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# A NOTE ON DECIDABLE CATEGORICITY AND INDEX SETS

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Abstract. A structure $S$ is decidably categorical if $S$ has a decidable copy, and for any decidable copies $A$ and $B$ of $S$, there is a computable isomorphism from $A$ onto $B$. Goncharov and Marchuk proved that the index set of decidably categorical graphs is $\Sigma^0_{\omega+2}$ complete. In this paper, we isolate two familiar classes of structures $K$ such that the index set for decidably categorical members of $K$ has a relatively low complexity in the arithmetical hierarchy. We prove that the index set of decidably categorical real closed fields is $\Sigma^0_3$ complete. We obtain a complete characterization of decidably categorical equivalence structures. We prove that decidably presentable equivalence structures have a $\Sigma^0_4$ complete index set. A similar result is obtained for decidably categorical equivalence structures.

**Keywords:** decidable categoricity, autostability relative to strong constructivizations, index set, real closed field, equivalence structure, strong constructivization, decidable structure.

## 1. Introduction

The paper studies algorithmic complexity for classes of computable algebraic structures. A computable structure $\mathcal{S}$ is *decidable* if its complete diagram $D^c(\mathcal{S})$ is computable — in other words, there is an algorithm which given a first-order formula $\psi(\bar{x})$ and a tuple $\bar{a}$ from $\mathcal{S}$, decides whether the formula $\psi(\bar{a})$ is satisfied in $\mathcal{S}$. A structure $\mathcal{S}$ is *decidably presentable* (or *strongly constructivizable*) if it has a decidable copy.

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Ershov [1] and Morley [2] initiated the systematic studies of decidably presentable structures. In the 1970s and 1980s, computable structure theory was mainly focused on the study of structures with natural model-theoretic properties. Ershov [1] proved that every decidable theory $T$ has a decidable model $\mathcal{M} \models T$. Goncharov and Nurtazin [3], and independently, Harrington [4] proved that a complete decidable theory $T$ has a decidable prime model if and only if $T$ has a prime model and the set of all principal types of $T$ is computable. Goncharov [5] and Peretyat'kin [6] obtained a characterization of decidably presentable homogeneous models. The reader is referred to, e.g., the survey [7] and the recent papers [8, 9, 10] for further results on decidable models of decidable theories.

One of the important results in this area was obtained by Nurtazin [11]. The result connects the complexity of isomorphisms with model-theoretic properties. A decidably presentable structure $\mathcal{S}$ is *decidably categorical* (or *autostable relative to strong constructivizations*) if for any decidable copies $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$, there is a computable isomorphism $f\colon \mathcal{A} \cong \mathcal{B}$. The paper [11] provides a characterization of decidably categorical structures. In particular, this result shows that for any decidably categorical structure $\mathcal{M}$, there is a tuple $\bar{c} \in \mathcal{M}$ such that $(\mathcal{M}, \bar{c})$ is a prime model of the theory $Th(\mathcal{M}, \bar{c})$ (see Section 2.1 for a detailed discussion).

Goncharov and Marchuk [12] proved that the characterization of [11] is optimal: Nurtazin's criterion implies that the index set of decidably categorical structures belongs to the class $\Sigma^0_{\omega+2}$ of the hyperarithmetical hierarchy. The article [12] establishes that this set is $\Sigma^0_{\omega+2}$ complete, hence, there is no simpler syntactic description of decidable categoricity than the one provided by [11].

Recall that a computable structure $\mathcal{S}$ is *computably categorical* if for any computable copy $\mathcal{A}$ of $\mathcal{S}$, there is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{S}$. In general, it turns out that the behavior of decidable categoricity differs significantly from that of computable categoricity. This phenomenon can be witnessed even for familiar classes of structures:

- The article [13] proves that the index set of computably categorical structures is $\Pi^1_1$ complete. Note that by employing the results of [14], it is not hard to show that the index set of computably categorical linear orders is $\Sigma^0_3$ complete — informally speaking, computable categoricity among linear orders admits a pretty simple description.
- On the other hand, the index set of decidably categorical linear orders is $\Sigma^0_{\omega+2}$ complete [15]. In other words, the property "being decidably categorical among linear orders" has *the same complexity* as "being decidably categorical among arbitrary structures".

These kinds of phenomena lead to the following natural problem:

**Problem.** Find familiar classes of structures $K$ such that the index set of decidably categorical members of $K$ is *not* $\Sigma^0_{\omega+2}$ complete.

In this paper, we isolate two such classes $K$: real closed fields and equivalence structures. The article is arranged as follows. Section 2 gives necessary preliminaries. Section 3 proves that the index set of decidably categorical real closed fields is $\Sigma^0_3$ complete (Theorem 3). Section 4 provides a complete characterization of decidably categorical equivalence structures (Theorem 6). This characterization and results of [16] allow to obtain the following result: decidably categorical equivalence structures and decidably presentable equivalence structures both have a $\Sigma^0_4$

complete index set (Theorem 7 and Corollary 2). Section 5 discusses further results on index sets among decidable structures.

## 2. Preliminaries

We consider only computable languages, and structures with domain contained in the set of natural numbers $\omega$. We identify first-order formulas with their Gödel numbers. If not specified otherwise, a formula is always a first-order formula. As usual, $\{\varphi_e\}_{e\in\omega}$ is the standard enumeration of all partial computable functions.

For a set $X \subseteq \omega$, $\overline{X}$ is the complement of $X$, and $\mathrm{card}(X)$ denotes the cardinality of $X$. Let $\alpha$ be a non-zero computable ordinal. Recall that a set $X$ is $\Sigma^0_\alpha$ *complete* if $X \in \Sigma^0_\alpha$ and every $\Sigma^0_\alpha$ set $A$ is $m$-reducible to $X$.

For a structure $\mathcal{S}$, $Th(\mathcal{S})$ is the first-order theory of $\mathcal{S}$, and $D(\mathcal{S})$ is the atomic diagram of $\mathcal{S}$. For a natural number $n$, $\Sigma^c_n$ formulas are computable infinitary $\Sigma_n$ formulas.

Let $L$ be a language. For a computable $L$-structure $\mathcal{S}$, its *computable index* is a number $e$ such that the characteristic function $\chi_{D(\mathcal{S})}$ of the atomic diagram $D(\mathcal{S})$ is equal to $\varphi_e$.

For $e \in \omega$, by $\mathcal{M}_e$ we denote the structure with computable index $e$. Suppose that $K$ is a class of $L$-structures. The *index set* of the class $K$ is the set

$$I(K) = \{e : \mathcal{M}_e \in K\}.$$

The reader is referred to the monographs [17, 18] for further background on computable structure theory.

2.1. **Decidable categoricity.** A structure $\mathcal{M}$ is a *prime model* (of the theory $Th(\mathcal{M})$) if $\mathcal{M}$ is elementarily embeddable into any model of $Th(\mathcal{M})$. A structure $\mathcal{M}$ is an *almost prime model* if there is a finite tuple $\bar{c}$ from $\mathcal{M}$ such that $(\mathcal{M}, \bar{c})$ is a prime model.

An $L$-structure $\mathcal{M}$ is an *atomic model* if for any tuple $\bar{a} = a_0, \dots, a_n$ from $\mathcal{M}$, there exists an $L$-formula $\psi(x_0, \dots, x_n)$ such that $\mathcal{M} \models \psi(\bar{a})$ and any $L$-formula $\xi(x_0, \dots, x_n)$ satisfies the following: if $\mathcal{M} \models \xi(\bar{a})$, then $\mathcal{M} \models \forall \bar{x}(\psi(\bar{x}) \to \xi(\bar{x}))$. Such a formula $\psi$ is called a *complete formula* of the theory $Th(\mathcal{M})$.

**Proposition 1** (Vaught, see [19])**.** *An $L$-structure $\mathcal{M}$ is a prime model if and only if $\mathcal{M}$ is a countable atomic model.*

Nurtazin [11] obtained the following characterization of decidably categorical structures:

**Theorem 1** (Theorem 1 of [11])**.** *Let $L$ be a computable language. A decidable $L$-structure $\mathcal{M}$ is decidably categorical if and only if there is a finite tuple $\bar{c}$ from $\mathcal{M}$ with the following properties:*

(1) *$(\mathcal{M}, \bar{c})$ is a prime model, and*
(2) *given an $(L \cup \{\bar{c}\})$-formula $\psi(\bar{x})$, one can effectively check whether $\psi$ is a complete formula of the theory $Th(\mathcal{M}, \bar{c})$.*

Theorem 1 implies that the index set of decidably categorical $L$-structures is $\Sigma^0_{\omega+2}$. It is known that for any of the classes $K$ given below, the index set of decidably categorical members of $K$ is $\Sigma^0_{\omega+2}$ complete:

- graphs [12, 20],
- linear orders [15],

- Boolean algebras [21],
- 2-step nilpotent groups [22],
- structures with two equivalence relations [23].

## 3. Real closed fields

3.1. **Background on real closed fields.** A field $F$ is *real closed* if $F(\sqrt{-1})$ is algebraically closed and $F \neq F(\sqrt{-1})$. Note that every real closed field has a unique ordering: $a < b$ if and only if $(a - b)$ has a nonzero square root in the field. We will consider real closed fields in the language $L_{RC} = \langle 0, 1, +, \cdot, < \rangle$.

Let $F$ be a field of characteristic 0. A subset $S$ of a field $F$ is *algebraically dependent* if for some $n \in \omega$ there exist distinct $s_1, \ldots, s_n \in S$ and a nonzero polynomial $p \in \mathbb{Q}[x_1, \ldots, x_n]$ such that $p(s_1, \ldots, s_n) = 0$. A set $S$ is *algebraically independent* if it is not algebraically dependent. A maximal algebraically independent set in $F$ is called *transcendence basis* of $F$ over $\mathbb{Q}$. The *transcendence degree* of a field $F$ is the cardinality of some transcendence basis. For more background on real closed fields, the reader is referred to [24, 25]

Note that as the theory of real closed fields has effective quantifier elimination, every computable real closed field is decidable. Nurtazin [11] obtained the following characterization of decidably categorical real closed fields.

**Theorem 2** (Nurtazin, see [11])**.** *Let $\mathcal{R}$ be a computable real closed field. Then $\mathcal{R}$ is decidably categorical if and only if $\mathcal{R}$ has finite transcendence degree.*

In computable structure theory there are various results connected with real closed fields. R. Miller and Ocasio [26] investigated the degree spectra of real closed fields. The estimates for the complexity of index sets of Archimedean real closed fields with different transcendence degrees were obtained by Calvert, Harizanov, Knight, and S. Miller [27]. Ocasio investigated computability properties of real closed fields in his dissertation [28] .

3.2. **Index set of decidably categorical real closed fields.** Marker in unpublished work introduced a uniform way of computing, from the atomic diagram of a linear order $\mathcal{L}$, the atomic diagram of a real closed field $\mathcal{R}_{\mathcal{L}}$ such that $\mathcal{L} \cong \mathcal{L}'$ if and only if $\mathcal{R}_{\mathcal{L}} \cong \mathcal{R}'_{\mathcal{L}}$. Recall the construction of Marker's embedding, described by Ocasio in [28].

Let $\mathcal{L}$ be a linear order. Then there exists a Turing operator taking $\mathcal{L}$ to an embedding of $\mathcal{L}$ into $(\mathbb{Q}, <)$. Let $\mathcal{R}_{\mathbb{Q}}$ be a computable real closed field, such that $\mathrm{dom}(\mathcal{R}_{\mathbb{Q}}) \supseteq \{a_q : q \in \mathbb{Q}\}$, $\mathcal{R}_{\mathbb{Q}} \models n < a_q$ for all $n \in \omega$, and if $q < q'$, then $\mathcal{R}_{\mathbb{Q}} \models a_q^n < a'_q$ for all $n \in \omega$. There is a Turing operator taking the subset of $S$ of $\mathbb{Q}$ to a copy of the real closure of $\{a_q : q \in S\}$ in $\mathcal{R}_{\mathbb{Q}}$. Thus $\mathcal{R}_{\mathcal{L}}$ denotes the real closure of $\{a_l : l \in \mathcal{L}\}$ in $\mathcal{R}_{\mathbb{Q}}$.

**Lemma 1** (Ocasio [28])**.** *Let $\mathcal{L}$ be a $\Delta_2^0$ linear order. Then there is a computable copy $\mathcal{R}$ of $\mathcal{R}_{\mathcal{L}}$ and a $\Delta_2^0$ $\mathcal{L}$-embedding $i\colon \mathcal{L} \to \mathcal{R}$ such that $i$ extends to an isomorphism from $\mathcal{R}_{\mathcal{L}}$ onto $\mathcal{R}$.*

**Proposition 2** (Ocasio [28])**.** *Let $\mathcal{L}$ be a computable linear order. If $\mathcal{L}$ is not computably categorical, then $\mathcal{R}_{\mathcal{L}}$ is not $\Delta_2^0$-categorical.*

**Lemma 2.** *For any $\Sigma^0_3$ set $S$, there exists a sequence of $\mathbf{0}'$-computable linear orders $\{\mathcal{A}_n\}_{n\in\omega}$ such that:*

$$\mathcal{A}_n \cong \begin{cases} \gamma_n, \text{ such that } \gamma_n < \omega, & \text{if } n \in S, \\ \omega, & \text{if } n \notin S. \end{cases}$$

*Furthermore, given a number $n$, we can uniformly compute a $\Delta^0_2$-computable index for $\mathcal{A}_n$*

*Proof.* Let $A$ be a $\Sigma^0_2$ set. Using the presentation of $\Pi^0_2$ set with $(\exists^\infty)$ quantifier, i.e.

$$n \notin A \;\Leftrightarrow\; \exists^\infty x P(n,x), \text{ for a computable predicate } P,$$

we can build a computable sequence of computable linear orders $\{\mathcal{B}_n\}_{n\in\omega}$ such that:

$$\mathcal{B}_n \cong \begin{cases} \gamma_n, \text{ such that } \gamma_n < \omega, & \text{if } n \in A, \\ \omega, & \text{if } n \notin A. \end{cases}$$

Then we can relativize this construction to the $\mathbf{0}'$ to get the result from the statement of the lemma. □

**Theorem 3.** *The index set of decidably categorical real closed fields is $\Sigma^0_3$ complete.*

*Proof.* Let $I_{RC}$ be the index set of real closed fields, and let $I_{DC(RC)}$ be the index set of decidably categorical real closed fields.

First, we establish that $I_{DC(RC)}$ is a $\Sigma^0_3$ set. It is not hard to see that $I_{RC}$ is a $\Pi^0_2$ set (see, e.g., Theorem 5 of [29]). By Theorem 2, a real closed field $\mathcal{R}$ is decidably categorical if and only if it has finite transcendence degree. This condition can be presented as

$$\exists x_1,\dots,x_k \in R\left[\forall y \in R\left(\mathop{\&}_{i=1}^{k}[y \neq x_i] \to \exists p(u_1,\dots,u_k,v)[p(x_1,\dots,x_k,y)=0]\right)\right],$$

where $p(u_1,\dots,u_k,v)$ is a nonzero polynomial with coefficients from $\mathbb{Z}$. This is a $\Sigma^0_3$ condition, therefore $I_{DC(RC)}$ is a $\Sigma^0_3$ set.

Let $S$ be a $\Sigma^0_3$ set, and let $\{\mathcal{A}_n\}_{n\in\omega}$ be the sequence of $\mathbf{0}'$-computable linear orders, constructed for $S$, from Lemma 2. By Lemma 1, there exists a computable sequence of real closed fields $\{\mathcal{R}_n\}_{n\in\omega}$ such that $\mathcal{R}_n$ is a computable copy of $\mathcal{R}_{\mathcal{A}_n}$. If $n \in S$, then $\mathcal{A}_n$ is finite, therefore $\mathcal{R}_n$ has finite transcendence degree, so $\mathcal{R}_n$ is decidably categorical. If $n \notin S$, then $\mathcal{A}_n \cong \omega$ and $\mathcal{R}_n$ is not decidably categorical by Proposition 2. Then $\{\mathcal{R}_n\}_{n\in\omega}$ is a computable sequence of computable real closed field such that:

$n \in S \Leftrightarrow \mathcal{R}_n$ is decidably categorical;
$n \notin S \Leftrightarrow \mathcal{R}_n$ is not decidably categorical. □

## 4. Equivalence structures

The section is arranged as follows. Subsection 4.1 gives a brief overview of the necessary preliminary results. Subsection 4.2 provides a complete characterization of decidably categorical equivalence structures (Theorem 6). In Subsection 4.3, we prove that the index set of decidably categorical equivalence structures is $\Sigma^0_4$ complete (Theorem 7). A similar result is obtained for decidably presentable equivalence structures (Corollary 2).

4.1. **Background on equivalence structures.** An equivalence structure $\mathcal{A} = (A, E^{\mathcal{A}})$ consists of a set with an equivalence relation on this set. If $A$ is countably infinite, then without loss of generality, we may assume that $A = \omega$. For $a \in A$, $[a]_{\mathcal{A}}$ denotes the $E^{\mathcal{A}}$-equivalence class of $a$. Recall that

$$Inf^{\mathcal{A}} = \{a : [a]_{\mathcal{A}} \text{ is infinite}\},$$
$$Fin^{\mathcal{A}} = \{a : [a]_{\mathcal{A}} \text{ is finite}\}.$$

We say that a sequence $\{a_n\}_{n\in\omega}$ is a *transversal* of $\mathcal{A}$ if all $a_n$ are pairwise not $E^{\mathcal{A}}$-equivalent, and for any $b \in \mathcal{A}$, there is $n$ such that $a_n$ and $b$ are $E^{\mathcal{A}}$-equivalent.

The *character* of an equivalence structure $\mathcal{A}$ is the set

$$\chi(\mathcal{A}) = \{(n, k) : n, k > 0, \text{ and } \mathcal{A} \text{ has at least } k \text{ equivalence classes of size } n\}.$$

The character of $\mathcal{A}$ is *bounded* if there is a number $n_0 \in \omega$ such that all finite equivalence classes of $\mathcal{A}$ have size at most $n_0$.

**Theorem 4** (follows from § 5 of [16])**.** *A computable equivalence structure $\mathcal{A}$ is decidable if and only if the character $\chi(\mathcal{A})$ is computable and the set*

$$K(\mathcal{A}) := \{(a, k) \in \omega^2 : \operatorname{card}([a]_{\mathcal{A}}) \geq k\}$$

*is computable.*

**Theorem 5** (Theorem 5.6 of [16])**.** *If $\mathcal{A}$ is a countable equivalence structure with computable character $\chi(\mathcal{A})$, then $\mathcal{A}$ has a decidable copy.*

It is not difficult to establish the following fact.

**Lemma 3.** *Suppose that $\mathcal{A}$ is a decidable equivalence structure. Then the set $Fin^{\mathcal{A}}$ is c.e., and $Inf^{\mathcal{A}}$ is co-c.e. Furthermore, the function*

$$size^{\mathcal{A}}(x) := \begin{cases} \operatorname{card}([x]_{\mathcal{A}}), & \text{if } x \in Fin^{\mathcal{A}}, \\ \text{undefined}, & \text{if } x \in Inf^{\mathcal{A}}, \end{cases}$$

*is partial computable.*

For more background on computable equivalence structures, the reader is referred to [16, 30].

4.2. **Decidable categoricity.** Let $\mathbf{d}$ be a Turing degree. A decidably presentable structure $\mathcal{S}$ is *decidably* $\mathbf{d}$*-categorical* if for any decidable copies $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$, there is a $\mathbf{d}$-computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. A Turing degree $\mathbf{c}$ is the *degree of decidable categoricity* (or *degree of autostability relative to strong constructivizations*) for $\mathcal{S}$ if $\mathbf{c}$ is the least degree such that $\mathcal{S}$ is decidably $\mathbf{c}$-categorical.

The studies of degrees of decidable categoricity were initiated by Goncharov [31]: he proved that any c.e. degree $\mathbf{d}$ is a degree of decidable categoricity for some prime model. The article [32] shows that for any computable successor ordinal $\alpha$, every degree $\mathbf{d}$, which is c.e. in $\mathbf{0}^{(\alpha)}$ and above $\mathbf{0}^{(\alpha)}$, is a degree of decidable categoricity for some structure. The paper [33] establishes that the index set of decidably $\mathbf{0}'$-categorical structures is $\Pi^1_1$ complete. We refer the reader to, e.g., the papers [34, 35, 36] for further results on decidable $\mathbf{d}$-categoricity.

**Theorem 6.** *A computable equivalence structure $\mathcal{A}$ is decidably categorical if and only if its character $\chi(\mathcal{A})$ is computable, and $\mathcal{A}$ satisfies one of the following condions:*

(A) *the character* $\chi(A)$ *is bounded, or*
(B) *the character* $\chi(A)$ *is unbounded and* $\mathcal{A}$ *contains only finitely many infinite equivalence classes.*

*Proof.* ($\Leftarrow$). Since the character $\chi(\mathcal{A})$ is computable, by Theorem 5, our structure $\mathcal{A}$ has a decidable copy. Without loss of generality, we may assume that the structure $\mathcal{A}$ is infinite. Let $\mathcal{B}$ and $\mathcal{C}$ be two decidable copies of $\mathcal{A}$. We build a computable isomorphism $f$ from $\mathcal{B}$ onto $\mathcal{C}$.

*Case (A).* Fix a non-zero number $n_0$ such that all finite equivalence classes of $\mathcal{A}$ have size at most $(n_0 - 1)$. Notice that for any $b \in \mathcal{B}$, we have $b \in Inf^{\mathcal{B}}$ if and only if $\mathrm{card}([b]_{\mathcal{B}}) \geq n_0$. Hence, the set $Inf^{\mathcal{B}}$ is computable. Using this fact and decidability of $\mathcal{B}$, one can build a computable transversal $\{b_n\}_{n\in\omega}$ in $\mathcal{B}$ with the following property: There is a computable function $\theta^{\mathcal{B}}\colon \omega \to \omega \cup \{\omega\}$ such that for any $n$, $\theta^{\mathcal{B}}(n) = \mathrm{card}([b_n]_{\mathcal{B}})$.

We also build a computable transversal $\{c_n\}_{n\in\omega}$ in $\mathcal{C}$ and a computable function $\theta^{\mathcal{C}}(n) = \mathrm{card}([c_n]_{\mathcal{C}})$. Using the obtained transversals $\{b_n\}_{n\in\omega}$ and $\{c_n\}_{n\in\omega}$, it is straightforward to construct a desired computable isomorphism.

*Case (B).* Suppose that $\mathcal{A}$ has precisely $m$ infinite classes. Fix elements $d_1, d_2, \dots, d_m$ from $\mathcal{B}$ such that all $d_i$ belong to $Inf^{\mathcal{B}}$ and they are pairwise non-equivalent. Note that $b \in Inf^{\mathcal{B}}$ if and only if $(bE^{\mathcal{B}}d_1) \vee (bE^{\mathcal{B}}d_2) \vee \dots \vee (bE^{\mathcal{B}}d_m)$. This observation implies that the set $Inf^{\mathcal{B}}$ is computable. Therefore, one can build a computable isomorphism from $\mathcal{B}$ onto $\mathcal{C}$, via an argument similar to that of Case (A).

($\Rightarrow$). Since $\mathcal{A}$ has a decidable copy, by Theorem 4, the character $\chi(\mathcal{A})$ is computable. Now it is sufficient to prove the following claim.

**Lemma 4.** *Let* $\mathcal{A}$ *be a decidable equivalence structure such that the character* $\chi(\mathcal{A})$ *is unbounded and* $\mathcal{A}$ *has infinitely many infinite classes. Then* $\mathbf{0}'$ *is the degree of decidable categoricity for* $\mathcal{A}$.

*Proof.* By Lemma 3, the sets $Fin^{\mathcal{A}}$ and $Inf^{\mathcal{A}}$ are $\mathbf{0}'$-computable. Moreover, one can build a computable transversal $\{a_n\}_{n\in\omega}$ in $\mathcal{A}$ such that the function $\theta^{\mathcal{A}}(n) := \mathrm{card}([a_n]_{\mathcal{A}})$ is $\mathbf{0}'$-computable. Hence, for a decidable copy $\mathcal{D}$ of $\mathcal{A}$, it is easy to construct a $\mathbf{0}'$-computable isomorphism $f$ from $\mathcal{A}$ onto $\mathcal{D}$. Thus, the structure $\mathcal{A}$ is decidably $\mathbf{0}'$-categorical.

We build two decidable copies $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ with the following property: For any isomorphism $f\colon \mathcal{C} \cong \mathcal{B}$, $f$ computes $\mathbf{0}'$.

Let $\mathcal{B}_0$ be a decidable equivalence structure such that $\chi(\mathcal{B}_0) = \chi(\mathcal{A})$ and $\mathcal{B}_0$ has no infinite equivalence classes. Let $\mathcal{B}_1$ be a decidable equivalence structure such that every equivalence class of $\mathcal{B}_1$ is infinite and $\mathcal{B}_1$ has infinitely many classes. (One can employ Theorem 5 to construct these structures.) We define $\mathcal{B}$ as the disjoint sum of $\mathcal{B}_0$ and $\mathcal{B}_1$. More formally, we set $xE^{\mathcal{B}}y$ iff

$$(x = 2u \,\&\, y = 2v \,\&\, uE^{\mathcal{B}_0}v) \vee (x = 2u + 1 \,\&\, y = 2v + 1 \,\&\, uE^{\mathcal{B}_1}v).$$

Note that $\mathcal{B}$ is isomorphic to $\mathcal{A}$. Furthermore, Theorem 4 implies that $\mathcal{B}$ is decidable.

Fix a computable transversal $\{b_n\}_{n\in\omega}$ in $\mathcal{B}_0$ and a computable function $r\colon \omega \to \omega$ such that for any $n$, $r(n) = \mathrm{card}([b_n]_{\mathcal{B}_0})$. We choose a strongly computable sequence of finite sets $\{V[s]\}_{s\in\omega}$ such that $\bigcup_{s\in\omega} V[s] = \emptyset'$, $V[0]$ is empty, and $V[s] \subset V[s+1]$ for all $s$.

The construction of $\mathcal{C}$ proceeds in stages. For $e \in \omega$, let $a_e = 2e$. At a stage $s$, we define a computable equivalence relation $E_s$ and a finite set $R_s$.

*Stage 0.* Set $E_0 = \{(a_e, a_e) : e \in \omega\}$ and $R_0 = \emptyset$.

*Stage $s+1 = 2t+1$.* Find the least number $n$ such that $n \notin R_s$. We use fresh odd numbers $b_1, b_2, \dots, b_{r(n)}$ to form a new $E_{s+1}$-equivalence class of size $r(n)$. Enumerate $n$ into $R_{s+1}$.

*Stage $s+1 = 2t+2$.* Suppose that $t = \langle e, k+1\rangle$. If $e \in V[k+1] \setminus V[k]$, then find the least $n \notin R_s$ such that $r(n) \geq \mathrm{card}([a_e]_{E_s})$. Since the character $\chi(\mathcal{A})$ is unbounded, we can always find such $n$. Enumerate $n$ into $R_{s+1}$ and use fresh odd numbers to grow the equivalence class of $a_e$ to the size $r(n)$. If $e \notin V[k+1]$, then add a fresh odd number $b$ to the equivalence class of $a_e$. If $e \in V[k+1] \cap V[k]$, then proceed to the next stage.

We set $E^{\mathcal{C}} := \bigcup_{s\in\omega} E_s$. It is easy to see that $\mathcal{C} := (\omega, E^{\mathcal{C}})$ is a computable equivalence structure. Furthermore, the construction guarantees that $\chi(\mathcal{C}) = \chi(\mathcal{A})$ and $\mathcal{C}$ has infinitely many infinite classes. Thus, $\mathcal{C}$ is isomorphic to $\mathcal{A}$. Note that for a non-zero $k$, we have $\mathrm{card}([a_e]_{\mathcal{C}}) \geq k$ if and only if $\mathrm{card}([a_e]_{E_{2\langle e,k-1\rangle+2}}) \geq k$. This implies that the set $K(\mathcal{C})$ is computable. By Theorem 4, the structure $\mathcal{C}$ is decidable.

Notice that $e \notin \emptyset'$ if and only if $a_e \in Inf^{\mathcal{C}}$. Since $Inf^{\mathcal{C}}$ is a co-c.e. set, we have $Inf^{\mathcal{C}} \equiv_m \overline{\emptyset'}$. On the other hand, the set $Inf^{\mathcal{B}}$ is computable. Suppose that $f$ is an arbitrary isomorphism from $\mathcal{C}$ onto $\mathcal{B}$. Then $\overline{\emptyset'} \equiv_m Inf^{\mathcal{C}} \leq_T Inf^{\mathcal{B}} \oplus f \equiv_T f$. Hence, $f$ computes $\mathbf{0}'$. This concludes the proofs of Lemma 4 and Theorem 6. □

The proof of Theorem 6 implies the following fact.

**Corollary 1.** *If $\mathcal{A}$ is a decidable equivalence structure, then $\mathcal{A}$ has degree of decidable categoricity $\mathbf{d} \in \{\mathbf{0}, \mathbf{0}'\}$.*

4.3. **Index sets.**

**Theorem 7.** *The index set of decidably categorical equivalence structures is $\Sigma^0_4$ complete.*

*Proof.* Let $I_{DC}$ be the index set of decidably categorical equivalence structures. First, we establish that $I_{DC}$ is a $\Sigma^0_4$ set. By Theorem 6, a structure $\mathcal{M}_e$ is a decidably categorical equivalence structure if and only if:

(a) $\mathcal{M}_e$ is an equivalence structure (this is a $\Pi^0_2$ condition — see, e.g., Proposition 4.1 in [37]), and

(b1) either $\mathcal{M}_e$ has a bounded character, i.e.

$$\bigvee_{k\in\omega} \forall x \left[(\mathrm{card}([x]_{\mathcal{M}_e}) \geq k) \to x \in Inf^{\mathcal{M}_e}\right]$$

(since $Inf^{\mathcal{A}}$ is definable by a $\Pi^c_2$ formula, this is a $\Sigma^0_3$ condition),

(b2) or the character $\chi(\mathcal{M}_e)$ is computable and $\mathcal{M}_e$ has only finitely many infinite classes, i.e.

$$\exists u\big[(\text{the function } \varphi_u(x,y) \text{ is total } \{0,1\}\text{-valued}) \mathbin{\&} \bigwedge_{k,l\in\omega} \big(\exists z_0 \dots \exists z_l \big(z_j \text{ are pairwise not } E^{\mathcal{M}_e}\text{-equivalent} \mathbin{\&} \mathrm{card}([z_0]_{\mathcal{M}_e}) = \dots = \mathrm{card}([z_l]_{\mathcal{M}_e}) = k\big) \Leftrightarrow \varphi_u(k, l+1) = 1\big)\big] \mathbin{\&} \bigvee_{n\in\omega} \exists a_0 \dots \exists a_n \forall x \big[x \in Inf^{\mathcal{M}_e} \to (x E^{\mathcal{M}_e} a_0 \vee \dots \vee x E^{\mathcal{M}_e} a_n)\big]$$

(this is a $\Sigma^0_4$ condition).

Therefore, we deduce that the set $I_{DC}$ is $\Sigma^0_4$.

Suppose that $S \subseteq \omega$ is a $\Sigma^0_4$ set. We need to prove that $S$ is $m$-reducible to $I_{DC}$. It is sufficient to build a uniformly computable sequence of equivalence structures $\{\mathcal{A}_n\}_{n\in\omega}$ with the following property: $n \in S$ if and only if $\mathcal{A}_n$ is decidably categorical.

Fix a computable ternary relation $R$ such that for any $n$,

$$n \notin S \;\Leftrightarrow\; \exists^\infty x \exists^\infty y R(n,x,y). \tag{1}$$

Let $W$ be a non-computable c.e. set such that $0 \in W$. Choose a strongly computable sequence of finite sets $\{W[s]\}_{s\in\omega}$ such that $W = \bigcup_{s\in\omega} W[s]$ and $W[s] \subseteq W[s+1]$ for all $s$.

For $n \in \omega$, we build a computable equivalence structure $\mathcal{A}_n$ in stages. At a stage $s$, we construct a computable equivalence structure $\mathcal{A}_n[s]$.

*Stage 0.* Let $\mathcal{A}_n[0]$ be a decidable equivalence structure such that the domain of $\mathcal{A}_n[0]$ is the set of all even numbers, $\mathcal{A}_n[0]$ has no infinite equivalence classes, and

$$\chi(\mathcal{A}_n[0]) = \{(2m+2, k+1) : m,k \in \omega\} \cup \{(2\langle t,j\rangle + 1, k+1) : t,k \in \omega,\ j \le t\}.$$

*Stage $s+1$.* Suppose that $s = \langle t,u\rangle$. If we have $R(n,t,u)$, then for every $j \le t$, we do the following: If $j \in W[s]$, then find the least even element $a$ such that $[a]_{\mathcal{A}_n[s]} = [a]_{\mathcal{A}_n[0]}$ and $\mathrm{card}([a]_{\mathcal{A}_n[0]}) = 2\langle t,j\rangle + 1$. Choose a fresh odd number $b$ and add it to the equivalence class of $a$. If $R(n,t,u)$ does not hold, then proceed to the next stage.

This completes the description of the construction. It is easy to show that the sequence of equivalence structures $\{\mathcal{A}_n[s]\}_{n,s\in\omega}$ is uniformly computable. We set $\mathcal{A}_n := \bigcup_{s\in\omega} \mathcal{A}_n[s]$. It is not difficult to verify the following claim:

**Lemma 5.** *For $n,t \in \omega$, let $Q_{n,t} := \{u \in \omega \colon R(n,t,u)\}$. For any $j \le t$, the following holds:*

1. *If $j \notin W$, then $\mathcal{A}_n$ has infinitely many equivalence classes of size $2\langle t,j\rangle + 1$ and infinitely many classes of size $2\langle t,j\rangle + 2$.*
2. *If $j \in W$ and the set $Q_{n,t}$ is infinite, then $\mathcal{A}_n$ has infinitely many classes of size $2\langle t,j\rangle + 2$, and $\mathcal{A}_n$ has no classes of size $2\langle t,j\rangle + 1$.*
3. *If $j \in W$ and the set $Q_{n,t}$ is finite, then $\mathcal{A}_n$ has infinitely many classes of size $2\langle t,j\rangle + 1$ and infinitely many classes of size $2\langle t,j\rangle + 2$.*

The next lemma shows that the constructed structure $\mathcal{A}_n$ has the desired properties.

**Lemma 6.** *(1) If $n \in S$, then there is a finite set $F \subseteq \omega^2$ such that*

$$\chi(\mathcal{A}_n) = \{(2m+2, k+1) : m,k \in \omega\} \cup \{(2\langle t,j\rangle+1, k+1) : t,k \in \omega,\ j \le t,\ (t,j) \notin F\}.$$

*In particular, the character $\chi(\mathcal{A}_n)$ is computable.*

*(2) If $n \notin S$, then $W \le_T \chi(\mathcal{A}_n)$.*

*Proof.* If $n \in S$, then by (1), there are only finitely many numbers $t$ such that the set $Q_{n,t}$ is infinite. Suppose that $t_0 < t_1 < \cdots < t_m$ are all such numbers. Using Lemma 5, it is not hard to show that the desired finite set $F$ is equal to

$$\{(t_i, j_i) : i \le m,\ j_i \le t_i,\ j_i \in W\}.$$

Now suppose that $n \notin S$. For $x \in \omega$, we describe how to check (effectively in $\chi(\mathcal{A}_n)$) whether $x$ belongs to $W$.

First, we find the least number $t_0 \geq x$ such that

(2) $$(2\langle t_0, 0\rangle + 1, 1) \notin \chi(\mathcal{A}_n).$$

Why does such $t_0$ exist? Lemma 5 and the fact that $0 \in W$ together imply that the condition (2) holds iff the set $Q_{n,t_0}$ is infinite. Since $n \notin S$, we apply (1) and deduce that there is a number $t_0 \geq x$ with infinite $Q_{n,t_0}$.

By Lemma 5, we have that $x \in W$ if and only if $(2\langle t_0, x\rangle + 1, 1) \notin \chi(\mathcal{A}_n)$. Lemma 6 is proved. □

If $n \in S$, then by Lemma 6, the character $\chi(\mathcal{A}_n)$ is computable. Since $\mathcal{A}_n$ has no infinite equivalence classes, by Theorem 6, $\mathcal{A}_n$ is decidably categorical.

If $n \notin S$, then Lemma 6 says that $\chi(\mathcal{A}_n)$ is not computable. Henceforth, $\mathcal{A}_n$ has no decidable copies, and $\mathcal{A}_n$ is not decidably categorical. Therefore, the set $I_{DC}$ is $\Sigma^0_4$ complete. This concludes the proof of Theorem 7. □

**Corollary 2.** *The index set of decidably presentable equivalence structures is $\Sigma^0_4$ complete.*

*Proof Sketch.* First, note that Theorem 5 easily implies the following fact: a countable equivalence structure is decidably presentable if and only if its character is computable. After that, one just follows the lines of the proof of Theorem 7:

(a) The index set of equivalence structures with a computable character is $\Sigma^0_4$.
(b) For a given $\Sigma^0_4$ set $S$, the construction of Theorem 7 produces a uniformly computable sequence of equivalence structures $\{\mathcal{A}_n\}_{n\in\omega}$. The sequence satisfies the following: $n \in S$ if and only if the character $\chi(\mathcal{A}_n)$ is computable.

This shows that our index set is $\Sigma^0_4$ complete. □

## 5. Index sets among decidable structures

In this section, we study how the results of the previous section can be transferred into the setting of index sets among decidable copies.

Let $L$ be a computable language. For a number $e \in \omega$, by $\mathcal{D}_e$ we denote the decidable $L$-structure $\mathcal{M}$ such that the characteristic function $\chi_{D^c(\mathcal{M})}$ of the complete diagram $D^c(\mathcal{M})$ is equal to $\varphi_e$.

For a class $K$, its *index set among decidable structures* is the set

$$I_D(K) = \{e : \mathcal{D}_e \in K\}.$$

**Theorem 8.** *Let $K_{dc\text{-}eq}$ be the class of decidably categorical equivalence structures. Then the set $I_D(K_{dc\text{-}eq})$ is $\Sigma^0_3$ complete.*

*Proof Sketch.* By Theorem 6, an index $e$ belongs to $I_D(K_{dc\text{-}eq})$ if and only if the following conditions hold:

(a) The function $\varphi_e$ computes the complete diagram of the structure $\mathcal{D}_e$ (i.e. $\varphi_e$ is total $\{0,1\}$-valued, and $\varphi_e$ is well-defined on the indices of first-order formulas). This is a $\Pi^0_2$ condition (see, e.g., Lemma 5.1 of [38] for a similar argument).
(b) $\mathcal{D}_e$ is an equivalence structure. This is a $\Pi^0_1$ condition.
(c) The structure $\mathcal{D}_e$ has a bounded character, or $\mathcal{D}_e$ has only finitely many infinite classes. Similarly to the argument of Theorem 7, this condition can be rewritten in a $\Sigma^0_3$ way (note that since $\mathcal{D}_e$ is decidable, the set $Inf^{\mathcal{D}_e}$ is $\Pi^0_1$).

Therefore, the index set $I_D(K_{dc\text{-}eq})$ is $\Sigma^0_3$.

Let $S$ be a $\Sigma^0_3$ set. Fix a computable ternary relation $R$ such that

$$n \notin S \ \Leftrightarrow\ \exists^\infty x \forall y R(n, x, y).$$

By employing Theorem 4, it is not hard to build a uniformly decidable sequence $\{\mathcal{A}_n\}_{n\in\omega}$ of equivalence structures such that for every $n \in \omega$, the structure $\mathcal{A}_n$ has the following properties:

(1) Every element of $\mathcal{A}_n$ is equivalent to some number $\langle k, 0\rangle$, $k \in \omega$. The elements $\langle k, 0\rangle$, $k \in \omega$, are pairwise non-equivalent.
(2) For $k, l \in \omega$, the size of the class $[\langle 2\langle k, l\rangle, 0\rangle]_{\mathcal{A}_n}$ equals $k + 1$.
(3) The size of the class $[\langle 2k+1, 0\rangle]_{\mathcal{A}_n}$ equals $1+\mathrm{card}(\{t : (\forall y \leq t)R(n, k, y)\})$.

Note that for every $n$, the character $\chi(\mathcal{A}_n)$ is equal to $\{(m, k) : m, k > 0\}$.

One can show that $\mathcal{A}_n$ is decidably categorical if and only if $n \in S$. Hence, the set $I_D(K_{dc\text{-}eq})$ is $\Sigma^0_3$ complete. □

A technique similar to that of Theorem 8 can be applied to the case of abelian $p$-groups of Ulm type 1.

For a number $n \in \omega$, the formula $(p^n \mid x)$ means that $\exists y(p^n y = x)$. It is known that a computable abelian $p$-group $\mathcal{A}$ is decidable if and only if the theory $Th(\mathcal{A})$ is decidable and the relations $(p^n \mid \cdot)$, $n \in \omega$, are uniformly computable inside $\mathcal{A}$ [39]. For more preliminaries on abelian groups, we refer the reader to, e.g., [36, 40].

**Proposition 3** (Lemma 3.2 of [36])**.** *Let $\mathcal{A}$ be a decidable abelian $p$-group of Ulm type 1. Then $\mathcal{A}$ is decidably categorical if and only if $\mathcal{A}$ is reduced or the character $\chi(\mathcal{A})$ is bounded.*

Let $K_{dc\text{-}ut1}$ be the class of decidably categorical abelian $p$-groups of Ulm type 1. An index $e$ belongs to the set $I_D(K_{dc\text{-}ut1})$ if and only if the following conditions hold:

(a) The function $\varphi_e$ computes the complete diagram of the structure $\mathcal{D}_e$. This is a $\Pi^0_2$ condition.
(b) The structure $\mathcal{D}_e$ is an abelian $p$-group. This is also a $\Pi^0_2$ condition.
(c) $\mathcal{D}_e$ satisfies one of the following:
  (c.1) $\mathcal{D}_e$ is reduced and has Ulm type 1. By Prüfer's Theorem, this is equivalent to the formula

$$\forall x \left[ x \neq 0 \to \bigvee_{k\in\omega} \neg(p^k \mid x) \right].$$

  Since $\mathcal{D}_e$ is decidable, this formula is equivalent to a $\Pi^0_2$ condition.
  (c.2) $\mathcal{D}_e$ has Ulm type 1 and possesses a bounded character. This is equivalent to

$$\bigvee_{k\geq 1} \forall x[(p^k \mid x) \to (p^{2k} \mid x)]. \tag{3}$$

  Since the condition $\forall x[(p^k \mid x) \to (p^{2k} \mid x)]$ is a first-order one, the formula (3) can be rewritten in a $\Sigma^0_1$ way.

Therefore, the set $I_D(K_{dc\text{-}ut1})$ is $\Pi^0_2$.

**Theorem 9.** *The index set $I_D(K_{dc\text{-}ut1})$ is $\Pi^0_2$ complete.*

*Proof Sketch.* Let $A$ be a $\Pi^0_2$ set. Fix a computable binary relation $R$ such that

$$n \in A \ \Leftrightarrow\ \exists^{\infty} x R(n,x).$$

Let $\mathcal{H}$ be a decidable copy of the quasicyclic group $\mathbb{Z}(p^{\infty})$. Inside $\mathcal{H}$ choose a computable sequence of elements $\{g_j\}_{j\in\omega}$ such that $g_0 \neq 0$ and $pg_{j+1} = g_j$ for all $j$. Consider a group

$$\mathcal{G} = \bigoplus_{i\in\omega} \mathcal{H}.$$

Without loss of generality, we may assume that $\mathcal{G}$ is decidable. For $i,j \in \omega$, by $g[i,j]$ we denote the (copy of the) element $g_j$ inside the $i$-th copy of $\mathcal{H}$.

Fix a decidable group

$$\mathcal{U} \cong \bigoplus_{i\in\omega} \Big( \bigoplus_{j\in\omega} \mathbb{Z}(p^{i+1}) \Big).$$

We build a computable sequence $(\mathcal{A}_n)_{n\in\omega}$ of computable subgroups of $\mathcal{G}$. For $n \in \omega$, the group $\mathcal{A}_n$ will be generated by the set $C_n$. This $C_n$ is constructed as follows.

At stage 0, set $C_n[0] = \{g[0,0]\}$, and let $c[0] = g[0,0]$.

*Stage* $s+1$. Suppose that $c[s] = g[i,j]$. If $R(n,s)$ holds, then define $C_n[s+1] = C_n[s] \cup \{g[i+1,0]\}$ and $c[s+1] = g[i+1,0]$. Otherwise, $C_n[s+1] = C_n[s] \cup \{g[i,j+1]\}$ and $c[s+1] = g[i,j+1]$.

For $n \in \omega$, we define $\mathcal{B}_n = \mathcal{A}_n \oplus \mathcal{U}$. The described construction ensures the following properties:

(1) Recall that if $n \in A$, then $\exists^{\infty} x R(n,x)$. This implies that the group $\mathcal{A}_n$ is reduced, and $\mathcal{B}_n \cong \mathcal{U}$. Hence, by Proposition 3, the group $\mathcal{B}_n$ is decidably categorical.

(2) If $n \notin A$, then $\exists^{<\infty} x R(n,x)$, and we have

$$\mathcal{A}_n \cong F \oplus \mathbb{Z}(p^{\infty}), \quad \mathcal{B}_n \cong \mathcal{U} \oplus \mathbb{Z}(p^{\infty}),$$

where $F$ is a finite $p$-group. Therefore, the group $\mathcal{B}_n$ is not reduced, and it has unbounded character. By Proposition 3, $\mathcal{B}_n$ is not decidably categorical.

By employing the techniques of [36], one can show that the sequence $(\mathcal{B}_n)_{n\in\omega}$ can be built in a uniformly decidable way. Here we only note the following:

- The relations $(p^m \mid \cdot)$, where $m \in \omega$, can be uniformly computed in all groups $\mathcal{B}_n$, $n \in \omega$.
- A straightforward analysis of the Szmielew invariants [41] shows that all groups $\mathcal{B}_n$, $n \in \omega$, are elementarily equivalent.

Since the sequence $(\mathcal{B}_n)_{n\in\omega}$ is uniformly decidable, we deduce that our index set is $\Pi^0_2$ complete. □

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