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ON SUFFICIENT CONDITIONS FOR Q-UNIVERSALITY

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Abstract. If a quasivariety $\mathbf{K}$ contains a B^*-class then $\mathbf{K}$ satisfies sufficient conditions for Q-universality found by V. A. Gorbunov.

Keywords: B-class, quasivariety, Q-universal.

1. Introduction

G. Birkhoff [3] and A. I. Maltsev [19] raised a problem by asking which lattices are isomorphic to quasivariety lattices; this problem is now referred to as the *Birkhoff–Maltsev problem*. Many results were obtained on such lattices which demonstrate their highly complex inner structure; some of them are presented in the monograph of V. A. Gorbunov [10]. M. V. Sapir introduced in [22] the notion of Q-universal quasivariety and constructed the first example of such quasivariety—his example was a quasivariety generated by a certain semigroup. In [1], M. E. Adams and W. Dziobiak found a sufficient condition for a quasivariety to be Q-universal, see Theorem 1. In [11], V. A. Gorbunov established some other sufficient conditions for Q-universality, see Theorem 2.

In the paper [13] by A. V. Kravchenko, A. M. Nurakunov, and the author, the notion of B-class was introduced. It was shown in [13] that if a quasivariety $\mathbf{K}$ contains a B-class then it contains uncountably many subquasivarieties with no independent quasi-equational basis relative to some subquasivariety of $\mathbf{K}$. Some other results which demonstrate the high complexity of the inner structure of quasivariety lattices for quasivarieties containing B-classes were obtained in [2, 12, 14, 15, 16, 17, 18, 20, 21]. Interesting results in the same direction concerning

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quasivarieties of groups were obtained recently by A. I. Budkin in [7, 8], see also his earlier articles [4]–[6].

It was shown in [13] that if a quasivariety $\mathbf{K}$ contains a B-class then $\mathbf{K}$ satisfies the Adams–Dziobiak condition for Q-universality. In [24], the notion of B-class was generalized, and it was shown in particular that if a quasivariety $\mathbf{K}$ contains a finite generalized B-class then $\mathbf{K}$ also satisfies the Adams–Dziobiak condition.

We prove here in Theorem 6 that if a quasivariety $\mathbf{K}$ contains a generalized B-class then $\mathbf{K}$ satisfies the Gorbunov conditions for Q-universality.

2. Basic definitions

For a semilattice $\mathcal{P}$, let $\operatorname{Sub}\mathcal{P}$ denote the lattice of all subsemilattices of $\mathcal{P}$. For nonzero $n < \omega$, let $\mathcal{B}_n$ denote the $\cap$-semilattice and $\mathcal{B}'_n$ denote the $\cup$-semilattice of all subsets of an n-element set. Let also $\mathrm{I}(FL(\omega))$ denote the ideal lattice of the free lattice of countable rank.

Let $\mathcal{E}$ denote the trivial structure of type σ and let $\mathbf{T} = \{\mathcal{E}\}$.

Let $\mathbf{K}(\sigma)$ denote the class of all structures of similarity type σ and let $\mathbf{K} \subseteq \mathbf{K}(\sigma)$. By $\mathbf{Q}(\mathbf{K})$, we denote the quasivariety generated by $\mathbf{K}$. By $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$, $\mathbf{P}_s$, $\mathbf{L}_s$, we denote the operators of taking homomorphic images, substructures, Cartesian products, subdirect products, and superdirect limits, respectively.

For a class operator $\mathbf{O}$ and a class $\mathbf{M} \subseteq \mathbf{K}(\sigma)$, we put

$$(\mathbf{O} \cap \mathbf{K})(\mathbf{M}) = \mathbf{O}(\mathbf{M}) \cap \mathbf{K}.$$

A subclass $\mathbf{K}' \subseteq \mathbf{K}$ is a $\mathbf{K}$-*quasivariety*, if $\mathbf{K}' = \mathbf{Q}(\mathbf{K}') \cap \mathbf{K}$. The set of all $\mathbf{K}$-quasivarieties forms a complete lattice under inclusion; we denote this lattice by $\mathrm{Lq}(\mathbf{K})$ and call a $\mathbf{K}$-*quasivariety lattice* or just a *quasivariety lattice*.

Let $\mathbf{K} \subseteq \mathbf{M} \subseteq \mathbf{K}(\sigma)$. The class $\mathbf{K}$ is a *homogeneous quasi-Birkhoff subclass in* $\mathbf{M}$ if for each $\mathbf{M}' \subseteq \mathbf{M}$, the equality

$$\mathbf{Q}(\mathbf{M}') \cap \mathbf{K} = (\mathbf{L}_s \cap \mathbf{K})(\mathbf{P}_s \cap \mathbf{K})(\mathbf{S} \cap \mathbf{K})(\mathbf{M}')$$

holds. A family $\{\mathbf{K}_i \subseteq \mathbf{K}(\sigma) \mid i \in I\}$ is *homomorphically disconnected* if $\mathbf{K}_i \cap \mathbf{S}(\mathbf{K}_j) = \mathbf{T}$ for each distinct $i, j \in I$. Let $\mathbf{T} \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. A nontrivial structure $\mathcal{A} \in \mathbf{K}(\sigma)$ is *homomorphically disconnected in* $\mathbf{K}$ if the family $\{\mathbf{K}_\theta \mid \theta \in \operatorname{Con}_{\mathbf{K}} \mathcal{A}\}$ is homomorphically disconnected, where $\mathbf{K}_\theta = \{\mathcal{A}/\theta, \mathcal{E}\}$ for each $\theta \in \operatorname{Con}_{\mathbf{K}} \mathcal{A}$. Equivalently, a nontrivial structure $\mathcal{A} \in \mathbf{K}(\sigma)$ is homomorphically disconnected in $\mathbf{K}$ if $\theta \in \{\theta', 1_{\mathcal{A}}\}$ for each $\theta, \theta' \in \operatorname{Con}_{\mathbf{K}} \mathcal{A}$ such that $\mathcal{A}/\theta$ embeds into $\mathcal{A}/\theta'$. A structure $\mathcal{A} \in \mathbf{K}$ is $\mathbf{K}$-*prime*, if $\operatorname{Con}_{\mathbf{K}} \mathcal{A}$ is a two-element lattice.

For all other definitions and notation concerning algebraic structures and quasivarieties, we refer to the monograph [10, Ch. 1] as well as to the papers [13, 14, 24].

3. Sufficient conditions for Q-universality

The following conditions were found in W. Dziobiak [9] and M. E. Adams and W. Dziobiak [1]. In the present form they appeared in [23].

Definition 1. If a class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{fin}(\omega)\}$ of structures of a finite similarity type σ possesses the following properties:

- (P_0) for each $X \in \mathcal{P}_{fin}(\omega)$, the structure $\mathcal{A}_X$ is l-projective in $\mathbf{Q}(\mathbf{A})$ and the trivial congruence is a dually compact element in the relative congruence lattice $\operatorname{Con}_{\mathbf{Q}(\mathbf{A})} \mathcal{A}_X$;
- (P_1) $\mathcal{A}_\varnothing$ is a trivial structure;

(P_2) if $X = Y \cup Z$ in $\mathcal{P}_{fin}(\omega)$, then $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y, \mathcal{A}_Z)$;
(P_3) if $\varnothing \neq X \in \mathcal{P}_{fin}(\omega)$ and $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y)$, then $X = Y$;
(P_4) if $\mathcal{A}_X \leq \mathcal{B}_0 \times \mathcal{B}_1$ for some structures $\mathcal{B}_0$, $\mathcal{B}_1 \in \mathbf{Q}(\mathbf{A})$, then there are Y_0, $Y_1 \in \mathcal{P}_{fin}(\omega)$ such that $\mathcal{A}_{Y_0} \in \mathbf{Q}(\mathcal{B}_0)$, $\mathcal{A}_{Y_1} \in \mathbf{Q}(\mathcal{B}_1)$, and $X = Y_0 \cup Y_1$

then $\mathbf{A}$ is called an *Adams–Dziobiak class* or simply an AD-*class*.

For the following statement, we refer to [1, Theorem 3.3] as well as to [23, Corollary 3.5].

Theorem 1. *Let a quasivariety* $\mathbf{K}$ *contain an* AD-*class. Then* $\mathbf{K}$ *is Q-universal and the lattice* $\mathrm{I}(FL(\omega))$ *embeds into* $\mathrm{Lq}(\mathbf{K})$.

The following definition is essentially due to V. A. Gorbunov [11], see also [10].

Definition 2. Let σ be finite, let $\mathbf{A} \subseteq \mathbf{K}(\sigma)$ be a prevariety, and let a class

$$\mathbf{G} = \{\mathcal{G}_n \mid n < \omega\} \subseteq \mathbf{B} \subseteq \mathbf{A}$$

possess the following properties:

(E_1) $\{\mathbf{H}(\mathcal{G}_n) \cap \mathbf{B} \mid n < \omega\}$ is a disconnected family of homogeneous quasi-Birkhoff subclasses of $\mathbf{A}$;
(E_2) for each $n < \omega$, the lattice $\mathrm{Lq}(\mathbf{H}(\mathcal{G}_n) \cap \mathbf{B})$ is finite and $\mathrm{Sub}\,\mathcal{B}_n$ is a homomorphic image of a sublattice in $\mathrm{Lq}(\mathbf{H}(\mathcal{G}_n) \cap \mathbf{B})$;
(E_3) for each $n < \omega$, the structure $\mathcal{G}_n$ is homomorphically disconnected in $\mathbf{B}$ and $\mathrm{Con}_{\mathbf{B}}\,\mathcal{G}_n$ is a complete meet-subsemilattice in $\mathrm{Con}\,\mathcal{G}_n$ which contains $\mathcal{B}_n$ as a subsemilattice.

If $\mathbf{G}$ satisfies (E_1) and (E_2) or (E_1) and (E_3) then $\mathbf{G}$ is called a *Gorbunov class* or simply a G-*class with respect to* $\mathbf{B} \subseteq \mathbf{A}$.

We note that condition (E_2) is weaker than the corresponding condition in [11, 10]. Nonetheless the proof of the following theorem is identical to the proof of [10, Theorem 5.4.26], see also [11, Theorem 5.19].

Theorem 2. *Let a prevariety* $\mathbf{K}$ *contain a* G-*class with respect to some class* $\mathbf{B} \subseteq \mathbf{K}$. *Then* $\mathbf{K}$ *is Q-universal and the lattice* $\mathrm{I}(FL(\omega))$ *embeds into* $\mathrm{Lq}(\mathbf{K})$.

4. B*-classes and the main result

The following definition was introduced in [24]. The definition of a B-class is due to [13].

Definition 3. Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a quasivariety of a finite similarity type σ and let $\mathbf{V} \subseteq \mathbf{K}(\sigma)$ be a nonempty homomorphically closed class. A class $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{M}$ is called a B*-*class with respect to* $\mathbf{M}$ *and* $\mathbf{V}$ if $\mathbf{A}$ satisfies the following conditions:

(B_0) for each nonempty $F \in \mathcal{P}_{fin}(\omega)$, the structure $\mathcal{A}_F$ is finitely presented in $\mathbf{M}$; $\mathcal{A}_\varnothing$ is a trivial structure;
(B_1) if $F = G \cup H$ in $\mathcal{P}_{fin}(\omega)$ then $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G, \mathcal{A}_H)$;
(B_2^*) for each $F, G \in \mathcal{P}_{fin}(\omega)$, if $F \neq \varnothing$ and $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G, \mathbf{V})$ then $F = G$;
(B_3^*) for every $F \in \mathcal{P}_{fin}(\omega)$ and every $i < \omega$, if $f \in \mathrm{Hom}(\mathcal{A}_F, \mathcal{A}_{\{i\}})$ then either $f(\mathcal{A}_F) \in \mathbf{V}$ or $i \in F$;
(B_4^*) for each $F \in \mathcal{P}_{fin}(\omega)$, $\big(\mathbf{H}(\mathcal{A}_F) \cap \mathbf{M}\big) \setminus \mathbf{V} \subseteq \mathbf{A}$.

If $\mathbf{V} = \mathbf{T}$ then we call $\mathbf{A}$ a B-*class with respect to* $\mathbf{M}$.

Consider also the following conditions:

(B_5^*) for every $n<\omega$, the structure $\mathcal{A}_{\{n\}}$ is $\mathbf{M}^*$-simple, where $\mathbf{M}^*=(\mathbf{M}\backslash\mathbf{V})\cup\{\mathcal{E}\}$.

(B^*) for every $F,G\in\mathcal{P}_{fin}(\omega)$ such that $\varnothing\neq G\subseteq F$, for an arbitrary $\mathcal{B}\in\mathbf{V}$ and arbitrary homomorphisms $f\in\mathrm{Hom}(\mathcal{A}_F,\mathcal{B})$ and $g\in\mathrm{Hom}(\mathcal{A}_F,\mathcal{A}_G)$, there is a homomorphism $h\in\mathrm{Hom}(\mathcal{A}_G,\mathcal{B})$ such that $f=hg$.

We cite some results from [24] which we use in the proof of our main result.

Lemma 3. [24, Lemma 1.3] *Let $\mathbf{A}=\{\mathcal{A}_F\mid F\in\mathcal{P}_{fin}(\omega)\}$ be a* B^**-class with respect to some quasivariety $\mathbf{M}\subseteq\mathbf{K}(\sigma)$ and to some variety $\mathbf{V}\subseteq\mathbf{K}(\sigma)$. The following statements hold.*

(i) *If $\mathcal{A}_F\in\mathbf{V}$ for some $F\in\mathcal{P}_{fin}(\omega)$ then $F=\varnothing$.*

(ii) *If $G\subseteq F\in\mathcal{P}_{fin}(\omega)$ then $\mathcal{A}_G\in\mathbf{H}(\mathcal{A}_F)$.*

(iii) *If $f\in\mathrm{Hom}(\mathcal{A}_F,\mathcal{A}_G)$ for some F, $G\in\mathcal{P}_{fin}(\omega)$ then either $f(\mathcal{A}_F)\in\mathbf{V}$ or $G\subseteq F$ and $f(\mathcal{A}_F)\cong\mathcal{A}_G$.*

Lemma 4. *For a quasivariety $\mathbf{M}\subseteq\mathbf{K}(\sigma)$ of finite type σ containing a* B^**-class $\mathbf{A}\subseteq\mathbf{M}$ satisfying* (B_5^*) *and* (B^*) *with respect to $\mathbf{M}$ and some variety $\mathbf{V}\subseteq\mathbf{K}(\sigma)$, the following statements hold.*

(i) *For each $F\in\mathcal{P}_{fin}(\omega)$, there is an isomorphism $\xi\colon 2^F\to\mathrm{Con}_{\mathbf{M}^*}\mathcal{A}_F$ such that $\mathcal{A}_F/\xi(G)\cong\mathcal{A}_{F\backslash G}$ for all $G\subseteq F$, where $\mathbf{M}^*=(\mathbf{M}\backslash\mathbf{V})\cup\{\mathcal{E}\}$.*

(ii) *$F=G_0\cup\ldots\cup G_k$ in $\mathcal{P}_{fin}(\omega)$ if and only if $\mathcal{A}_F\leq_s\mathcal{A}_{G_0}\times\ldots\times\mathcal{A}_{G_k}$.*

Proof. Statement (i) follows from Lemma 2.1 in [24].

We prove (ii). If $F=G_0\cup\ldots\cup G_k$ in $\mathcal{P}_{fin}(\omega)$ then $\mathcal{A}_F\leq_s\mathcal{A}_{G_0}\times\ldots\times\mathcal{A}_{G_k}$ by [24, Lemma 1.6(i)].

Conversely, suppose that $\mathcal{A}_F\leq_s\mathcal{A}_{G_0}\times\ldots\times\mathcal{A}_{G_k}$ for some F, $G_0,\ldots,G_k\in\mathcal{P}_{fin}(\omega)$. We have $G=G_0\cup\ldots\cup G_k\subseteq F$ by Lemma 3(iii). By Lemma 3(ii), there is a surjective homomorphism $f\colon\mathcal{A}_F\to\mathcal{A}_G$.

We prove that $\mathcal{A}_G\leq_s\mathcal{A}_{G_0}\times\ldots\times\mathcal{A}_{G_k}$. Indeed, $\mathcal{A}_G\in\mathbf{Q}(\mathcal{A}_{G_0}\times\ldots\times\mathcal{A}_{G_k})$ by (B_1). Since $\mathcal{A}_G$ is an l-projective structure by (B_0), we conclude that $\mathcal{A}_G\in\mathbf{SP}(\mathcal{A}_{G_0},\ldots,\mathcal{A}_{G_k})$. Thus, there are structures $\mathcal{B}_t\in\mathbf{S}(\mathcal{A}_{G_0},\ldots,\mathcal{A}_{G_k})$, $t\in T$, such that $\mathcal{A}_G\leq_s\prod_{t\in T}\mathcal{B}_t$. Applying ($B_4^*$) and ($B_2^*$), we conclude that for each $t\in T$, either $\mathcal{B}_t\in\mathbf{V}$ or $\mathcal{B}_t\cong\mathcal{A}_{G_i}$ for some $i\leqslant k$. Applying statement (i), we obtain that there is a set $I\subseteq\{0,\ldots,k\}$ and a structure $\mathcal{B}\in\mathbf{V}$ such that $\mathcal{A}_G\leq_s\mathcal{B}\times\prod_{i\in I}\mathcal{A}_{G_i}$. Let $\pi\colon\mathcal{A}_G\to\mathcal{B}$ and $\pi_i\colon\mathcal{A}_G\to\mathcal{A}_{G_i}$, $i\in I$, denote the projection homomorphisms in the above subdirect decomposition. If $I\neq\varnothing$ then $\mathcal{A}_G\in\mathbf{V}$, whence $G=G_0=\ldots=G_k=\varnothing$ by Lemma 3(i) and $\mathcal{A}_G\leq_s\mathcal{A}_{G_0}\times\ldots\times\mathcal{A}_{G_k}$ holds trivially. Suppose therefore that $I\neq\varnothing$ and fix an element $j\in I$. Using (B^*), we conclude that there is a homomorphism $f\colon\mathcal{A}_{G_j}\to\mathcal{B}$ such that $f\pi_j=\pi$. Therefore $\ker\pi_j\subseteq\ker\pi$ and this inclusion implies that $0_{\mathcal{A}_G}=\ker\pi\cap\bigcap_{i\in I}\ker\pi_i=\bigcap_{i\in I}\ker\pi_i$. According to Lemma 3(ii), for each $i\in\{0,\ldots,n\}\backslash I$, there is a surjective homomorphism $\pi_i\colon\mathcal{A}_G\to\mathcal{A}_{G_i}$. This implies that $0_{\mathcal{A}_G}=\bigcap_{i\leqslant n}\ker\pi_i$ and $\mathcal{A}_G\leq_s\mathcal{A}_{G_0}\times\ldots\times\mathcal{A}_{G_k}$ which is our desired conclusion.

By what we have just proved,

$$\mathcal{A}_G\leq_s\mathcal{A}_{G_0}\times\ldots\times\mathcal{A}_{G_k};\quad \mathcal{A}_F\leq_s\mathcal{A}_{G_0}\times\ldots\times\mathcal{A}_{G_k},$$

Hence for each $i\leqslant k$, there is a surjective homomorphism $\pi_i\colon\mathcal{A}_F\to\mathcal{A}_{G_i}$ such that $\ker\pi_0\cap\ldots\cap\ker\pi_k=0_{\mathcal{A}_F}$. Moreover, for each $i\leqslant k$, there is a surjective homomorphism $\rho_i\colon\mathcal{A}_G\to\mathcal{A}_{G_i}$. Fix an index $i\leqslant k$ and consider the congruence

$\theta = \ker \pi_i \cap \ker(\rho_i f)$. It is clear that $\theta \in \mathrm{Con}_{\mathbf{M}^*} \mathcal{A}_F$, where $\mathbf{M}^* = (\mathbf{M} \backslash \mathbf{V}) \cup \{\mathcal{E}\}$. Thus in view of ($\mathrm{B}_4^*$) and Lemma 3(iii), there is a set $H \subseteq F$ such that

$$\mathcal{A}_H \cong \mathcal{A}_F/\theta \leq_s \mathcal{A}_F/\ker \pi_i \times \mathcal{A}_F/\ker(\rho_i f) \cong \mathcal{A}_{G_i} \times \mathcal{A}_{G_i}.$$

This implies that $\mathcal{A}_H \in \mathbf{Q}(\mathcal{A}_{G_i})$, whence $H \subseteq G_i$ by (B_2^*). Moreover, in view of (B_3^*) and Lemma 3(i), $G_i \subseteq H$ whence $H = G_i$. We conclude that

$$\mathcal{A}_F/\theta \cong \mathcal{A}_{G_i} \cong \mathcal{A}_F/\ker \pi_i \cong \mathcal{A}_F/\ker(\rho_i f).$$

Therefore by statement (i), $\theta = \ker \pi_i = \ker(\rho_i f)$ for all $i \leqslant k$.

Suppose that $(a_1, \ldots, a_m) \in \ker f(p)$ for some $p^m \in \sigma^P \cup \{=\}$. Then for all $i \leqslant k$, we have $(a_1, \ldots, a_m) \in \ker(\rho_i f)(p)$. Therefore,

$$(a_1, \ldots, a_m) \in \ker(\rho_0 f)(p) \cap \ldots \cap \ker(\rho_k f)(p) = \ker \pi_0(p) \cap \ldots \cap \ker \pi_k(p) = 0_{\mathcal{A}_F}(p).$$

This means that f is an isomorphism and $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G)$. Hence we get by ($\mathrm{B}_2^*$) that $F = G = G_0 \cup \ldots \cup G_k$. □

Lemma 5. *Let a quasivariety* $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *of finite type* σ *contain a* B^**-class*

$$\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{M}$$

satisfying (B^*) *with respect to* $\mathbf{M}$ *and some variety* $\mathbf{V} \subseteq \mathbf{K}(\sigma)$. *If* $\mathcal{A}_F \in \mathbf{Q}(\mathbf{X})$ *for some* $F \in \mathcal{P}_{fin}(\omega)$ *and some* $\mathbf{X} \subseteq \mathbf{A}$ *then* $\mathcal{A}_F \in \mathbf{P}_s(\mathbf{X})$

Proof. Without loss of generality, we may assume that $F \neq \varnothing$. Since $\mathbf{V}$ is a variety, the structure $\mathcal{A}_F$ is l-projective in $\mathbf{M}$, and $\mathcal{A}_F \in \mathbf{Q}(\mathbf{X}) = \mathbf{L}_s\mathbf{SP}(\mathbf{X})$, there are a set I and a family $\{\mathcal{A}_{F_i} \in \mathbf{X} \mid i \in I\}$ such that $\mathcal{A}_F \leq \prod_{i \in I} \mathcal{A}_{F_i}$. For each $i \in I$, let $\pi_i \colon \mathcal{A}_F \to \mathcal{A}_{F_i}$ denote the canonical projection. Then we have $\pi_i(\mathcal{A}_F) \in \mathbf{S}(\mathcal{A}_{F_i}) \subseteq \mathbf{S}(\mathbf{X}) \subseteq \mathbf{Q}(\mathbf{A})$ for all $i \in I$. Therefore by (B_4^*), either $\pi_i(\mathcal{A}_F) \in \mathbf{V}$ or $\pi_i(\mathcal{A}_F) \in \mathbf{A}$. In the second case, we have $\pi_i(\mathcal{A}_F) \in \mathbf{S}(\mathcal{A}_{F_i})$, whence $\pi_i(\mathcal{A}_F) \cong \mathcal{A}_{F_i} \in \mathbf{X}$ by (B_2^*). Thus, there are a set $J \subseteq I$ a structure $\mathcal{V} \in \mathbf{V}$ such that $\mathcal{A}_F \leq_s \mathcal{V} \times \prod_{i \in J} \mathcal{A}_{F_i}$, $\mathcal{A}_{F_i} \in \mathbf{X}$, and $F_i \neq \varnothing$ for each $i \in J$. Let $\pi \colon \mathcal{A}_F \to \mathcal{V}$ denote the canonical projection which is a surjective homomorphism. By Lemma 3(i), $\mathcal{A}_F \notin \mathbf{V}$, and we conclude that $J \neq \varnothing$; fix an element $j \in J$. According to (B^*), there is a homomorphism $h \colon \mathcal{A}_{F_j} \to \mathcal{V}$ such that $\pi = h\pi_j$; in particular, $\ker \pi_j \subseteq \ker \pi$ whence $\bigcap_{i \in J} \ker \pi_i = \ker \pi \cap \bigcap_{i \in J} \ker \pi_i = 0_{\mathcal{A}_F}$. This implies that $\mathcal{A}_F \leq_s \prod_{i \in J} \mathcal{A}_{F_i}$ whence $\mathcal{A}_F \in \mathbf{P}_s(\mathbf{X})$. □

The following theorem is our main result.

Theorem 6. *If a quasivariety* $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *of finite type* σ *contains a* B^**-class* $\mathbf{A} \subseteq \mathbf{M}$ *satisfying* (B_5^*) *and* (B^*) *with respect to* $\mathbf{M}$ *and some variety* $\mathbf{V} \subseteq \mathbf{K}(\sigma)$, *then there is a class* $\mathbf{A}' \subseteq \mathbf{A}$ *which satisfies* (E_1)–(E_3) *with respect to some class* $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$, *whence* $\mathbf{A}'$ *is a* G*-class with respect to* $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$.

Proof. Let $\cup_{n<\omega} P_n = \omega$ be a partition of the set ω such that $|P_n| = n$ for each $n < \omega$. Let $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\}$ be a B^*-class satisfying (B^*) with respect to $\mathbf{M}$ and $\mathbf{V}$. For each $n < \omega$, we put $\mathcal{G}_n = \mathcal{A}_{P_n}$. We prove that the class $\mathbf{G} = \{\mathcal{G}_n \mid n < \omega\}$ is a G-class with respect to $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$, where $\mathbf{B} = (\mathbf{Q}(\mathbf{A}) \backslash \mathbf{V}) \cup \{\mathcal{E}\}$. We have in particular that $\mathbf{A} \subseteq \mathbf{B}$.

Claim 1. *The class* $\mathbf{G}$ *satisfies* (E_1) *with respect to* $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$.

Proof of Claim. Assume first that $m, n < \omega$ are distinct and $\mathcal{A} \in \mathbf{H}(\mathcal{G}_m) \cap \mathbf{B}$ embeds into $\mathcal{B} \in \mathbf{H}(\mathcal{G}_n) \cap \mathbf{B}$. According to (B_4^*) and Lemma 3, there are sets $F \subseteq P_m$, $G \subseteq P_n$ such that $\mathcal{A} \cong \mathcal{A}_F$ and $\mathcal{B} \cong \mathcal{A}_G$. Since $\mathcal{A}_F$ embeds into $\mathcal{A}_G$, we conclude by (B_2^*) that either $F = \varnothing$ or $F = G$. If $F = G$ then $F = G \subseteq P_m \cap P_n = \varnothing$ whence $F = \varnothing$. This proves that the family $\{\mathbf{H}(\mathcal{G}_n) \cap \mathbf{B} \mid n < \omega\}$ is disconnected.

Let now $\mathbf{K}_n = \mathbf{H}(\mathcal{G}_n) \cap \mathbf{B}$ for some $n < \omega$, let $\mathbf{K} \subseteq \mathbf{Q}(\mathbf{A})$, and let $\mathcal{A} \in \mathbf{Q}(\mathbf{K}) \cap \mathbf{K}_n$. This implies that $\mathcal{A} \in \mathbf{H}(\mathcal{G}_n) \cap \mathbf{B} \subseteq \mathbf{A}$. Lemma 3(iii) yields that there is $F \subseteq P_n$ such that $\mathcal{A} \cong \mathcal{A}_F$. Since $\mathcal{A}_F \in \mathbf{B}$, we conclude that $\mathcal{A}_F \notin \mathbf{V}$, whence $F \neq \varnothing$ by (B_0). Since $\mathbf{V}$ is a variety, the structure $\mathcal{A}_F$ is l-projective in $\mathbf{M}$, and $\mathcal{A}_F \in \mathbf{Q}(\mathbf{K}) \subseteq \mathbf{Q}(\mathbf{A})$, there are a set I and a family $\{\mathcal{C}_i \in \mathbf{K} \mid i \in I\}$ such that $\mathcal{A}_F \leq \prod_{i \in I} \mathcal{C}_i$. For each $i \in I$, let $\pi_i \colon \mathcal{A}_F \to \mathcal{C}_i$ denote the canonical projection. Then we have $\pi_i(\mathcal{A}_F) \in \mathbf{S}(\mathcal{C}_i) \subseteq \mathbf{S}(\mathbf{K}) \subseteq \mathbf{Q}(\mathbf{A})$, $i \in I$. Therefore by (B_4^*), either $\pi_i(\mathcal{A}_F) \in \mathbf{V}$ or $\pi_i(\mathcal{A}_F) \in \mathbf{A}$. Thus, there is a set $J \subseteq I$, a family $\{F_i \subseteq F \mid i \in J\}$, and a structure $\mathcal{V} \in \mathbf{V}$ such that $\mathcal{A}_F \leq_s \mathcal{V} \times \prod_{i \in J} \mathcal{A}_{F_i}$, $\mathcal{A}_{F_i} \in \mathbf{S}(\mathcal{C}_i)$ and $F_i \neq \varnothing$ for each $i \in J$. We have therefore that $\mathcal{A}_{F_i} \in (\mathbf{S} \cap \mathbf{K}_n)(\mathbf{K})$ for each $i \in J$. Let $\pi \colon \mathcal{A}_F \to \mathcal{V}$ denote the canonical projection which is a surjective homomorphism. As $\mathcal{A}_F \notin \mathbf{V}$, we conclude that $J \neq \varnothing$; fix an element $j \in J$. According to (B^*), there is a homomorphism $h \colon \mathcal{A}_{F_j} \to \mathcal{V}$ such that $\pi = h\pi_j$; in particular, $\ker \pi_j \subseteq \ker \pi$ whence $\bigcap_{i \in J} \ker \pi_i = \ker \pi \cap \bigcap_{i \in J} \ker \pi_i = 0_{\mathcal{A}_F}$. As $\mathcal{A}_F \in \mathbf{B}$, this implies that $\mathcal{A}_F \in (\mathbf{P}_s \cap \mathbf{K}_n)(\mathbf{S} \cap \mathbf{K}_n)(\mathbf{K})$. Therefore $\mathbf{Q}(\mathbf{K}) \cap \mathbf{K}_n \subseteq (\mathbf{P}_s \cap \mathbf{K}_n)(\mathbf{S} \cap \mathbf{K}_n)(\mathbf{K})$ which proves that $\mathbf{K}_n$ is a homogeneous quasi-Birkhoff subclass of $\mathbf{Q}(\mathbf{A})$. □

Claim 2. *The class* $\mathbf{G}$ *satisfies* (E_2) *with respect to* $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$.

Proof of Claim. Let $n < \omega$ and let $\mathbf{K}_n = \mathbf{H}(\mathcal{G}_n) \cap \mathbf{B}$. Then according to (B_4^*) and Lemma 3, we have

$$\mathbf{K}_n = \mathbf{H}(\mathcal{G}_n) \cap \mathbf{B} = \{\mathcal{A}_F \mid F \subseteq P_n\}.$$

Consider the mapping

$$\psi \colon \mathrm{Lq}(\mathbf{K}_n) \to \mathrm{Sub}\, \mathcal{B}_n', \quad \psi \colon \mathbf{X} \mapsto \{F \subseteq P_n \mid \mathcal{A}_F \in \mathbf{X}\}.$$

Let $F, G \in \psi(\mathbf{X})$ and let $H = F \cup G$; then $H \subseteq P_n$. Hence $\mathcal{A}_H \in \mathbf{K}_n$. According to (B_1), we have $\mathcal{A}_H \in \mathbf{Q}(\mathcal{A}_F, \mathcal{A}_G) \cap \mathbf{K}_n \subseteq \mathbf{Q}(\mathbf{X}) \cap \mathbf{K}_n = \mathbf{X}$ as $\mathbf{X} \in \mathrm{Lq}(\mathbf{K}_n)$. Therefore $H \in \psi(\mathbf{X})$, $\psi(\mathbf{X}) \in \mathrm{Sub}\, \mathcal{B}_n'$, and the mapping ψ is well-defined.

If $\mathbf{X}_0, \mathbf{X}_1 \in \mathrm{Lq}(\mathbf{K}_n)$ are such that $\mathbf{X}_0 \nsubseteq \mathbf{X}_1$ then $\mathcal{A}_F \in \mathbf{X}_0 \backslash \mathbf{X}_1$ for some $F \subseteq P_n$. Hence $F \in \psi(\mathbf{X}_0) \backslash \psi(\mathbf{X}_1)$, and ψ is one-to-one.

It is clear that ψ preserves meets, whence ψ preserves the ordering. In order to prove that ψ preserves joins, it suffices to show that $\psi(\mathbf{X}_0 \vee \mathbf{X}_1) \subseteq \psi(\mathbf{X}_0) + \psi(\mathbf{X}_1)$ for all $\mathbf{X}_0, \mathbf{X}_1 \in \mathrm{Lq}(\mathbf{K}_n)$. Indeed, let $F \in \psi(\mathbf{X}_0 \vee \mathbf{X}_1)$. This means that

$$\mathcal{A}_F \in \mathbf{X}_0 \vee \mathbf{X}_1 = \mathbf{Q}(\mathbf{X}_0 \cup \mathbf{X}_1) \cap \mathbf{K}_n.$$

By Lemma 5, $\mathcal{A}_F \in \mathbf{P}_s(\mathbf{X}_0 \cup \mathbf{X}_1) \cap \mathbf{K}_n$. Thus, there is a family $\{F_i \mid i \in I\} \subseteq \psi(\mathbf{X}_0) \cup \psi(\mathbf{X}_1)$ such that $\mathcal{A}_F \leq_s \prod_{i \in I} \mathcal{A}_{F_i}$. By Lemma 4(i), $\mathcal{A}_F \leq_s \prod_{i \in J} \mathcal{A}_{F_i}$ for some finite set $J \subseteq I$. According to Lemma 4(ii), $F = \bigcup_{i \in J_0} F_i \in \psi(\mathbf{X}_0) + \psi(\mathbf{X}_1)$, which is our desired conclusion.

Finally, let $\mathcal{S} = \{F_i \mid i \in I\} \in \mathrm{Sub}\, \mathcal{B}_n'$ and let $\mathbf{X} = \{\mathcal{A}_{F_i} \mid i \in I\}$. In order to prove that ψ is onto, it suffices to show that $\mathbf{X} = \mathbf{Q}(\mathbf{X}) \cap \mathbf{K}_n \in \mathrm{Lq}(\mathbf{K}_n)$. To prove this, we consider an arbitrary structure $\mathcal{A}_F \in \mathbf{Q}(\mathbf{X}) \cap \mathbf{K}_n$. By Lemma 5, $\mathcal{A}_F \in \mathbf{P}_s(\mathbf{X})$. By Lemma 4(i), $\mathcal{A}_F \leq_s \prod_{i \in I} \mathcal{A}_{F_i}$, where I is a finite set and $\mathcal{A}_{F_i} \in \mathbf{X}$ for all $i \in I$. According to Lemma 4(ii), $F = \bigcup_{i \in J_0} F_i \in \mathcal{S}$ whence $\mathcal{A}_F \in \mathbf{X}$. Inclusion $\mathbf{X} \subseteq \mathbf{Q}(\mathbf{X}) \cap \mathbf{K}_n$ is obvious.

Therefore, ψ is an isomorphism. It remains to note that $\operatorname{Sub}\mathcal{B}'_n \cong \operatorname{Sub}\mathcal{B}_n$. □

Claim 3. *The class* $\mathbf{G}$ *satisfies* (E_3) *with respect to* $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$.

Proof of Claim. Let $n < \omega$ and let $\theta, \theta' \in \operatorname{Con}_{\mathbf{B}} \mathcal{G}_n$ be such that $\mathcal{G}_n/\theta$ embeds into $\mathcal{G}_n/\theta'$. Since $\mathcal{G}_n/\theta$, $\mathcal{G}_n/\theta' \in \mathbf{H}(\mathcal{G}_n) \cap \mathbf{B}$, using ($\mathrm{B}_4^*$) and Lemma 3(iii), we get that $\mathcal{G}_n/\theta \cong \mathcal{A}_F$, $\mathcal{G}_n/\theta' \cong \mathcal{A}_{F'}$ for some sets $F, F' \subseteq P_n$. As $\mathcal{A}_F$ embeds into $\mathcal{A}_{F'}$, we conclude by (B_2^*) that either $F = \varnothing$ or $F = F'$. In the first case, $\theta = 1_{\mathcal{G}_n}$; in the second case, $\theta = \theta'$ by Lemma 4(i). This proves that the structure $\mathcal{G}_n$ is homomorphically disconnected.

We prove now that $\operatorname{Con}_{\mathbf{B}} \mathcal{G}_n$ is a complete meet-subsemilattice in $\operatorname{Con} \mathcal{G}_n$. Indeed, $\mathcal{E} \in \mathbf{B}$ whence $1_{\mathcal{G}_n} \in \operatorname{Con}_{\mathbf{B}} \mathcal{G}_n$. Let $I \neq \varnothing$, let $\{\theta_i \mid i \in I\} \subseteq \operatorname{Con}_{\mathbf{B}} \mathcal{G}_n$ and let $\theta = \bigcap_{i \in I} \theta_i$ in $\operatorname{Con} \mathcal{G}_n$. If $\theta \notin \operatorname{Con}_{\mathbf{B}} \mathcal{G}_n$ then $\mathcal{G}_n/\theta \in \mathbf{V}$. Fix an element $i \in I$. Since $\theta \leq \theta_i$, we have $\mathcal{G}_n/\theta_i \in \mathbf{H}(\mathcal{G}_n/\theta) \subseteq \mathbf{H}(\mathbf{V}) \subseteq \mathbf{V}$ which contradicts the choice of θ_i. This contradiction shows that $\theta \in \operatorname{Con}_{\mathbf{B}} \mathcal{G}_n$.

Finally, we get from Lemma 4(i) that $\operatorname{Con}_{\mathbf{B}} \mathcal{G}_n \cong \mathcal{B}_n$. □

The proof is complete. □

From Definition 3 and Theorem 6, we get the following statement.

Corollary 7. *If a quasivariety* $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *of finite type* σ *contains a* B*-class* $\mathbf{A}$ *with respect to* $\mathbf{M}$ *then there is a class* $\mathbf{A}' \subseteq \mathbf{A}$ *which satisfies* (E_1)–(E_3) *with respect to some class* $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$*, whence* $\mathbf{A}'$ *is a* G*-class with respect to* $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$.

It follows from [24, Remark 1.5] that a finite B^*-class satisfies (B_5^*). Therefore Theorems 2 and 6 yield the following statement which generalizes [24, Corollary 3.4].

Corollary 8. *Let a quasivariety* $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *of finite type* σ *contain a* B^**-class satisfying* (B_5^*) *and* (B^*) *with respect to* $\mathbf{M}$ *and some variety* $\mathbf{V} \subseteq \mathbf{K}(\sigma)$*. Then* $\mathbf{M}$ *is* Q*-universal and* $\mathrm{I}(FL(\omega))$ *embeds into* $\mathrm{Lq}(\mathbf{M})$.

It is clear that a B-class $\mathbf{A}$ with respect to a quasivariety $\mathbf{M}$ satisfies (B^*) with respect to $\mathbf{M}$ and $\mathbf{T}$. Moreover according to [13, Remark 2.4], $\mathbf{A}$ satisfies (B_5^*) with respect to $\mathbf{M}$ and $\mathbf{T}$. Therefore we get the following

Corollary 9. *Let a quasivariety* $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *of finite type* σ *contain a* B*-class* $\mathbf{A}$ *with respect to* $\mathbf{M}$*. Then there is a class* $\mathbf{A}' \subseteq \mathbf{A}$ *which satisfies* (E_1)–(E_3) *with respect to some class* $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$*, whence* $\mathbf{A}'$ *is a* G*-class with respect to* $\mathbf{B} \subseteq \mathbf{Q}(\mathbf{A})$.

As demonstrated in [14], many well-known quasivarieties contain B-classes. Moreover, it is shown in [24] that many quasivarieties contain B^*-classes (but do not contain B classes). In particular, the variety $\mathbf{Dm}$ of differential groupoids contains a B^*-class satisfying (B_5^*) and (B^*) with respect to $\mathbf{Dm}$ and the variety $\mathbf{V}(\mathcal{D}_1)$, where $\mathcal{D}_1 = \langle \{a, b\}; \cdot \rangle$ and

$$a \cdot a = a \cdot b = a, \quad b \cdot a = b \cdot b = b.$$

Moreover, according to [24, Corollary 6.11] each almost finite-to-finite universal quasivariety contains a B^*-class satisfying (B_5^*) and (B^*).

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