S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 17, стр. 1027–1042 (2020) УДК 512.554

DOI 10.33048/semi.2020.17.077 MSC 17C70

CENTRAL ORDERS IN SIMPLE FINITE DIMENSIONAL SUPERALGEBRAS

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Abstract. The well-known Formanek's module finiteness theorem states that every unital prime PI-algebra (i.e. a central order in a matrix algebra by Posner's theorem) embeds into a finitely generated module over its center. An analogue of this theorem for alternative and Jordan algebras was earlier proved by V.N. Zhelyabin and the author. In this paper we discuss this problem for associative, classical Jordan and some alternative superalgebras.

Keywords: central order, associative superalgebra, alternatve superalgebra, Jordan superalgebra, simple superalgebra.

1. Introduction

The study of simple and prime (super)algebras in different useful varieties is an important part of the structure theory of these varieties. The prime algebras with polynomial identities are especially interesting. The first point of this theory is Posner's Theorem [1]. It states that all associative prime PI-rings coincide with central orders in matrix algebras over finite dimensional division algebras. After this result, the similar theory was constructed in other varieties of nearly associative algebras. M. Slater [2] proved that an alternative prime non-degenerate algebra is a central order in a matrix algebra or in the Cayley–Dickson algebra. Later, E.I. Zel'manov [3] obtained a description of prime non-degenerate Jordan PI-algebras: they are central orders in finite dimensional central simple Jordan algebras or in the algebra of a non-degenerate symmetric bilinear form.

For alternative superalgebras the similar result was obtained by I.P. Shestakov [4]. He proved that every alternative non-associative prime superalgebra under some

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Panasenko, A.S., Central orders in simple finite dimensional superalgebras.

Received December, 25, 2019, published July, 30, 2020.

restriction on the even part (for example, if it is non-degenerate) is a central order in a finite dimensional alternative superalgebra.

These results motivate one to study central orders in simple (super)algebras more carefully. For example, E. Formanek in [5] proved that prime PI-algebra embeds into a finitely generated module over its center. It is important to notice that the proof of this result is essentially based on the Posner's Theorem.

In paper [6] the author proved that an alternative prime non-degenerate algebra embeds into a finitely generated module over its center. In [7], V.N. Zhelyabin and the author showed that a central order in a finite dimensional simple Jordan algebra embeds into a finitely generated module over its center.

It should be noticed that the study of Jordan superalgebras have been continued. For example, in works [8–10] V.N. Zhelyabin and A.S. Zakharov studied simple Jordan superalgebras with an associative even part. In works [11–14] F.A. Gomez Gonzalez proved analogues of the Wedderburn Principal Theorem for the cases, when the algebra A/N is one of the classical simple Jordan superalgebras (N is the solvable radical of a superalgebra A).

In this paper, we study central orders in simple finite dimensional associative, alternative and Jordan superalgebras. We prove an analogue of Formanek's Theorem for associative superalgebras and (with some restrictions) for alternative and Jordan superalgebras.

2. Central orders in simple finite dimensional associative superalgebras

The definition of $\mathcal{M}$-superalgebra in an arbitrary variety $\mathcal{M}$ of nonassociative algebras may be found in a lot of papers (for example, in [15]).

Let us remind some notations. If A is a (super)algebra and $x, y, z \in A$ then

$$(x, y, z) = (xy)z - x(yz), \qquad [x, y] = xy - yx.$$

Definition 1. *If A is a (super)algebra then a subalgebra*

$$Z(A) = \{x \in A \mid (x, a, b) = (a, x, b) = (a, b, x) = [a, x] = 0 \quad \forall a, b \in A\}$$

is called a ***center*** *of a (super)algebra A.*

Example 1. Let $B = M_{n+m}(A)$ be the $(n + m) \times (n + m)$ matrix algebra with elements from an associative k-algebra A. Then B is a Z_2-graded algebra with the following grading for every $X \in B$:

$$X = X_0 + X_1, \qquad X_0 = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \qquad X_1 = \begin{pmatrix} 0 & D_3 \\ D_4 & 0 \end{pmatrix},$$

where $D_1 \in M_n(A)$, $D_2 \in M_m(A)$, $D_3 \in M_{n,m}(A)$, $D_4 \in M_{m,n}(A)$.

It is easy to notice that in the example above $M_0 \simeq M_p(A) \oplus M_q(A)$, where $\oplus$ denotes the direct sum of ideals of M_0.

Let B be an algebra. We will use a notation $C_B(u) = \{x \in B \mid xu = ux\}$ for the centralizer of an element u in B. Similarly, $S_B(u) = \{x \in B \mid xu = -ux\}$.

Definition 2. *A superalgebra A is called* ***central*** *over a field k, if $Z(A)_0$ is isomorphic to k.*

M.L. Racine in [16] proved that an associative Artinian simple superalgebra is an algebra of linear maps of some vector superspace over some division superalgebra. In the same paper all division superalgebras were described. Let us give a part of this description for finite dimensional algebras:

Theorem (M.L. Racine, [16]). *If $D = D_0 + D_1$ is a central finite dimensional division superalgebra over a field k, then one and only one of the following statements holds (everywhere below ε stands for some division algebra over a field k):*

1) $D = D_0 = \varepsilon$, $D_1 = 0$;

2) $D = \varepsilon \otimes_k k[u]$, $u^2 = \lambda \in k$, $\lambda \neq 0$, $D_0 = \varepsilon \otimes_k k1$, $D_1 = \varepsilon \otimes_k ku$;

3) $D = \varepsilon$, $D_0 = C_\varepsilon(u)$, where $k[u] \subset \varepsilon$ is a quadratic extension of the field k;

4) $D = M_2(\varepsilon) = \varepsilon \otimes_k M_2(k)$, $D_0 = \varepsilon \otimes_k k[u]$, $D_1 = \varepsilon \otimes_k k[u]w$, where $u = \begin{pmatrix} 0 & 1 \\ \lambda & p \end{pmatrix}$, $w = \begin{pmatrix} 1 & 0 \\ p & -1 \end{pmatrix}$, and $k[u]$ is not contained in ε. Here $p = 0$ if $\operatorname{char}(k) \neq 2$ and $p = 1$ if $\operatorname{char}(k) = 2$.

We will prove the following theorem by using this classification.

Theorem 1. *Let $B = B_0 + B_1$ be a unital associative superalgebra, $Z = Z(B)_0$ does not have zero divisors of B, $A = Z^{-1}B$ is the central closure of a superalgebra B. If the superalgebra A is simple and finite dimensional, then B embeds into a free finitely generated Z-module.*

Proof. In [5] E. Formanek proved this theorem for the case $B_1 = A_1 = 0$. By Racine's Theorem [16] $A \simeq End_D(V)$ (algebra isomorphism), where D is a division superalgebra and V is a superspace over D. We will use the following notations: $k = Z^{-1}Z$, k_1 means the odd part of the center of the superalgebra A.

1) Case $D = D_0 = \varepsilon$. Then $End_D(V) \simeq M_{p+q}(D)$. $Z(M_{p+q}(D)) \subset A_0$ implies $Z(B) \subset B_0$ and $A \simeq Z(B)^{-1}B$. Since $M_{p+q}(D)$ is a simple algebra, we have our statement by Formanek' Theorem

2) Case $D = \varepsilon \otimes_k k[u]$, $u^2 = \lambda \in k$ and $\lambda \neq 0$. Then $A \simeq M_n(D_0) + M_n(D_1)$.

Firstly, we will consider the case $\varepsilon = k$, i.e. $D \simeq k[u]$. Let $a \in B$. Then $a = \sum_{i,j=1}^{n} (\alpha_{ij} + \beta_{ij}u)e_{ij}$. There exists $z \in Z$, $z \neq 0$, such that $e_{ij} = f_{ij}/z$, $u = v/z, \lambda = \mu/z$, where $f_{ij}, v \in B$, $\mu, z \in Z$. Note that

$$\sum_{t=1}^{n} f_{ti} a f_{jt} = z^2(\alpha_{ij} + \beta_{ij}u) \in B \cap Z(A) = Z(B),$$

that imply $\alpha_{ij}z^2 \in Z$. But

$$z^2(\alpha_{ij} + \beta_{ij}u)v = z^2(\alpha_{ij}v + \beta_{ij}\mu) \in B \cap Z(A) = Z(B),$$

hence $\beta_{ij}\mu z^2 \in Z$. It means that B embeds into a free finitely generated Z-module $\sum Ze_{ij} + \sum Zue_{ij}$.

Now, let ε be an arbitrary finite dimensional division algebra. Then, according to ([17], p.105), ε contains a maximal subfield $P = k[b]$, $m = \dim_k P$. Moreover, we can assume that $b \in B_0$. Note, that

$$Z^{-1}(B \otimes_Z Z[b]) \simeq M_n(D) \otimes_Z Z^{-1}Z[b] \simeq M_n(D) \otimes_Z k[b] \simeq M_n(D) \otimes_k k[b] \simeq$$
$$\simeq M_n(\varepsilon \otimes_k k[u]) \otimes_k P \simeq k[u] \otimes_k M_n(\varepsilon) \otimes_k P$$

By ([17], p.96) we have an isomorphism $M_n(\varepsilon)\otimes_k P\simeq M_{mn}(P)$. So,

$$k[u]\otimes_k M_n(\varepsilon)\otimes_k P\simeq M_{mn}(P)\otimes_k k[u]\simeq$$
$$\simeq M_{mn}(P)\otimes_P k[u]\simeq M_{mn}(P)\otimes_P P[u]\simeq M_{mn}(P\otimes_P P[u])\simeq M_{mn}(P[u]).$$

It means that $B\otimes_Z Z[b]$ satisfies the conditions of case 1 and hence $B\otimes_Z Z[b]$ embeds into a free finitely generated $Z[b]$-module. Since $Z[b]$ is finitely generated over Z, $B\otimes_Z Z[b]$ embeds into a finitely generated Z-module. It remains to notice that $B\subset B\otimes_Z Z[b]$.

3) Case $D=\varepsilon$, $D_0=C_\varepsilon(u)$, $k[u]\subset\varepsilon$. Let us prove that $Z(\varepsilon)=k$. It is enough to show that $Z(\varepsilon)_1=0$. If $0\neq x\in Z(\varepsilon)_1$, then $ux=xu$, that implies $x\in D_0$, a contradiction. Now we can use the Formanek's Theorem for non-graded algebras.

4) Case $D=M_2(\varepsilon)=\varepsilon\otimes_k M_2(k)$, $D_0=\varepsilon\otimes_k k[u]$, $D_1=\varepsilon\otimes k[u]w$, where $w\in M_2(k)$, and $k[u]$ is not contained in ε.

Let us consider the case $\varepsilon=k$. In this case $D=M_2(k)$, hence $Z(A)_1=0$. The algebra $M_n(M_2(k))$ is simple, so we can use the Formanek's Theorem.

Now, let ε be an arbitrary division algebra. It is obvious that $Z(M_n(M_2(\varepsilon)))\simeq Z(\varepsilon)\simeq k$ like in case 3. Hence, we can use the Formanek's Theorem. □

3. Central orders in simple finite dimensional alternative superalgebras

In [18] E.I. Zel'manov and I.P. Shestakov proved that all simple alternative superalgebras are either associative or trivial, if a field characteristic is not 2 or 3. Let us state examples of simple alternative superalgebras over fields with a characteristic equal to 2 or 3.

Example 2. a) $\mathrm{Char}(k)=2$. Let $\mathbf{O}=\mathbf{H}+v\mathbf{H}$ be the Calyley–Dickson algebra with a natural Z_2-grading, $\mathbf{H}$ are quaternions over a field k. Then $\mathbf{O}=\mathbf{H}+v\mathbf{H}$ is a simple alternative superalgebra.

b) $\mathrm{Char}(k)=2$. We need to apply the Cayley–Dickson process to algebra $\mathbf{O}$. We have $k[u]=k+ku$, $u^2=\alpha\neq 0\in k$. Then $\mathbf{O}[u]=k[u]\otimes_k\mathbf{O}=\mathbf{O}+\mathbf{O}u$ is a simple alternative superalgebra.

c) $\mathrm{Char}(k)=3$. Let $A=k1$ be one-dimensional space and let $M=kx+ky$ be a two-dimensional space over a field k. We will use a notation $\mathbf{B(1,2)}=A+M$ for the commutative superalgebra over k, where 1 is a unit in $\mathbf{B(1,2)}$ and $xy=-yx=1$. Then $\mathbf{B(1,2)}$ is a simple alternative superalgebra.

d) $\mathrm{Char}(k)=3$. Let $A=M_2(k)$ be a 2×2 matrix algebra over a field k, $M=km_1+km_2$ be a two-dimensional space over k. We will use a notation $\mathbf{B(4,2)}=A+M$ for Z_2-graded algebra over k, where $e_{ij}m_k=\delta_{ij}m_j$, $m_1m_2=e_{11}$, $m_2m_1=-e_{22}$, $m_1^2=-e_{21}$, $m_2^2=e_{12}$ and $ma=\overline{a}m$ for $a\in A$. Here $\overline{a}$ is the symplectic involution. Then $\mathbf{B(4,2)}$ is a simple alternative superalgebra.

e) $\mathrm{Char}(k)=3$. Let Γ be an associative and commutative ∂-simple algebra, ∂ is a derivation on Γ, $\gamma\in\Gamma$ and $\overline{\Gamma}$ is an isomorpic copy of linear space Γ. Consider $B(\Gamma,\partial,\gamma)=\Gamma+\overline{\Gamma}$ is a direct sum of linear spaces with multiplication below:

$$x\cdot y=xy,$$

$$x\cdot\overline{y}=\overline{x}\cdot y=\overline{xy},$$

$$\overline{x}\cdot\overline{y}=(\gamma xy+2\partial(x)y+x\partial(y)),$$

where $x, y \in \Gamma$, xy is the product in Γ. Then $B(\Gamma, \partial, \gamma)$ is a simple alternative superalgebra.

In paper [4] I.P. Shestakov classified all simple alternative superalgebras over a field of characteristic 2 or 3.

Theorem (I.P. Shestakov, [4]). *Let $B = A + M$ be a simple alternative non-associative central superalgebra over a field k. If $M \neq 0$, then B is isomorphic to one of five algebras from Example 2.*

Now, using this classification, we prove the following

Theorem 2. *Let B be a unital alternative non-associative superalgebra and $Z = Z(B)_0$. If $Z^{-1}B$ is a finite dimensional central simple superalgebra, then either B embeds (as a Z-submodule) into a free finitely generated Z-module or $Z^{-1}B$ is isomorphic to $B(\Gamma, \partial, \gamma)$.*

Proof. Let $B = A + M$ be a unital alternative non-associative superalgebra and $Z = Z(B)_0$. We may define $k = Z^{-1}Z$, $\overline{B} = Z^{-1}B$, $\overline{A} = \overline{B}_0$, $\overline{M} = \overline{B}_1$. By Shestakov's theorem we have to consider one of the following cases:

1. $charF = 2$, $\overline{B} = \mathbf{O} = \mathbf{H} + v\mathbf{H}$ is the Cayley–Dickson algebra over k with natural grading, generated by the Cayley–Dickson process, $\mathbf{H}$ is the quaternion algebra. There exist $\mu \in k, \beta, \gamma \in (k \setminus 0)$ such that the multiplication table of $\overline{B}$ in a certain basis $e_0, e_1, ..., e_7$, $e_0 = 1$, has the following form (see., for example, [19], §2.2):

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$e_1+\mu$	e_2+e_3	μe_2	e_4+e_5	μe_4	e_7	$e_7+\mu e_6$
e_2	e_3	β	βe_1	e_6	e_7	βe_4	βe_5
e_3	$e_3+\mu e_2$	$\beta+\beta e_1$	$\mu\beta$	e_7	αe_6+e_7	$\beta(e_5+e_4)$	$\mu\beta e_4$
e_4	e_5	e_6	e_7	γ	γe_1	γe_2	γe_3
e_5	$e_5+\mu e_4$	e_7	$e_7+\mu e_6$	$\gamma+\gamma e_1$	$\mu\gamma$	$\gamma(e_3+e_2)$	$\mu\gamma e_2$
e_6	e_6+e_7	βe_4	$\beta(e_5+e_4)$	γe_2	$\gamma(e_2+e_3)$	$\beta\gamma$	$\beta\gamma(e_1-1)$
e_7	μe_6	βe_5	$\beta\gamma e_4$	γe_3	$\mu\gamma e_2$	$\beta\gamma e_1$	$\mu\beta\gamma$

There exists $z \in Z$, $z \neq 0$, such that $e_i = \frac{f_i}{z}, \mu = \frac{\nu}{z}, \beta = \frac{\lambda}{z}, \gamma = \frac{\delta}{z}$, where $f_i \in \overline{B}$, $\delta, \lambda, \in Z \setminus \{0\}$, $\nu \in Z$. Let $a \in B$, $a = \sum_{i=0}^{7} \alpha_i e_i$. We will use a notation $a \circ b = ab + ba$. We have two cases:

1a. $\mu \neq 0$. Then the following relations hold:

$$\nu z \alpha_1 = (az - a \circ f_1)(f_1 - z),$$
$$z\nu\lambda\alpha_2 = \nu(a \circ f_3)z - (az - a \circ f_1)(f_1 - z)f_3,$$
$$z\nu\lambda\alpha_3 = \nu(a \circ f_2)z - (az - a \circ f_1)(f_1 - z)f_2,$$
$$z\nu\delta\alpha_4 = \nu(a \circ f_5)z - (az - a \circ f_1)(f_1 - z)f_5,$$
$$z\nu\delta\alpha_5 = \nu(a \circ f_4)z - (az - a \circ f_1)(f_1 - z)f_4,$$
$$\nu\lambda\delta\alpha_6 = \nu(a \circ f_7)z - (az - a \circ f_1)(f_1 - z)f_7,$$
$$\nu\lambda\delta\alpha_7 = \nu(a \circ f_6)z - (az - a \circ f_1)(f_1 - z)f_6.$$

So, if $z_0 = \nu\lambda\delta z$ then $az_0 \in D = \sum_{i=0}^{7} Z \cdot e_i$. As $Bz_0 \simeq B$, the superalgebra B embeds into the free finitely generated Z-module D.

1b. $\mu = 0$. Then

$$\lambda\delta^2 z^2 \alpha_0 = ((((f_4 - f_5)(az - a \circ f_1))f_4) \circ f_6)f_6.$$

It means that $\lambda\delta^2 z^2 \alpha_0 \in Z$. Therefore,

$$\lambda^2\delta^4\alpha_1 z = ((\lambda\delta^2 az^2 - \lambda\delta^2 za \circ f_1 - \lambda\delta^2\alpha_0 z^2) \circ f_6)f_6,$$

hence $\lambda^2\delta^4\alpha_1 z \in Z$.

$$\lambda^3\delta^4\alpha_2 z = \lambda^2\delta^4 za \circ f_3 - \lambda^2\delta^4 z\alpha_1 f_3$$
$$\lambda^3\delta^4 z\alpha_3 = \lambda^2\delta^2 za \circ f_2 - \lambda^2\delta^4 z\alpha_1 f_2$$
$$\lambda^2\delta^5\alpha_4 z = \lambda^2\delta^4 za \circ f_5 - \lambda^2\delta^4 z\alpha_1 f_5$$
$$\lambda^2\delta^5\alpha_5 z = \lambda^2\delta^4 za \circ f_4 - \lambda^2\delta^4 z\alpha_1 f_4$$
$$\lambda^3\delta^5\alpha_6 = \lambda^2\delta^4 za \circ f_7 - \lambda^2\delta^4 z\alpha_1 f_7$$
$$\lambda^3\delta^5\alpha_7 = \lambda^2\delta^4 za \circ f_6 - \lambda^2\delta^4 z\alpha_1 f_6$$

As $\lambda^2\delta^4 z\alpha_1 \in Z$, we have $\lambda^3\delta^5\alpha_i z^2 \in Z$. So if $z_0 = \lambda^3\delta^5 z^2$ then Bz_0 embeds into $D = \sum_{i=0}^{7} Z \cdot e_i$. As $Bz_0 \simeq B$, the superalgebra B embeds into the free finitely generated Z-module D.

2. $M = 0$. Then $B = A$ is a prime nondegenerate alternative algebra, so it is a central order in the Cayley–Dickson algebra by [2]. Using the proof in [6] and case 1 we have that B embeds into a free finitely generated Z-module.

3. $charF = 3$, $\overline{A} = k \cdot 1$, $\overline{M} = k \cdot x + k \cdot y$, 1 is the unit, $x^2 = y^2 = 0$ and $xy = -yx = 1$. There exists $z \in Z$, $z \neq 0$, such that $x = \frac{x_0}{z}$, $y = \frac{y_0}{z}$. Let $a \in B$. Then $a = \alpha x + \beta y + \gamma \cdot 1$ and

$$((ay_0)y_0)x_0 = -\alpha z^3,$$
$$((ax_0)x_0)y_0 = -\beta z^3,$$
$$(((ay_0)x_0)x_0)y_0 = -\gamma z^4.$$

So, if $z_0 = z^4$ then $az_0 \in D = Z \cdot x + Z \cdot y + Z \cdot 1$. As $Bz_0 \simeq B$, hence B embeds into the free finitely generated Z-module D.

4. $charF = 3$, $\overline{A} = M_2(k)$ is the matrix algebra over k, $\overline{M} = k \cdot m_1 + k \cdot m_2$,

$$m_1^2 = -e_{21}, m_2^2 = e_{12}, m_1 m_2 = e_{11}, m_2 m_1 = -e_{22},$$
$$e_{ij} \cdot m_k = \delta_{ik} m_j,$$
$$m \cdot a = \overline{a} m, \text{ if } a \in A,$$

where $\overline{a}$ is the symplectic involution on $M_2(k)$. There exists $z \in Z$, $z \neq 0$, such that $e_{ij} = \frac{f_{ij}}{z}$, $m_i = \frac{n_i}{z}$. Let $a \in B$. Then $a = \sum_{i,j} \alpha_{ij} e_{ij} + \beta_1 m_1 + \beta_2 m_2$ and the following relations hold:

$$(f_{12}(((an_1)n_1)f_{12}))f_{21} + z^2((an_1)n_1)f_{12} = -\alpha_{11} z^5,$$
$$f_{12}(((an_1)n_1)f_{21}) + (((an_1)n_1)f_{21})f_{12} = -\alpha_{12} z^4,$$
$$f_{21}(((an_2)n_2)f_{12}) + (((an_2)n_2)f_{12})f_{21} = \alpha_{21} z^4,$$
$$(f_{21}(((an_2)n_2)f_{21}))f_{12} + z^2((an_2)n_2)f_{21} = \alpha_{22} z^5,$$

$$((f_{22}(f_{12}a))n_2)f_{21}+f_{21}((f_{22}(f_{12}a))n_2)=\beta_1 z^4,$$
$$((f_{11}(f_{21}a))n_1)f_{12}+f_{12}((f_{11}(f_{21}a))n_1)=-\beta_2 z^4.$$

So, if $z_0=z^5$ then $az_0\in D=\sum_{i,j} Z\cdot e_{ij}+Z\cdot m_1+Z\cdot m_2$. As $Bz_0\simeq B$, hence B embeds into the free finitely generated Z-module D.

5. $char F=2$, $k[u]=k+ku$, $u^2=\alpha\neq 0\in k$ is a two-dimensional superalgebra and $\overline{B}=k[u]\otimes_k \mathbb{O}=\mathbb{O}+\mathbb{O}u$. Consider the same basis of $\mathbb{O}$ as in case 1.

There exists $z\in Z$, $z\neq 0$, such that $e_i=\frac{f_i}{z},\mu=\frac{\nu}{z},\beta=\frac{\lambda}{z},\gamma=\frac{\delta}{z},\alpha=\frac{\varepsilon}{z},u=\frac{v}{z}$, where $f_i\in B$, $\delta,\lambda,\alpha\in Z\setminus\{0\}$, $\nu\in Z$, $v\in Z(B)\cap M$. Let $a\in B$. Then $a=\sum_{i=0}^{7}\alpha_i e_i+\sum_{i=0}^{7}\beta_i e_i u$. We have two cases:

5a. $\mu\neq 0$. Then

$$\nu z(\alpha_1+\beta_1 u)=(az-a\circ f_1)(f_1-z),$$
$$z\nu\lambda(\alpha_2+\beta_2 u)=\nu(a\circ f_3)z-(az-a\circ f_1)(f_1-z)f_3,$$
$$z\nu\lambda(\alpha_3+\beta_3 u)=\nu(a\circ f_2)z-(az-a\circ f_1)(f_1-z)f_2,$$
$$z\nu\delta(\alpha_4+\beta_4 u)=\nu(a\circ f_5)z-(az-a\circ f_1)(f_1-z)f_5,$$
$$z\nu\delta(\alpha_5+\beta_5 u)=\nu(a\circ f_4)z-(az-a\circ f_1)(f_1-z)f_4,$$
$$\nu\lambda\delta(\alpha_6+\beta_6 u)=\nu(a\circ f_7)z-(az-a\circ f_1)(f_1-z)f_7,$$
$$\nu\lambda\delta(\alpha_7+\beta_7 u)=\nu(a\circ f_6)z-(az-a\circ f_1)(f_1-z)f_6.$$

So, if $z_0=\nu\lambda\delta z$ then $az_0\in D_1=\sum_{i=0}^{7}(B\cap(k[u]))\cdot e_i$.

5b. $\mu=0$. Then

$$\lambda\delta^2 z^2(\alpha_0+\beta_0 u)=((((f_4-f_5)(az-a\circ f_1))f_4)\circ f_6)f_6.$$

It means that $\lambda\delta^2 z^2(\alpha_0+\beta_0 u)\in(B\cap k[u])$. We also have

$$\lambda^2\delta^4(\alpha_1+\beta_1 u)z=((\lambda\delta^2 az^2-\lambda\delta^2 za\circ f_1-\lambda\delta^2(\alpha_0+\beta_0 u)z^2)\circ f_6)f_6,$$

hence $\lambda^2\delta^4(\alpha_1+\beta_1 u)z\in Z$. As above, the following relations hold:

$$\lambda^3\delta^4\alpha_2 z=\lambda^2\delta^4 za\circ f_3-\lambda^2\delta^4 z(\alpha_1+\beta_1 u)f_3,$$
$$\lambda^3\delta^4 z\alpha_3=\lambda^2\delta^2 za\circ f_2-\lambda^2\delta^4 z(\alpha_1+\beta_1 u)f_2,$$
$$\lambda^2\delta^5\alpha_4 z=\lambda^2\delta^4 za\circ f_5-\lambda^2\delta^4 z(\alpha_1+\beta_1 u)f_5,$$
$$\lambda^2\delta^5\alpha_5 z=\lambda^2\delta^4 za\circ f_4-\lambda^2\delta^4 z(\alpha_1+\beta_1 u)f_4,$$
$$\lambda^3\delta^5\alpha_6=\lambda^2\delta^4 za\circ f_7-\lambda^2\delta^4 z(\alpha_1+\beta_1 u)f_7,$$
$$\lambda^3\delta^5\alpha_7=\lambda^2\delta^4 za\circ f_6-\lambda^2\delta^4 z(\alpha_1+\beta_1 u)f_6.$$

As $\lambda^2\delta^4 z(\alpha_1+\beta_1 u)\in Z$, then $\lambda^3\delta^5(\alpha_i+\beta_i u)z^2\in Z$. So if $z_0=\lambda^3\delta^5 z^2$ then Bz_0 embeds into $D_1=\sum_{i=0}^{7}(B\cap k[u])\cdot e_i$.

In both cases we have $Bz_0\subset\sum_{i=0}^{7}(B\cap k[u])e_i$. If $a\in A$ then $az_0\in\sum_{i=0}^{7}(B\cap k)e_i=\sum_{i=0}^{7}Ze_i$ and $Az_0\subset\sum_{i=0}^{7}Z\cdot e_i$. If $a\in M$ then $az_0v\in\sum_{i=0}^{7}(B\cap k)e_i=\sum_{i=0}^{7}Ze_i$ and $Mz_0v\in\sum_{i=0}^{7}Z\cdot e_i$. So, Az_0 and Mz_0v are finite Z-modules, $Az_0\simeq A$ and $Mz_0v\simeq M$, hence $B=A+M$ is a finite Z-module. Theorem is proved.

□

In [4], I.P. Shestakov also classified all prime alternative superalgebras under some restriction.

Definition 3. *Let A be a Z_2-graded algebra. It is called* **prime**, *if for every two nonzero ideals I and J the statement $IJ = 0$ implies $I = 0$ or $J = 0$.*

Theorem (I.P. Shestakov, [4]). *Let $B = A+M$ be a prime alternative nonassociative superalgebra over a field k, $M \neq 0$. Let one of the following two statements hold:*
1) there exist $a, b \in A$ such that $(ab - ba)^4 \neq 0$;
2) A does not contain nonzero nil-ideals I satisfying condition $(I, M, M) \subset I$.
Then the even part $Z = Z(B)_0$ of a center of superalgebra B is not zero, Z does not have zero divisors of B, and $Z^{-1}B$ is a central simple alternative superalgebra over the field $Z^{-1}Z$.

So we trivially have the following corollary.

Corollary 1. *Let B be a unital prime alternative nonassociative superalgebra with a restriction 1) or 2) from Shestakov's Theorem and $Z = Z(B)_0$. Then either B is an order in $B(\Gamma, d, \gamma)$ or B embeds into a finitely generated Z-module.*

4. Central orders in classical simple Jordan superalgebras with a semi-simple even part.

In this section all fields are assumed to be of characteristic not equal 2.

If $A = A_0 + A_1$ is a Z_2-graded algebra, then we may define an algebra A^+ with the same linear structure, but with a new multiplication:

$$x \circ_s y = \frac{1}{2}(xy + (-1)^{|x||y|}yx)$$

for every $x, y \in A_0 \cup A_1$. If A is an associative superalgebra then A^+ is a Jordan superalgebra.

Let A be an associative superalgebra with a superinvolution $x \to x^*$. Then the algebra of symmetric elements relative to this superinvolution is a subsuperalgebra of A^+ and denotes by $H(A, *)$. Let the notation $H_n(A)$ stand for the subsuperalgebra of symmetric elements of $M_n(A)$ with a natural grading relative to the $*$-transpose involution.

In [15] M.L. Racine and E.I. Zel'manov classified all finite dimensional simple Jordan superalgebras with a semi-simple even part. Let us give examples of simple Jordan superalgebras.

Example 3. a) If $B = M_{n+m}(k)$ then B^+ is denoted by $M_{n,m}(k)$.

b) We will use the following notation:

$$Q_n(k) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in M_n(k) \right\}^+.$$

Then $Q_n(k)$ is a simple Jordan superalgebra. It is easy to see that $Q_n(k) = M_n(k) + \overline{M_n(k)} = A + M$, where $A = M_n(k)$ acts on $M = \overline{M_n(k)}$ like the Jordan matrix multiplication on $M_n(k)$ and for $\overline{x}, \overline{y} \in \overline{M_n(k)}$ we have $\overline{x}\overline{y} = \frac{1}{2}(x \cdot y - y \cdot x)$, where $x \cdot y$ is the multiplication in the associative algebra $M_n(k)$.

c) We will use the following notation:

$$P_n(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{n+n}(k) \mid a^T = d, b^T = -b, c^T = c \right\}.$$

Then $P_n(k)$ is a simple Jordan superalgebra, if $n > 1$. Let the notation $S_n(k)$ stand for the subspace of skewsymmetric elements of $M_n(k)$. It is easy to see that $P_n(k) = M_n(k) + (\overline{S_n(k)} + \overline{H_n(k)}) = A + M$, where $A = M_n(k)$ acts on $M = \overline{S_n(k)} + \overline{H_n(k)}$ in the following way:

$$a\overline{s} = \frac{1}{2}(as + sa^T), \quad a \in M_n(k), s \in S_n(k),$$

$$a\overline{h} = \frac{1}{2}(a^T h + ha), \quad a \in M_n(k), h \in H_n(k).$$

For $\overline{x}_1, \overline{x}_2 \in \overline{S_n(k)}$, $\overline{y}_1, \overline{y}_2 \in \overline{H_n(k)}$ we have

$$\overline{x}_1\overline{x}_2 = \overline{y}_1\overline{y}_2 = 0,$$

$$\overline{x}_1\overline{y}_1 = \frac{1}{2}(x_1 \cdot y_1 - y_1 \cdot x_1),$$

where $x \cdot y$ is the multiplication in $M_n(k)$.

d) We will use the following notation:

$$osp_{n,2m}(k) = \left\{ \begin{pmatrix} a & b_1 & b_2 \\ -b_2^T & c_1 & c_2 \\ b_1^T & c_3 & c_1^T \end{pmatrix} \mid a = a^T, c_2 = -c_2^T, c_3 = -c_3^T \right\},$$

where $a \in M_n(k)$, $b_i \in M_{n\times m}(k)$ (matrices with n lines and m rows), $d_i \in M_m(k)$.

Then $osp_{n,2m}(k)$ is a simple Jordan superalgebra with Z_2-grading

$$\begin{pmatrix} a & b_1 & b_2 \\ -b_2^T & c_1 & c_2 \\ b_1^T & c_3 & c_1^T \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & c_1 & c_2 \\ 0 & c_3 & c_1^T \end{pmatrix} + \begin{pmatrix} 0 & b_1 & b_2 \\ -b_2^T & 0 & 0 \\ b_1^T & 0 & 0 \end{pmatrix},$$

i.e. $osp_{n,2m}(k)$ is a subsuperalgebra of $M_{n,2m}(k)$.

e) Let $A = ke_1 + ke_2$, $M = kx + ky$, where e_1, e_2 are orthogonal idempotents, $e_i x = \frac{1}{2}x$, $e_i y = \frac{1}{2}y$, $xy = e_1 + te_2$, where $0 \neq t \in k$. Then $D_t = A + M$ is a Jordan simple superalgebra.

f) Let $V = V_0 \oplus V_1$ be a Z_2-graded vector space over k and let $(\cdot,\cdot)$ be a non-degenerate superform on V such that its restriction on $V_0 \times V_0$ is symmetric, but its restriction on $V_1 \times V_1$ is skew-symmetric. Let $J = A + M$, where $A = V_0 + ke$, $M = V_1$, where e is a unit of J and $xy = (x, y)e$ for $x, y \in V$. Then J is a simple Jordan superalgebra.

g) Let $A = H_3(k)$. Then there is a Jordan action of $H_3(k)$ on $S_3(k)$. We will use notations $\overline{S_3(k)}$ and $\overline{\overline{S_3(k)}}$ for two isomorphic copies of $H_3(k)$-modules $S_3(k)$, so $M = \overline{S_3(k)} \oplus \overline{\overline{S_3(k)}}$. Let us define a multiplication on $J = A + M$. If $\overline{x}_1, \overline{x}_2 \in \overline{S_3(k)}$ and $\overline{\overline{y}}_1, \overline{\overline{y}}_2 \in \overline{\overline{S_3(k)}}$ then

$$\overline{x}_1\overline{x}_2 = \overline{\overline{y}}_1\overline{\overline{y}}_2 = 0,$$

$$\overline{x}_1\overline{\overline{y}}_1 = x_1 \cdot y_1,$$

where $x_1 \cdot y_1$ is the multiplication in the Jordan algebra $M_3(k)^+$. Then $J = H_3(k) + (\overline{S_3(k)} \oplus \overline{\overline{S_3(k)}})$ is a simple Jordan superalgebra.

h) Let $B(4,2) = A + M$ from Example 2.d. There is an involution $*$ on $B(4,2)$, $(a + m)^* = \overline{a} - m$, where $a \in A$, $m \in M$ and $\overline{a}$ is the symplectic involution on $A = M_2(k)$. Then define a superinvolution on $M_3(B(4,2))$, which is the *-transpose superinvolution. The set of symmetric elements with respect to this superinvolution is denoted by $H_3(B(4,2))$, it is a simple Jordan superalgebra.

Definition 4. *We will call algebras from Example 3 as* ***classical simple Jordan superalgebras****.*

Theorem (M.L. Racine, E.I. Zel'manov, [15]). *Every unital simple finite dimensional Jordan superalgebra with a semisimple even part over an algebraically closed field is isomorphic to one of the classical simple Jordan superalgebras.*

Theorem 3. *Let $J = A + M$ be a classical simple Jordan superalgebra and $Z = Z(J)_0$. Then a unital central order in J embeds into a free finitely generated Z-module.*

Proof. Let $B = B_0 + B_1$ be a unital Jordan superalgebra, $Z = Z(B)_0$, $k = Z^{-1}Z$, $J = Z^{-1}B$, $A = Z^{-1}B_0$, $M = Z^{-1}B_1$ and let $J = A + M$ be a classical simple Jordan superalgebra over a field k. We will consider all posible cases according to Example 3.

a) $J = M_{n,m}(k)$. Then J has a basis e_{ij}, $1 \le i \le n+m$, $1 \le j \le n+m$. If $a \in B$ then

$$a = \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \alpha_{ij} e_{ij}.$$

There exists an element $z \in Z$, $z \neq 0$, such that $e_{ij} = \frac{f_{ij}}{z}$ and $f_{ij} \in B$.

We have

$$2(a \circ_s f_{kk}) \circ_s f_{kk} - a \circ_s f_{kk} = z^2 \circ_s \alpha_{kk} e_{kk}$$

Let $i \neq k$. Then we have

$$z^2 \alpha_{kk} (e_{kk} \circ_s f_{ik}) \circ_s f_{ki} = \frac{1}{4} z^4 \alpha_{kk} (e_{ii} \pm e_{kk}) \in B.$$

So, we have that $z^4 \alpha_{kk} e_{ii} \in B$ for $1 \le i \le n+m$ and

$$z^4 \alpha_{kk} = \sum_{i=1}^{n+m} z^4 \alpha_{kk} e_{ii} \in B \cap Z(J)_0 = Z.$$

In other hand,

$$\frac{1}{8} \alpha_{lk} (e_{ll} \pm e_{kk}) = ((a \circ_s f_{kk}) \circ_s f_{ll}) \circ_s f_{kl} \in B.$$

Hence

$$z^4 \alpha_{lk} e_{ll} = 16(((a \circ_s f_{kk}) \circ_s f_{ll}) \circ_s f_{kl}) \circ_s f_{ll} \in B$$

for $1 \le l \le n+m$. Then for $1 \le i \le n+m$ we have

$$z^6 \alpha_{lk} (e_{ii} \pm e_{ll}) = 4 z^4 \alpha_{lk} (e_{ll} \circ_s f_{il}) \circ_s f_{li} \in B.$$

Then $z^6 \alpha_{lk} e_{ii} \in B$ and $z^6 \alpha_{lk} = \sum\limits_{i=1}^{n+m} z^6 \alpha_{lk} e_{ii} \in B \cap Z(J)_0 = Z$. Hence, we have $z^6 a \in \sum_{i,j=1}^{n+m} Z e_{ij}$. So the Z-module $z^6 B \simeq B$ embeds into the free finitely generated Z-module $\sum_{i,j=1}^{n+m} Z e_{ij}$.

b) $J = Q_n(k)$. Then J has a basis $e_{ij}, \overline{e}_{ij}$, $1 \le i, j \le n$. If $a \in B$ then

$$a = \sum_{i,j=1}^{n} \alpha_{ij} e_{ij} + \sum_{i,j=1}^{n} \beta_{ij} \overline{e}_{ij}.$$

There exists an element $z \in Z$, $z \neq 0$, such that $e_{ij} = \frac{f_{ij}}{z}$, $\overline{e}_{ij} = \frac{\overline{f}_{ij}}{z}$ and $f_{ij}, \overline{f}_{ij} \in B$. As in the case (a),

$$z^6(\alpha_{ij} \cdot 1 + \beta_{ij} \cdot \overline{1}) \in B \cap Z(J).$$

Hence $z^6 \alpha_{ij} \in B \cap Z(J)_0 = Z$, $z^6 \beta_{ij} \cdot \overline{1} \in B$. But

$$z^{11}\beta_{ij} = \sum_{k=2}^{n} 8(((((z^6\beta_{ij} \cdot \overline{1}) \circ_s f_{12}) \circ_s \overline{f}_{21}) \circ_s f_{11}) \circ_s f_{1k}) \circ_s f_{k1} - \\ - 2(n-2)z^2(((z^6\beta_{ij} \cdot \overline{1}) \circ_s f_{12}) \circ_s \overline{f}_{21}) \circ_s f_{11}.$$

So $z^{11}\beta_{ij} \in Z$. If $z_0 = z^{11}$ then the Z-module $z_0 B \simeq B$ embeds into the free finitely generated Z-module $\sum\limits_{i,j=1}^{n} Z e_{ij} + \sum\limits_{i,j=1}^{n} \overline{e}_{ij}$.

c) $J = P_n(k)$. Then J has a basis $e_{ij} \in M_n(k)$, $1 \leq i, j \leq n$, $\overline{e}_{ij} \in \overline{H_n(k)}$, $1 \leq i \leq j \leq n$, $\overline{\overline{e}}_{ij} \in \overline{S_n(k)}$, $1 \leq i < j \leq n$. If $a \in B$ then

$$a = \sum_{i,j=1}^{n} \alpha_{ij} e_{ij} + \sum_{1 \leq i \leq j \leq n} \beta_{ij} \overline{e}_{ij} + \sum_{1 \leq i < j \leq n} \gamma_{ij} \overline{\overline{e}}_{ij}.$$

There exists an element $z \in Z$, $z \neq 0$, such that $e_{ij} = \frac{f_{ij}}{z}$, $\overline{e}_{ij} = \frac{\overline{f}_{ij}}{z}$, $\overline{\overline{e}}_{ij} = \frac{\overline{\overline{f}}_{ij}}{z}$ and $f_{ij}, \overline{f}_{ij} \in B$. In the same way as in the case (a), one may show that $z^6 \alpha_{ij} + b \in B$, where b is a some element from M. Then $z^6 \alpha_{ij} \in Z$.

Let $Y = \begin{pmatrix} a & s \\ h & a^T \end{pmatrix} \in J$, i.e. $a \in M_n(k)$, $h \in H_n(k)$, $s \in S_n(k)$. We have the following identity:

$$Y \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} s & d \\ x & -s \end{pmatrix}$$

for some $d \in \overline{S_n(k)}$, $x \in \overline{H_n(k)}$. Hence, like with α_{ij} we have $z^7 \gamma_{ij} \in Z$ (since $\begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix} = \frac{1}{z}(f_{11} + \cdots + f_{nn})$).

Using a multiplication by $\begin{pmatrix} 0 & f_{1i} - f_{i1} \\ 0 & 0 \end{pmatrix}$, we take a matrix with $z\beta_{kj}$ on some place in the upper left quarter. So, we have $z^7 \beta_{kj} \in Z$. If $z_0 = z^7$ then the Z-module $z_0 B \simeq B$ embeds into the free finitely generated Z-module $\sum\limits_{i,j=1}^{n} Z e_{ij} + \sum\limits_{1 \leq i \leq j \leq n} \overline{e}_{ij} + \sum\limits_{1 \leq i < j \leq n} \overline{\overline{e}}_{ij}$.

d) $J = osp_{n,2m}(k)$. If $x \in B$ then

$$x = \begin{pmatrix} a & b_1 & b_2 \\ -b_2^T & c_1 & c_2 \\ b_1^T & c_3 & c_1^T \end{pmatrix}.$$

We will use the notation

$$S = \begin{pmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (2x \circ_s S - x) \circ_s S,$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & c_1 & c_2 \\ 0 & c_3 & c_1^T \end{pmatrix} = (2x \circ_s S - x) \circ_s (1 - S).$$

The first equation allows us to use case (a). The second one allows us to use case (c). Let

$$b_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{ij} e_{ij},$$

$$b_2 = \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} e_{ij}.$$

Then we have:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma_{ij} e_{kj} - \gamma_{ij} e_{jk} \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= 2\left(\left(\left(((x \circ_s S - x) \circ_s S) \circ_s \begin{pmatrix} e_{ii} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \circ_s \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{jj} & 0 \\ 0 & 0 & e_{jj} \end{pmatrix} \right) \circ_s \right.$$

$$\left. \circ_s \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{rj} & 0 \\ 0 & 0 & e_{jr} \end{pmatrix} \right) \circ_s \begin{pmatrix} 0 & 0 & e_{rk} \\ -e_{kr} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $r \neq i$, $k \neq j$.

This identity allows us to use case (c). Similarly, we can use it for β_{ij}. So, we can find $z \in Z$ such that $zB \simeq B$ embeds into a free finitely generated Z-module.

e) $J = D_t$, $0 \neq t \in k$. Then J has a basis e_1, e_2, x, y. If $a \in B$ then

$$a = \alpha_1 e_1 + \alpha_2 e_2 + \beta_1 x + \beta_2 y.$$

There exists an element $z \in Z$, $z \neq 0$, such that $e_i = \frac{f_i}{z}$, $x = \frac{v}{z}$, $y = \frac{w}{z}$, $t = \frac{u}{z}$ and $f_i, v, w \in B$, $0 \neq u \in Z$. We have the following identities:

$$uz^4 \beta_1 = 4(((af_1)f_2)w)(uf_1 + zf_2),$$

$$uz^4 \beta_2 = -4(((af_1)f_2)v)(uf_1 + zf_2),$$

$$uz^6 \alpha_1 = 4(((((af_1)v)f_1)f_2)w)(uf_1 + zf_2),$$

$$uz^6 \alpha_2 = 4(((((af_2)v)f_1)f_2)w)(uf_1 + zf_2).$$

If $z_0 = uz^6$ then $z_0 B \simeq B$ embeds into the free finitely generated Z-module $Ze_1 + Ze_2 + Zx + Zy$.

f) J is the superalgebra of non-degenerate supersymmetric superform on a superspace $V = V_0 \oplus V_1$. Then J has a basis $e, e_1, \ldots, e_n, g_1, \ldots, g_{2m}$, where e is a unit, $\{e_1, \ldots, e_n\}$ is an orthogonal basis of V_0, $(e_i, e_i) = \alpha_i$, $\{g_1, \ldots, g_{2m}\}$ is a basis of V_1, $(g_{2i-1}, g_{2i}) = \beta_i \neq 0$ and $(g_i, g_j) = 0$ otherwise. If $a \in B$ then

$$a = \gamma e + \sum_{i=1}^{n} \delta_i e_i + \sum_{i=1}^{2k} \varepsilon_i g_i.$$

There exists an element $z \in Z$, $z \neq 0$, such that $e_i = \frac{f_i}{z}$, $g_i = \frac{h_i}{z}$, $\alpha_i = \frac{\lambda_i}{z}$, $\beta_i = \frac{\mu_i}{z}$ and $f_i, h_i \in B$, $\lambda_i, \mu_i \in Z$. We have the following identities:

$$z^2\lambda_1\lambda_2\gamma = (((af_1)f_1)f_2)f_2,$$
$$z\lambda_1\lambda_i\delta_i = ((af_i)f_1)f_1, \quad i > 1,$$
$$z\lambda_1\lambda_2\delta_1 = ((af_1)f_2)f_2.$$

Therefore,

$$((ah_{2k})h_{2k})h_{2k-1} = -z\mu_k^2\varepsilon_{2k-1},$$
$$((ah_{2k-1})h_{2k-1})h_{2k} = -z\mu_k^2\varepsilon_{2k}.$$

If $z_0 = z^2\prod_{i=1}^n \lambda_i \prod_{i=1}^m \mu_i^2$ then $z_0B \simeq B$ embeds into the free finitely generated Z-module $Ze + \sum\limits_{i=1}^{n} Ze_i + \sum\limits_{i=1}^{2m} Zg_i$.

g) $J = H_3(k) + (\overline{S_3(k)} \oplus \overline{\overline{S_3(k)}})$. Then J has a basis e_{11}, e_{22}, e_{33}, e_{12}, e_{13}, e_{23}, $\overline{e}_{12}, \overline{e}_{13}, \overline{e}_{23}$, $\overline{\overline{e}}_{12}, \overline{\overline{e}}_{13}, \overline{\overline{e}}_{23}$. If $a \in B$ then

$$a = \sum_{1\le i\le j\le 3} \alpha_{ij}e_{ij} + \sum_{1\le i<j\le 3} \beta_{ij}\overline{e}_{ij} + \sum_{1\le i<j\le 3} \gamma_{ij}\overline{\overline{e}}_{ij}.$$

There exists an element $z \in Z$, $z \neq 0$, such that $e_{ij} = \frac{f_{ij}}{z}$, $\overline{e}_{ij} = \frac{\overline{f}_{ij}}{z}$, $\overline{\overline{e}}_{ij} = \frac{\overline{\overline{f}}_{ij}}{z}$ and $f_{ij}, \overline{f}_{ij}, \overline{\overline{f}}_{ij} \in B$. Let

$$F_1(a) = ((((a\overline{\overline{f}}_{12})\overline{\overline{f}}_{12})\overline{f}_{12})f_{11})f_{33},$$
$$F_2(a) = ((((a\overline{\overline{f}}_{12})\overline{\overline{f}}_{12})\overline{f}_{12})f_{22})f_{33},$$
$$F_3(a) = ((((a\overline{\overline{f}}_{13})\overline{\overline{f}}_{13})\overline{f}_{13})f_{33})f_{22},$$
$$G_1(a) = ((((a\overline{f}_{12})\overline{f}_{12})\overline{\overline{f}}_{12})f_{11})f_{33},$$
$$G_2(a) = ((((a\overline{f}_{12})\overline{f}_{12})\overline{\overline{f}}_{12})f_{22})f_{33},$$
$$G_3(a) = ((((a\overline{f}_{13})\overline{f}_{13})\overline{\overline{f}}_{13})f_{33})f_{22}.$$

Then we have the following identities:

$$z^7\beta_{23} = 16zF_1(a)f_{13} + 32(F_1(a)f_{23})f_{12} + 32(F_1(a)f_{12})f_{23},$$
$$z^7\beta_{13} = 16zF_2(a)f_{23} + 32(F_2(a)f_{13})f_{12} + 32(F_2(a)f_{12})f_{13},$$
$$z^7\beta_{12} = 16zF_3(a)f_{23} + 32(F_3(a)f_{12})f_{13} + 32(F_3(a)f_{13})f_{12},$$
$$z^7\gamma_{23} = 16zG_1(a)f_{13} + 32(G_1(a)f_{23})f_{12} + 32(G_1(a)f_{12})f_{23},$$
$$z^7\gamma_{13} = 16zG_2(a)f_{23} + 32(G_2(a)f_{13})f_{12} + 32(G_2(a)f_{12})f_{13},$$
$$z^7\gamma_{12} = 16zG_3(a)f_{23} + 32(G_3(a)f_{12})f_{13} + 32(G_3(a)f_{13})f_{12},$$
$$z^4\alpha_{12} = 2z((af_{11})f_{22})f_{12} + 4(((af_{11})f_{22})f_{13})f_{23} + 4(((af_{11})f_{22})f_{23})f_{13},$$
$$z^4\alpha_{13} = 2z((af_{11})f_{33})f_{13} + 4(((af_{11})f_{33})f_{12})f_{23} + 4(((af_{11})f_{33})f_{23})f_{12},$$
$$z^4\alpha_{23} = 2z((af_{22})f_{33})f_{23} + 4(((af_{22})f_{33})f_{12})f_{13} + 4(((af_{22})f_{33})f_{13})f_{12},$$
$$z^4\alpha_{11} = 2(((2af_{11}-za)f_{11})f_{12})f_{12} + 2(((2af_{11}-za)f_{11})f_{13})f_{13} - z^2(2af_{11}-za)f_{11},$$
$$z^4\alpha_{22} = 2(((2af_{22}-za)f_{22})f_{12})f_{12} + 2(((2af_{22}-za)f_{22})f_{23})f_{23} - z^2(2af_{22}-za)f_{22}$$
$$z^4\alpha_{33} = 2(((2af_{33}-za)f_{33})f_{23})f_{23} + 2(((2af_{33}-za)f_{33})f_{13})f_{13} - z^2(2af_{33}-za)f_{33}.$$

If $z_0 = z^4$ then $z_0B \simeq B$ embeds into the free finitely generated Z-module $\sum\limits_{1\le i\le j\le 3} Ze_{ij} + \sum\limits_{1\le i<j\le 3} Z\overline{e}_i + \sum\limits_{1\le i<j\le 3} Z\overline{\overline{e}}_{ij}$.

h) $J = H_3(B(4,2))$. If $a \in B$ then

$$a = \begin{pmatrix} \alpha_1 & b_1 & b_2 \\ \overline{b_1} & \alpha_2 & b_3 \\ \overline{b_2} & \overline{b_3} & \alpha_3 \end{pmatrix},$$

where

$$b_1 = \sum_{i,j=1}^{2} \alpha_{ij} e_{ij} + \delta_1 m_1 + \delta_2 m_2,$$

$$b_2 = \sum_{i,j=1}^{2} \beta_{ij} e_{ij} + \varepsilon_1 m_1 + \varepsilon_2 m_2,$$

$$b_3 = \sum_{i,j=1}^{2} \gamma_{ij} e_{ij} + \mu_1 m_1 + \mu_2 m_2.$$

Let us denote

$$b[12] = \begin{pmatrix} 0 & b & 0 \\ \overline{b} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b[13] = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ \overline{b} & 0 & 0 \end{pmatrix}, \quad b[23] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & \overline{b} & 0 \end{pmatrix},$$

if $b \in B(4,2)$. Also, $e_{ij} = 1[ij]$ as usual.

Then we have the following identities:

$$\alpha_1 = 4(((2ae_{11} - a)e_{11})e_{12})e_{12} + 4(((2ae_{11} - a)e_{11})e_{13})e_{13} - (2ae_{11} - a)e_{11},$$

$$\alpha_2 = 4(((2ae_{22} - a)e_{22})e_{12})e_{12} + 4(((2ae_{22} - a)e_{22})e_{23})e_{23} - (2ae_{22} - a)e_{22},$$

$$\alpha_3 = 4(((2ae_{33} - a)e_{33})e_{23})e_{23} + 4(((2ae_{33} - a)e_{33})e_{13})e_{13} - (2ae_{33} - a)e_{33},$$

$$\delta_1 = 32((((ae_{11})e_{22})m_2[12])e_{13})e_{13} + 8((ae_{11})e_{22})m_2[12] - 8(((ae_{11})e_{22})m_2[12])e_{11},$$

$$\delta_2 = 32((((ae_{11})e_{22})m_1[12])e_{13})e_{13} + 8((ae_{11})e_{22})m_1[12] - 8(((ae_{11})e_{22})m_1[12])e_{11},$$

$$\varepsilon_1 = 32((((ae_{11})e_{33})m_2[13])e_{12})e_{12} + 8((ae_{11})e_{33})m_2[13] - 8(((ae_{11})e_{33})m_2[13])e_{11},$$

$$\varepsilon_2 = 32((((ae_{11})e_{33})m_1[13])e_{12})e_{12} + 8((ae_{11})e_{33})m_1[13] - 8(((ae_{11})e_{33})m_1[13])e_{11},$$

$$\mu_1 = 32((((ae_{22})e_{33})m_2[23])e_{12})e_{12} + 8((ae_{22})e_{33})m_2[23] - 8(((ae_{22})e_{33})m_2[23])e_{22},$$

$$\mu_2 = 32((((ae_{22})e_{33})m_1[23])e_{12})e_{12} + 8((ae_{22})e_{33})m_1[23] - 8(((ae_{22})e_{33})m_1[23])e_{22},$$

$$-\alpha_{21} = 32((((ae_{11})e_{22})e_{12}[12])e_{13})e_{13} + 8((ae_{11})e_{22})e_{12}[12] - 8(((ae_{11})e_{22})e_{12}[12])e_{11},$$

$$-\alpha_{12} = 32((((ae_{11})e_{22})e_{21}[12])e_{13})e_{13} + 8((ae_{11})e_{22})e_{21}[12] - 8(((ae_{11})e_{22})e_{21}[12])e_{11},$$

$$-\beta_{21} = 32((((ae_{11})e_{33})e_{12}[13])e_{12})e_{12} + 8((ae_{11})e_{33})e_{12}[13] - 8(((ae_{11})e_{33})e_{12}[13])e_{11},$$

$$-\beta_{12} = 32((((ae_{11})e_{33})e_{21}[13])e_{12})e_{12} + 8((ae_{11})e_{33})e_{21}[13] - 8(((ae_{11})e_{33})e_{21}[13])e_{11},$$

$$-\gamma_{21} = 32((((ae_{22})e_{33})e_{12}[23])e_{12})e_{12} + 8((ae_{22})e_{33})e_{12}[23] - 8(((ae_{22})e_{33})e_{12}[23])e_{22},$$

$$-\gamma_{12} = 32((((ae_{22})e_{33})e_{21}[23])e_{12})e_{12} + 8((ae_{22})e_{33})e_{21}[23] - 8(((ae_{22})e_{33})e_{21}[23])e_{22},$$

$$\alpha_{11} = 32((((ae_{11})e_{22})e_{22}[12])e_{13})e_{13} + 8((ae_{11})e_{22})e_{22}[12] - 8(((ae_{11})e_{22})e_{22}[12])e_{11},$$

$$\alpha_{22} = 32((((ae_{11})e_{22})e_{11}[12])e_{13})e_{13} + 8((ae_{11})e_{22})e_{11}[12] - 8(((ae_{11})e_{22})e_{11}[12])e_{11},$$

$$\beta_{11} = 32((((ae_{11})e_{33})e_{22}[13])e_{12})e_{12} + 8((ae_{11})e_{33})e_{22}[13] - 8(((ae_{11})e_{33})e_{22}[13])e_{11},$$

$$\beta_{22} = 32((((ae_{11})e_{33})e_{11}[13])e_{12})e_{12} + 8((ae_{11})e_{33})e_{11}[13] - 8(((ae_{11})e_{33})e_{11}[13])e_{11},$$

$$\gamma_{11} = 32((((ae_{22})e_{33})e_{22}[23])e_{12})e_{12} + 8((ae_{22})e_{33})e_{22}[23] - 8(((ae_{22})e_{33})e_{22}[23])e_{22},$$

$$\gamma_{22} = 32((((ae_{22})e_{33})e_{11}[23])e_{12})e_{12} + 8((ae_{22})e_{33})e_{11}[23] - 8(((ae_{22})e_{33})e_{11}[23])e_{22}.$$

We have representations of basis elements of J as fractions with numerators $a_1, a_2, \ldots, a_{21}$ from B and the common denominator $z \in Z$. Then our identities imply that $z^5 B \simeq B$ embeds into the free finitely generated Z-module $\sum_{i=1}^{21} Z a_i$. $\square$

5. Acknowlegments

The work is supported by Mathematical Center in Akademgorodok, the agreement with Ministry of Science and High Education of the Russian Federation number 075-15-2019-1613.

The reported study was funded by RFBR, project number 19-31-90055.

The author is grateful to the referee for the useful comments.

The author also expresses his gratitude to P. S. Kolesnikov, whose remarks helped to improve this paper.

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