ON SUFFICIENT CONDITIONS FOR $Q$-UNIVERSALITY

M.V. SCHWIDEFSKY

Abstract. If a quasivariety $K$ contains a $B^*$-class then $K$ satisfies sufficient conditions for $Q$-universality found by V. A. Gorbunov.

Keywords: B-class, quasivariety, $Q$-universal.

1. Introduction

G. Birkhoff [3] and A. I. Maltsev [19] raised a problem by asking which lattices are isomorphic to quasivariety lattices; this problem is now referred to as the Birkhoff-Maltsev problem. Many results were obtained on such lattices which demonstrate their highly complex inner structure; some of them are presented in the monograph of V. A. Gorbunov [10]. M. V. Sapir introduced in [22] the notion of $Q$-universal quasivariety and constructed the first example of such quasivariety—his example was a quasivariety generated by a certain semigroup. In [1], M. E. Adams and W. Dziobiak found a sufficient condition for a quasivariety to be $Q$-universal, see Theorem 1. In [11], V. A. Gorbunov established some other sufficient conditions for $Q$-universality, see Theorem 2.

In the paper [13] by A. V. Kravchenko, A. M. Nurakunov, and the author, the notion of $B$-class was introduced. It was shown in [13] that if a quasivariety $K$ contains a $B$-class then it contains uncountably many subquasivarieties with no independent quasi-equational basis relative to some subquasivariety of $K$. Some other results which demonstrate the high complexity of the inner structure of quasivariety lattices for quasivarieties containing $B$-classes were obtained in [2, 12, 14, 15, 16, 17, 18, 20, 21]. Interesting results in the same direction concerning
quasivarieties of groups were obtained recently by A. I. Budkin in [7, 8], see also his earlier articles [4]-[6].

It was shown in [13] that if a quasivariety $K$ contains a $B$-class then $K$ satisfies the Adams-Dziobiak condition for $Q$-universality. In [24], the notion of $B$-class was generalized, and it was shown in particular that if a quasivariety $K$ contains a finite generalized $B$-class then $K$ also satisfies the Adams-Dziobiak condition.

We prove here in Theorem 6 that if a quasivariety $K$ contains a generalized $B$-class then $K$ satisfies the Gorbunov conditions for $Q$-universality.

2. Basic definitions

For a semilattice $P$, let $\text{Sub}P$ denote the lattice of all subsemilattices of $P$. For nonzero $n < \omega$, let $\mathcal{B}_n$ denote the $\cap$-semilattice and $\mathcal{B}_n'$ denote the $\cup$-semilattice of all subsets of an $n$-element set. Let also $I(FL(\omega))$ denote the ideal lattice of the free lattice of countable rank.

Let $\mathcal{E}$ denote the trivial structure of type $\sigma$ and let $T = \{\mathcal{E}\}$.

Let $K(\sigma)$ denote the class of all structures of similarity type $\sigma$ and let $K \subseteq K(\sigma)$. By $Q(K)$, we denote the quasivariety generated by $K$. By $H$, $S$, $P$, $P_s$, $L_s$, we denote the operators of taking homomorphic images, substructures, Cartesian products, subdirect products, and superdirect limits, respectively.

For a class operator $O$ and a class $M \subseteq K(\sigma)$, we put

$$(O \cap K)(M) = O(M) \cap K.$$ 

A subclass $K' \subseteq K$ is a $K$-quasivariety, if $K' = Q(K') \cap K$. The set of all $K$-quasivarieties forms a complete lattice under inclusion; we denote this lattice by $Lq(K)$ and call a $K$-quasivariety lattice or just a quasivariety lattice.

Let $K \subseteq M \subseteq K(\sigma)$. The class $K$ is a homogeneous quasi-Birkhoff sub-class in $M$ if for each $M' \subseteq M$, the equality

$$Q(M') \cap K = (L_s \cap K)(P_s \cap K)(S \cap K)(M')$$

holds. A family $\{K_i \subseteq K(\sigma) \mid i \in I\}$ is homomorphically disconnected if $K_i \cap S(K_j) = T$ for each distinct $i, j \in I$. Let $T \subseteq K \subseteq K(\sigma)$. A nontrivial structure $A \in K(\sigma)$ is homomorphically disconnected in $K$ if the family $\{K_\theta \mid \theta \in \text{Con}_K A\}$ is homomorphically disconnected, where $K_\theta = \{A/\theta, \mathcal{E}\}$ for each $\theta \in \text{Con}_K A$. Equivalently, a nontrivial structure $A \in K(\sigma)$ is homomorphically disconnected in $K$ if $\theta \in \{\theta', 1_A\}$ for each $\theta, \theta' \in \text{Con}_K A$ such that $A/\theta$ embeds into $A/\theta'$. A structure $A \in K$ is $K$-prime, if $\text{Con}_K A$ is a two-element lattice.

For all other definitions and notation concerning algebraic structures and quasivarieties, we refer to the monograph [10, Ch. 1] as well as to the papers [13, 14, 24].

3. Sufficient conditions for $Q$-universality

The following conditions were found in W. Dziobiak [9] and M. E. Adams and W. Dziobiak [1]. In the present form they appeared in [23].

Definition 1. If a class $A = \{A_X \mid X \in \mathcal{P}_{fin}(\omega)\}$ of structures of a finite similarity type $\sigma$ possesses the following properties:

- (P$_0$) for each $X \in \mathcal{P}_{fin}(\omega)$, the structure $A_X$ is $l$-projective in $Q(A)$ and the trivial congruence is a dually compact element in the relative congruence lattice $\text{Con}_{Q(A)} A_X$;
- (P$_1$) $A_\varnothing$ is a trivial structure;
- (P$_2$) for each $\vartheta \in \{\theta', 1_A\}$, the structure $A/\vartheta$ is a $K$-prime in $K$;
- (P$_3$) for each $X \in \mathcal{P}_{fin}(\omega)$, the structure $A_X$ is a trivial congruence in $\text{Con}_{Q(A)} A_X$;
- (P$_4$) the trivial congruence is a dually compact element in the relative congruence lattice $\text{Con}_{Q(A)} A_X$;
- (P$_5$) each $A_X$ is a $K$-prime in $K$;
- (P$_6$) for each $\theta \in \{\theta', 1_A\}$, the structure $A/\theta$ is a dually compact element in the relative congruence lattice $\text{Con}_{Q(A)} A_X$;
- (P$_7$) each $A_X$ is a trivial congruence in $\text{Con}_{Q(A)} A_X$;
- (P$_8$) for each $\vartheta \in \{\theta', 1_A\}$, the structure $A/\vartheta$ is a dually compact element in the relative congruence lattice $\text{Con}_{Q(A)} A_X$;
- (P$_9$) each $A_X$ is a trivial congruence in $\text{Con}_{Q(A)} A_X$;
- (P$_{10}$) for each $\vartheta \in \{\theta', 1_A\}$, the structure $A/\vartheta$ is a dually compact element in the relative congruence lattice $\text{Con}_{Q(A)} A_X$;
- (P$_{11}$) each $A_X$ is a trivial congruence in $\text{Con}_{Q(A)} A_X$;
- (P$_{12}$) for each $\vartheta \in \{\theta', 1_A\}$, the structure $A/\vartheta$ is a dually compact element in the relative congruence lattice $\text{Con}_{Q(A)} A_X$;
(P_2) if $X = Y \cup Z$ in $\mathcal{P}_{\text{fin}}(\omega)$, then $A_X \in \mathbf{Q}(A_Y, A_Z)$;
(P_3) if $\emptyset \neq X \subseteq \mathcal{P}_{\text{fin}}(\omega)$ and $A_X \in \mathbf{Q}(A_Y)$, then $X = Y$;
(P_4) if $A_X \leq B_0 \times B_1$ for some structures $B_0, B_1 \in \mathbf{Q}(A)$, then there are $Y_0, Y_1 \in \mathcal{P}_{\text{fin}}(\omega)$ such that $A_{Y_0} \in \mathbf{Q}(B_0), A_{Y_1} \in \mathbf{Q}(B_1)$, and $X = Y_0 \cup Y_1$ then $A$ is called an Adams-Dziobiak class or simply an AD-class.

For the following statement, we refer to [1, Theorem 3.3] as well as to [23, Corollary 3.5].

**Theorem 1.** Let a quasivariety $K$ contain an AD-class. Then $K$ is $Q$-universal and the lattice $\mathbf{L}(\omega)$ embeds into $\mathbf{L}_q(K)$.

The following definition is essentially due to V. A. Gorbunov [11], see also [10].

**Definition 2.** Let $\sigma$ be finite, let $A \subseteq K(\sigma)$ be a prevariety, and let a class $G = \{s_n \mid n < \omega\} \subseteq B \subseteq A$

possess the following properties:

- (E_1) $\{H(s_n) \cap B \mid n < \omega\}$ is a disconnected family of homogeneous quasi-Birkhoff subclasses of $A$;
- (E_2) for each $n < \omega$, the lattice $\mathbf{L}(H(s_n) \cap B)$ is finite and $\mathbf{Sub}_B$ is a homomorphic image of a sublattice in $\mathbf{L}(H(s_n) \cap B)$;
- (E_3) for each $n < \omega$, the structure $s_n$ is homomorphically disconnected in $B$ and $\mathbf{Con}_B s_n$ is a complete meet-subsemilattice in $\mathbf{Con}s_n$ which contains $B$, as a subsemilattice.

If $G$ satisfies (E_1) and (E_2) or (E_1) and (E_3) then $G$ is called a Gorbunov class or simply a G-class with respect to $B \subseteq A$.

We note that condition (E_2) is weaker than the corresponding condition in [11, 10]. Nonetheless the proof of the following theorem is identical to the proof of [10, Theorem 5.4.26], see also [11, Theorem 5.19].

**Theorem 2.** Let a prevariety $K$ contain a G-class with respect to some class $B \subseteq K$. Then $K$ is $Q$-universal and the lattice $\mathbf{L}(\omega)$ embeds into $\mathbf{L}_q(K)$.

4. B*-classes and the main result

The following definition was introduced in [24]. The definition of a B-class is due to [13].

**Definition 3.** Let $M \subseteq K(\sigma)$ be a quasivariety of a finite similarity type $\sigma$ and let $V \subseteq K(\sigma)$ be a nonempty homomorphically closed class. A class $A = \{A_F \mid F \in \mathcal{P}_{\text{fin}}(\omega)\} \subseteq M$ is called a B*-class with respect to $M$ and $V$ if $A$ satisfies the following conditions:

- (B_0) for each nonempty $F \in \mathcal{P}_{\text{fin}}(\omega)$, the structure $A_F$ is finitely presented in $M$; $A_\emptyset$ is a trivial structure;
- (B_1) if $F = G \cup H$ in $\mathcal{P}_{\text{fin}}(\omega)$ then $A_F \in \mathbf{Q}(A_G, A_H)$;
- (B_2) for each $F, G \in \mathcal{P}_{\text{fin}}(\omega)$, if $F \neq \emptyset$ and $A_F \in \mathbf{Q}(A_G, V)$ then $F = G$;
- (B_3) for every $F \in \mathcal{P}_{\text{fin}}(\omega)$ and every $i < \omega$, if $f \in \mathbf{Hom}(A_F, A_{\{i\}})$ then either $f(A_F) \in V$ or $i \in F$;
- (B_4) for each $F \in \mathcal{P}_{\text{fin}}(\omega)$, $(H(A_F) \cap M) \setminus V \subseteq A$.

If $V = T$ then we call $A$ a B-class with respect to $M$. 
Consider also the following conditions:

(B$_g^*$) for every $n < \omega$, the structure $A_{(n)}$ is M$^*$-simple, where M$^* = (M \setminus V) \cup \{\varepsilon\}.$

(B$^*$) for every $F, G \in \mathcal{P}_{fin}(\omega)$ such that $\emptyset \neq G \subseteq F$, for an arbitrary $B \in V$ and arbitrary homomorphisms $f \in \text{Hom}(A_F, B)$ and $g \in \text{Hom}(A_G, A_G)$, there is a homomorphism $h \in \text{Hom}(A_G, B)$ such that $f = hg$.

We cite some results from [24] which we use in the proof of our main result.

**Lemma 3.** [24, Lemma 1.3] Let $A = \{A_F \mid F \in \mathcal{P}_{fin}(\omega)\}$ be a B$^*$-class with respect to some quasivariety $M \subseteq K(\sigma)$ and to some variety $V \subseteq K(\sigma)$. The following statements hold.

(i) If $A_F \in V$ for some $F \in \mathcal{P}_{fin}(\omega)$ then $F = \emptyset$.

(ii) If $G \subseteq F \in \mathcal{P}_{fin}(\omega)$ then $A_G \in H(A_F)$.

(iii) If $f \in \text{Hom}(A_F, A_G)$ for some $F, G \in \mathcal{P}_{fin}(\omega)$ then either $f(A_F) \in V$ or $G \subseteq F$ and $f(A_F) \cong A_G$.

**Lemma 4.** For a quasivariety $M \subseteq K(\sigma)$ of finite type $\sigma$ containing (B$_g^*$) and (B$^*$) with respect to $M$ and some variety $V \subseteq K(\sigma)$, the following statements hold.

(i) For each $F \in \mathcal{P}_{fin}(\omega)$, there is an isomorphism $\xi: 2^F \to \text{Con}(M^*, A_F)$ such that $A_F/\xi(G) \cong A_F \cap G$ for all $G \subseteq F$. Hence $M^* = (M \setminus V) \cup \{\varepsilon\}$.

(ii) $F = G_0 \cup \ldots \cup G_k$ in $\mathcal{P}_{fin}(\omega)$ if and only if $A_F \leq_s A_{G_0} \times \ldots \times A_{G_k}$.

**Proof.** Statement (i) follows from Lemma 2.1 in [24].

We prove (ii). If $F = G_0 \cup \ldots \cup G_k$ in $\mathcal{P}_{fin}(\omega)$ then $A_F \leq_s A_{G_0} \times \ldots \times A_{G_k}$ by [24, Lemma 1.6(i)].

Conversely, suppose that $A_F \leq_s A_{G_0} \times \ldots \times A_{G_k}$ for some $F, G_0, \ldots, G_k \in \mathcal{P}_{fin}(\omega)$. We have $G = G_0 \cup \ldots \cup G_k \leq F$ by Lemma 3(iii). By Lemma 3(ii), there is a surjective homomorphism $f: A_F \to A_G$.

We prove that $A_G \leq_s A_{G_0} \times \ldots \times A_{G_k}$. Indeed, $A_G \in Q(A_{G_0} \times \ldots \times A_{G_k})$ by (B$_g$). Since $A_G$ is an l-projective structure by (B$_0$), we conclude that $A_G \in SP(A_{G_0}, \ldots, A_{G_k})$. Thus, there are structures $B_i \in S(A_{G_0}, \ldots, A_{G_k})$, $i \in T$, such that $A_G \leq \prod_{i \in T} B_i$. Applying (B$_1^*$) and (B$_2^*$), we conclude that for each $i \in T$, either $B_i \in V$ or $B_i \cong A_{G_i}$ for some $i \leq k$. Applying statement (i), we obtain that there is a set $J \subseteq \{0, \ldots, k\}$ and a structure $B \in V$ such that $A_G \leq B \times \prod_{i \in T} A_{G_i}$. Let $\pi: A_G \to B$ and $\pi_i: A_G \to A_{G_i}$, $i \in I$, denote the projection homomorphisms in the above subdirect decomposition. If $I \neq \emptyset$ then $A_G \in V$, whence $G = G_0 = \ldots = G_k = \emptyset$ by Lemma 3(i) and $A_G \leq_s A_{G_0} \times \ldots \times A_{G_k}$ holds trivially. Suppose therefore that $I \neq \emptyset$ and fix an element $j \in I$. Using (B$^*$), we conclude that there is a homomorphism $f: A_G \to B$ such that $f \pi_j = \pi$. Therefore $\ker \pi_j \subseteq \ker \pi$ and this inclusion implies that $0_{A_G} = \ker \pi \cap \prod_{i \in I} \ker \pi_i = \prod_{i \in \tilde{J}} \ker \pi_i$. According to Lemma 3(ii), for each $i \in \{0, \ldots, n\} \setminus I$, there is a surjective homomorphism $\pi_i: A_G \to A_{G_i}$. This implies that $0_{A_G} = \prod_{i \leq n} \ker \pi_i$ and $A_G \leq_s A_{G_0} \times \ldots \times A_{G_k}$ which is our desired conclusion.

By what we have just proved,

$$A_G \leq_s A_{G_0} \times \ldots \times A_{G_k}; \quad A_F \leq_s A_{G_0} \times \ldots \times A_{G_k},$$

Hence for each $i \leq k$, there is a surjective homomorphism $\pi_i: A_F \to A_G$ such that $\ker \pi_0 \cap \ldots \cap \ker \pi_k = 0_{A_F}$. Moreover, for each $i \leq k$, there is a surjective homomorphism $\rho_i: A_G \to A_{G_i}$. Fix an index $i \leq k$ and consider the congruence
This means that we have $(B^*_4)$ and Lemma 3(iii), there is a set $H \subseteq F$ such that

$$\mathcal{A}_H \cong \mathcal{A}_F / \theta \leq, \mathcal{A}_F / \ker \pi_i \times \mathcal{A}_F / \ker (\rho_i f) \cong \mathcal{A}_{G_i} \times \mathcal{A}_{G_i}.$$ 

This implies that $\mathcal{A}_H \in \mathcal{Q}(\mathcal{A}_{G_i})$, whence $H \subseteq G_i$ by $(B^*_3)$. Moreover, in view of $(B^*_4)$ and Lemma 3(i), $G_i \subseteq H$ whence $H = G_i$. We conclude that

$$\mathcal{A}_F / \theta \cong \mathcal{A}_{G_i} \cong \mathcal{A}_F / \ker \pi_i \cong \mathcal{A}_F / \ker (\rho_i f).$$

Therefore by statement (i), $\theta = \ker \pi_i = \ker (\rho_i f)$ for all $i \leq k$. Suppose that $(a_1, \ldots, a_m) \in \ker f(p)$ for some $p^m \in \sigma^p \cup \{\cdot\}$. Then for all $i \leq k$, we have $(a_1, \ldots, a_m) \in \ker (\rho_i f)(p)$. Therefore,

$$(a_1, \ldots, a_m) \in \ker (\rho_0 f)(p) \cap \ldots \cap \ker (\rho_k f)(p) = \ker \pi_0 (p) \cap \ldots \cap \ker \pi_k (p) = 0_{\mathcal{A}_F}(p).$$

This means that $f$ is an isomorphism and $\mathcal{A}_F \in \mathcal{Q}(\mathcal{A}_{G_i})$. Hence we get by $(B^*_3)$ that $F = G = G_0 \cup \ldots \cup G_k$. 

Lemma 5. Let a quasivariety $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ of finite type $\sigma$ contain a $\mathbf{B}^*$-class

$$\mathbf{A} = \{ \mathcal{A}_F \mid F \in \mathcal{P}_{\mathbf{fin}}(\omega) \} \subseteq \mathbf{M}$$

satisfying $(\mathbf{B}^*)$ with respect to $\mathbf{M}$ and some variety $\mathbf{V} \subseteq \mathbf{K}(\sigma)$. If $\mathcal{A}_F \in \mathcal{Q}(\mathbf{X})$ for some $F \in \mathcal{P}_{\mathbf{fin}}(\omega)$ and some $\mathbf{X} \subseteq \mathbf{A}$ then $\mathcal{A}_F \in \mathcal{P}_{\mathbf{s}}(\mathbf{X})$

Proof. Without loss of generality, we may assume that $F \neq \emptyset$. Since $\mathbf{V}$ is a variety, the structure $\mathcal{A}_F$ is $l$-projective in $\mathbf{M}$, and $\mathcal{A}_F \in \mathcal{Q}(\mathbf{X}) = \mathbf{L}_s \mathbf{SP}(\mathbf{X})$, there are a set $I$ and a family $\{ \mathcal{A}_{F_i} : \mathcal{A}_i \in \mathbf{X} \mid i \in I \}$ such that $\mathcal{A}_F \leq \prod_{i \in I} \mathcal{A}_{F_i}$. For each $i \in I$, let $\pi_i : \mathcal{A}_F \rightarrow \mathcal{A}_{F_i}$ denote the canonical projection. Then we have $\pi_i (\mathcal{A}_F) \in \mathbf{S}(\mathcal{A}_{F_i}) \subseteq \mathbf{S}(\mathbf{X}) \subseteq \mathbf{Q}(\mathbf{A})$ for all $i \in I$. Therefore by $(\mathbf{B}^*_3)$, either $\pi_i (\mathcal{A}_F) \in \mathbf{V}$ or $\pi_i (\mathcal{A}_F) \in \mathbf{A}$. In the second case, we have $\pi_i (\mathcal{A}_F) \in \mathbf{S}(\mathcal{A}_{F_i})$, whence $\pi_i (\mathcal{A}_F) \cong \mathcal{A}_{F_i} \in \mathbf{X}$ by $(\mathbf{B}^*_3)$. Thus, there are a set $J \subseteq I$ a structure $\mathbf{V} \in \mathbf{V}$ such that $\mathcal{A}_F \leq \mathbf{V} \times \prod_{i \in J} \mathcal{A}_{F_i}$, $\mathcal{A}_{F_i} \in \mathbf{X}$, and $F_i \neq \emptyset$ for each $i \in J$. Let $\pi : \mathcal{A}_F \rightarrow \mathbf{V}$ denote the canonical projection which is a surjective homomorphism. By Lemma 3(i), $\mathcal{A}_F \notin \mathbf{V}$, and we conclude that $J \neq \emptyset$; fix an element $j \in J$. According to $(\mathbf{B}^*)$, there is a homomorphism $h : \mathcal{A}_{F_j} \rightarrow \mathbf{V}$ such that $\pi = h \pi_j$; in particular, $\ker \pi_j \subseteq \ker \pi$ whence $\bigcap_{i \in J} \ker \pi_i = \ker \pi \cap \bigcap_{i \in J} \ker \pi_i = 0_{\mathcal{A}_F}$. This implies that $\mathcal{A}_F \leq \mathbf{V} \times \bigcap_{i \in J} \mathcal{A}_{F_i}$ whence $\mathcal{A}_F \in \mathcal{P}_{\mathbf{s}}(\mathbf{X})$. 

The following theorem is our main result.

Theorem 6. If a quasivariety $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ of finite type $\sigma$ contains a $\mathbf{B}^*$-class $\mathbf{A} \subseteq \mathbf{M}$ satisfying $(\mathbf{B}^*_4)$ and $(\mathbf{B}^*)$ with respect to $\mathbf{M}$ and some variety $\mathbf{V} \subseteq \mathbf{K}(\sigma)$, then there is a class $\mathbf{A}' \subseteq \mathbf{A}$ which satisfies $(\text{E}_1)$–$(\text{E}_3)$ with respect to some class $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$, whence $\mathbf{A}'$ is a $\mathbf{G}$-class with respect to $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$.

Proof. Let $\bigcup_{n < \omega} P_n = \omega$ be a partition of the set $\omega$ such that $|P_n| = n$ for each $n < \omega$. Let $\mathbf{A} = \{ \mathcal{A}_F \mid F \in \mathcal{P}_{\mathbf{fin}}(\omega) \}$ be a $\mathbf{B}^*$-class satisfying $(\mathbf{B}^*)$ with respect to $\mathbf{M}$ and $\mathbf{V}$. For each $n < \omega$, we put $\mathbb{S}_n = \mathcal{A}_{P_n}$. We prove that the class $\mathbf{G} = \{ \mathbb{S}_n \mid n < \omega \}$ is a $\mathbf{G}$-class with respect to $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$, where $\mathbf{B} = (\mathbf{Q}(\mathbf{A}) \setminus \mathbf{V}) \cup \{ \mathbf{E} \}$. We have in particular that $\mathbf{A} \subseteq \mathbf{B}$.

Claim 1. The class $\mathbf{G}$ satisfies $(\text{E}_1)$ with respect to $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$. 

Proof of Claim. Assume first that \( m, n < \omega \) are distinct and \( A \in \mathbf{H}(S_n) \cap B \) embeds into \( B \in \mathbf{H}(S_m) \cap B \). According to (B\(^2\)) and Lemma 3, there are sets \( F \subseteq P_m \), \( G \subseteq P_n \) such that \( A \cong A_F \) and \( B \cong A_G \). Since \( A_F \) embeds into \( A_G \), we conclude by (B\(^2\)) that either \( F = \emptyset \) or \( F = G \). If \( F = G \) then \( F = G \subseteq P_m \cap P_n = \emptyset \) whence \( F = \emptyset \). This proves that the family \( \{ \mathbf{H}(S_n) \cap B \mid n < \omega \} \) is disconnected.

Let now \( K_n = H(S_n) \cap B \) for some \( n < \omega \), let \( K \subseteq Q(A) \), and let \( A \in Q(K) \cap K_n \). This implies that \( A \in \mathbf{H}(S_n) \cap B \subseteq A \). Lemma 3(iii) yields that there is \( F \subseteq P_n \) such that \( A \cong A_F \). Since \( A_F \in B \), we conclude that \( A_F \notin V \), whence \( F \neq \emptyset \) by (B\(_0\)). Since \( V \) is a variety, the structure \( A_F \) is \( l \)-projective in \( M \), and \( A_F \in Q(K) \subseteq Q(A) \), there are a set \( I \) and a family \( \{ \mathcal{E}_i \in K \mid i \in I \} \) such that \( A_F \leq \prod_{i \in I} \mathcal{E}_i \).

For each \( i \in I \), let \( \pi_i : A_F \to \mathcal{E}_i \) denote the canonical projection. Then we have \( \pi_i(A_F) \in S(\mathcal{E}_i) \subseteq S(K) \subseteq Q(A) \), \( i \in I \). Therefore by (B\(^2\))\(_i\), either \( \pi_i(A_F) \in V \) or \( \pi_i(A_F) \in A \). Thus, there is a set \( J \subseteq I \), a family \( \{ F_i \subseteq F \mid i \in J \} \), and a structure \( V \) such that \( A_F \leq V \times \prod_{i \in J} A_F, A_F \in S(\mathcal{E}_i) \) and \( F_i \neq \emptyset \) for each \( i \in J \). We have therefore that \( A_F \in (S \cap K_n)(K) \) for each \( i \in J \). Let \( \pi : A_F \to V \) denote the canonical projection which is a surjective homomorphism.

As \( A_F \notin V \), we conclude that \( J \neq \emptyset \); fix an element \( j \in J \). According to (B\(^1\)), there is a homomorphism \( h : A_F \to V \) such that \( \pi = h \pi_j \); in particular, \( \pi_j \subseteq \pi \) whence \( \bigcap_{i \in J} \ker \pi_i = \ker \pi \cap \bigcap_{i \in J} \ker \pi_i = 0_{A_F} \). As \( A_F \in B \), this implies that \( A_F \in (P_n \cap K_n)(S \cap K_n)(K) \). Therefore \( Q(K) \cap K_n \subseteq (P_n \cap K_n)(S \cap K_n)(K) \) which proves that \( K_n \) is a homogeneous quasi-Birkhoff subclass of \( Q(A) \). □

Claim 2. The class \( G \) satisfies \((E_2)\) with respect to \( B \subseteq Q(A) \).

Proof of Claim. Let \( n < \omega \) and let \( K_n = \mathbf{H}(S_n) \cap B \). Then according to (B\(^2\))\(_i\) and Lemma 3, we have

\[
K_n = \mathbf{H}(S_n) \cap B = \{ A_F \mid F \subseteq P_n \}.
\]

Consider the mapping

\[
\psi : Lq(K_n) \to \text{Sub} B'_n, \quad \psi : X \mapsto \{ F \subseteq P_n \mid A_F \in X \}.
\]

Let \( F, G \in \psi(X) \) and let \( H = F \cup G \); then \( H \subseteq P_n \). Hence \( A_H \in K_n \). According to (B\(_i\)), we have \( A_H \in Q(A_F, A_G) \cap K_n \subseteq Q(X) \cap K_n = X \) as \( X \in Lq(K_n) \).

Therefore \( H \in \psi(X) \), \( \psi(X) \subseteq \text{Sub} B'_n \), and the mapping \( \psi \) is well-defined.

If \( X_0, X_1 \in Lq(K_n) \) are such that \( X_0 \nsubseteq X_1 \) then \( A_F \in X_0 \setminus X_1 \) for some \( F \subseteq P_n \).

Hence \( F \in \psi(X_0) \setminus \psi(X_1) \), and \( \psi \) is one-to-one.

It is clear that \( \psi \) preserves meets, whence \( \psi \) preserves the ordering. In order to prove that \( \psi \) preserves joins, it suffices to show that \( \psi(X_0 \cup X_1) \subseteq \psi(X_0) + \psi(X_1) \) for all \( X_0, X_1 \in Lq(K_n) \). Indeed, let \( F \in \psi(X_0 \cup X_1) \). This means that

\[
A_F \in X_0 \cup X_1 = Q(X_0 \cup X_1) \cap K_n.
\]

By Lemma 5, \( A_F \in P_n(X_0 \cup X_1) \cap K_n \). Thus, there is a family \( \{ F_i \mid i \in I \} \subseteq \psi(X_0) \cup \psi(X_1) \) such that \( A_F \leq \prod_{i \in I} A_{F_i} \); by Lemma 4(i), \( A_F \leq \prod_{i \in I} A_{F_i} \) for some finite set \( J \subseteq I \). According to Lemma 4(ii), \( F = \bigcup_{i \in J} F_i \in \psi(X_0) + \psi(X_1) \), which is our desired conclusion.

Finally, let \( S = \{ F_i \mid i \in I \} \subseteq \text{Sub} B'_n \) and let \( X = \{ A_{F_i} \mid i \in I \} \). In order to prove that \( \psi \) is onto, it suffices to show that \( X = Q(X) \cap K_n \subseteq Lq(K_n) \). To prove this, we consider an arbitrary structure \( A_F \in Q(X) \cap K_n \). By Lemma 5, \( A_F \in P_n(X) \). By Lemma 4(i), \( A_F \leq \prod_{i \in I} A_{F_i} \), where \( I \) is a finite set and \( A_{F_i} \in \text{X} \) for all \( i \in I \). According to Lemma 4(ii), \( F = \bigcup_{i \in J} F_i \in S \) whence \( A_F \in \text{X} \). Inclusion \( X \subseteq Q(X) \cap K_n \) is obvious.
Therefore, \( \psi \) is an isomorphism. It remains to note that \( \text{Sub} \ B'_n \cong \text{Sub} \ B_n \). \( \square \)

**Claim 3.** The class \( G \) satisfies \((E_3)\) with respect to \( B \subseteq Q(A) \).

**Proof of Claim.** Let \( n < \omega \) and let \( \theta, \theta' \in \text{Con}_B S_n \) be such that \( S_n/\theta \) embeds into \( S_n/\theta' \). Since \( S_n/\theta, S_n/\theta' \in H(S_n) \cap B \), using (B\(_n^2\)) and Lemma 3(iii), we get that \( S_n/\theta \cong A_F, S_n/\theta' \cong A_{F'} \) for some sets \( F, F' \subseteq P_n \). As \( A_F \) embeds into \( A_{F'} \), we conclude by (B\(_n^2\)) that either \( F = \emptyset \) or \( F = F' \). In the first case, \( \theta = 1_{S_n} \); in the second case, \( \theta = \theta' \) by Lemma 4(i). This proves that the structure \( S_n \) is homomorphically disconnected.

We prove now that \( \text{Con}_B S_n \) is a complete meet-subsemilattice in \( \text{Con} S_n \). Indeed, \( E \in B \) whence \( 1_{S_n} \in \text{Con}_B S_n \). Let \( I \neq \emptyset \), let \( \{ \theta_i \mid i \in I \} \subseteq \text{Con}_B S_n \) and let \( \theta = \bigcap_{i \in I} \theta_i \) in \( \text{Con} S_n \). If \( \theta \notin \text{Con}_B S_n \) then \( S_n/\theta \notin V \). Fix an element \( i \in I \). Since \( \theta \leq \theta_i \), we have \( S_n/\theta_i \in H(S_n/\theta) \subseteq H(V) \subseteq V \) which contradicts the choice of \( \theta_i \). This contradiction shows that \( \theta \in \text{Con}_B S_n \).

Finally, we get from Lemma 4(i) that \( \text{Con}_B S_n \cong B_n \). \( \square \)

The proof is complete. \( \square \)

From Definition 3 and Theorem 6, we get the following statement.

**Corollary 7.** If a quasivariety \( M \subseteq K(\sigma) \) of finite type \( \sigma \) contains a \( B \)-class \( A \) with respect to \( M \) then there is a class \( A' \subseteq A \) which satisfies \((E_1)-(E_4)\) with respect to some class \( B \subseteq Q(A) \), whence \( A' \) is a \( G \)-class with respect to \( B \subseteq Q(A) \).

It follows from [24, Remark 1.5] that a finite \( B^* \)-class satisfies \((B_n^*)\). Therefore Theorems 2 and 6 yield the following statement which generalizes [24, Corollary 3.4].

**Corollary 8.** Let a quasivariety \( M \subseteq K(\sigma) \) of finite type \( \sigma \) contain a \( B^* \)-class satisfying \((B_n^*)\) and \((B^*)\) with respect to \( M \) and some variety \( V \subseteq K(\sigma) \). Then \( M \) is \( Q \)-universal and \( I(FL(\omega)) \) embeds into \( L_q(M) \).

It is clear that a \( B \)-class \( A \) with respect to a quasivariety \( M \) satisfies \((B^*)\) with respect to \( M \) and \( T \). Moreover according to [13, Remark 2.4], \( A \) satisfies \((B_n^*)\) with respect to \( M \) and \( T \). Therefore we get the following

**Corollary 9.** Let a quasivariety \( M \subseteq K(\sigma) \) of finite type \( \sigma \) contain a \( B \)-class \( A \) with respect to \( M \). Then there is a class \( A' \subseteq A \) which satisfies \((E_1)-(E_3)\) with respect to some class \( B \subseteq Q(A) \), whence \( A' \) is a \( G \)-class with respect to \( B \subseteq Q(A) \).

As demonstrated in [14], many well-known quasivarieties contain \( B \)-classes. Moreover, it is shown in [24] that many quasivarieties contain \( B^* \)-classes (but do not contain \( B \) classes). In particular, the variety \( DM \) of differential groupoids contains a \( B^* \)-class satisfying \((B_n^*)\) and \((B^*)\) with respect to \( DM \) and the variety \( V(D_1) \), where \( D_1 = \langle \{a,b\}; \cdot \rangle \) and

\[
 a \cdot a = a \cdot b = a, \quad b \cdot a = b \cdot b = b.
\]

Moreover, according to [24, Corollary 6.11] each almost finite-to-finite universal quasivariety contains a \( B^* \)-class satisfying \((B_n^*)\) and \((B^*)\).
References

Marina Vladimirovna Schwidefsky
Sobolev Institute of Mathematics,
4, Acad. Koptyug ave.,
Novosibirsk, 630090, Russia
Novosibirsk State Technical University,
20, Karl Marx ave.,
Novosibirsk, 630073, Russia
Email address: semenova@math.nsc.ru