathbf{b}$ pack perfectly the d-cubes C_n^t $(n \geq 2)$ into B^-. Consequently, all the cubes are packed perfectly in B.

□

References

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