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MONOMIAL ROTA—BAXTER OPERATORS ON FREE COMMUTATIVE NON-UNITAL ALGEBRA

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Abstract. A Rota—Baxter operator defined on the polynomial algebra is called monomial if it maps each monomial to a monomial with some coefficient. We classify monomial Rota—Baxter operators defined on the algebra of polynomials in one variable with no constant term. We also describe injective monomial Rota—Baxter operators of nonzero weight on the algebra of polynomials in several variables with no constant term.

Keywords: Rota—Baxter operator, polynomial algebra.

1. Introduction

A linear operator R defined on a (non-asssociative, in general) algebra A over a field $\Bbbk$ is called a Rota—Baxter operator (RB-operator), if the relation

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy) \tag{1}$$

holds for all $x, y \in A$. Here λ is a fixed constant from $\Bbbk$ called a weight of R. An algebra equipped with a Rota—Baxter operator is called a Rota—Baxter algebra.

G. Baxter introduced the notion of a Rota—Baxter operator in 1960 [1] as a natural generalization of the integration by parts formula for the integral operator. Further, many authors including G.-C. Rota, P. Cartier, and L. Guo have studied RB-operators, see details in [2, 3]. There are deep connections of Rota—Baxter algebras with mathematical physics, number theory, operad theory, combinatorics, and others.

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In the theory of Rota—Baxter operators, one possible direction is to study RB-operators on polynomials [4, 5] and power series [6, 7]. Since both algebras are infinite-dimensional, it is hard to give a classification of all RB-operators on them. Thus, RB-operators of a special kind were introduced into consideration. One of such special classes of RB-operators is a class of monomial RB-operators [5], i.e., RB-operators that map every monomial to some monomial with (maybe zero) coefficient. L. Guo, M. Rosenkranz, and S.H. Zheng described all injective monomial RB-operators of weight zero on $\Bbbk[x]$ in 2015. In 2016 [4], H. Yu classified all monomial RB-operators of any weight on $\Bbbk[x]$.

In the study of RB-operators, it is crucial that the algebra has a unit (i.e., a multiplicative identity). As it is shown in [3], RB-operators on a unital algebra are subject to various additional restrictions compared to non-unital algebras.

In this paper, we describe the structure of all monomial RB-operators of weight zero (Theorem 1) and nonzero (Theorem 2) on $\Bbbk_0[x]$, which is a free commutative non-unital algebra generated by x. As a corollary, we obtain the complete classification by H. Yu (Corollaries 2 and 3). Further, we describe injective monomial RB-operators of nonzero weight on both $\Bbbk_0[X]$ (Theorem 3) and $\Bbbk[X]$ (Corollary 4). In relation to the results mentioned above, for a finite-dimensional algebra, a partial grading by the spectrum of its RB-operator (Propositions 2 and 3) is stated. The analogous grading in case of derivations and automorphisms is well-known [8]. As examples of such partial gradings, we consider monomial RB-operators on the quotient of $\Bbbk_0[x]$ by the ideal generated by x^{N+1} for some $N > 0$ (Examples 7 and 9).

Everywhere except for the last section we assume that a ground field $\Bbbk$ has characteristic zero.

2. Preliminaries

Let us start with some basic properties of Rota—Baxter operators.

Trivial RB-operators of weight λ are the zero operator and $-\lambda\mathrm{id}$.

Lemma 1 ([2, 9]). *Suppose that A is an algebra and P is an RB-operator of weight λ on A.*

a) The operator $\lambda^{-1}P$ is an RB-operator of weight 1 provided that $\lambda \neq 0$,

b) Given an automorphism $\psi \in \mathrm{Aut}(A)$, the operator $\psi^{-1}P\psi$ is an RB-operator of weight λ on A.

Lemma 2 ([2, 9]). *Let the algebra A split as a vector space into a direct sum of two subalgebras A_1 and A_2. An operator P defined as $P(a_1 + a_2) = -\lambda a_2$, $a_1 \in A_1$, $a_2 \in A_2$, is an RB-operator of weight λ on A.*

We call an RB-operator from Lemma 1 a *splitting RB-operator* with subalgebras A_1 and A_2.

Lemma 3 ([2, 9]). *Suppose that A is a unital algebra and P is an RB-operator of weight λ on A.*

a) If $\lambda \neq 0$ and $P(1) \in \Bbbk$, then $P(1) \in \{0, -\lambda\}$ and P is splitting.

b) If $\lambda = 0$ and $P(x) \in \Bbbk$, then $P(x) = 0$.

Let R be an RB-operator of weight λ on the algebra A. Then $\mathrm{Im}(R)$ is a subalgebra of A. If $\lambda \neq 0$, then $\ker(R)$ is also a subalgebra of A. If $\lambda = 0$, then $\ker(R)$ is an $\mathrm{Im}(R)$-module.

For $\lambda = 0$, the relation

$$R(x_1)R(x_2)\dots R(x_k) = R(x_1R(x_2)\dots R(x_k)) + R(x_1)x_2R(x_3)\dots R(x_k) + \dots + R(x_1)R(x_2)\dots R(x_{k-1})x_k) \tag{2}$$

holds in an associative RB-algebra as a direct consequence of (1). In particular, when RB-algebra is commutative, $\lambda = 0$, and $x = x_1 = x_2 = \dots = x_k$, we get

$$(R(x))^k = kR(x(R(x))^{k-1}). \tag{3}$$

From now on, we focus on RB-operators defined on either polynomial algebra or free commutative non-unital algebras, i.e., polynomials with no constant term.

Example 1 [2]. A linear map P defined on $\Bbbk[x]$ by the formula $J_a(x^n) = \frac{x^{n+1}-a^{n+1}}{n+1}$, where a is a fixed element from $\Bbbk$, is an RB-operator of weight zero on $\Bbbk[x]$.

Given an algebra A and an element $r \in A$, we denote by l_r the linear operator on A defined by $l_r(x) = rx$, $x \in A$.

Example 2 [5]. Suppose that A is a commutative algebra, $r \in A$, and R is an RB-operator of weight zero on A. Then the linear map $R \circ l_r$ is also an RB-operator of weight zero on A. Here $\circ$ denotes the composition of operators.

Thus, the linear operator $P = J_a \circ l_{x^k}$ on $\Bbbk[x]$ as $P(x^n) = \frac{x^{n+k+1}-a^{n+k+1}}{n+k+1}$ is an RB-operator of weight zero.

For an algebra A, it is known that every solution $r = \sum a_i \otimes b_i \in A \otimes A$ of the associative Yang—Baxter equation of weight λ (AYBE) [10, 11]

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = \lambda r_{13} \tag{4}$$

induces an RB-operator of weight $-\lambda$ on A [10, 12] defined by the formula

$$R(x) = \sum a_i x b_i. \tag{5}$$

In (4), $r_{12} = \sum a_i \otimes b_i \otimes 1$, $r_{13} = \sum a_i \otimes 1 \otimes b_i$, and $r_{23} = \sum 1 \otimes a_i \otimes b_i$.

Proposition 1. *The only nonzero solution of AYBE of weight $\lambda \neq 0$ on $\Bbbk[x]$ is $\lambda(1 \otimes 1)$.*

Proof. Let $r = \sum\limits_{i,j\geq 0} \alpha_{ij} x^i \otimes x^j$ be a solution of (4), so we have

$$\sum_{i,j,k,l\geq 0} \alpha_{ij}\alpha_{kl}(x^{i+k} \otimes x^l \otimes x^j - x^i \otimes x^{j+k} \otimes x^l + x^i \otimes x^k \otimes x^{j+l}) - \lambda \sum_{i,j\geq 0} x^i \otimes 1 \otimes x^j = 0. \tag{6}$$

Consider a maximal N such that $\alpha_{Nj} \neq 0$ for some j. If $N > 0$, then the left-hand side of (6) is nonzero because of the summand $\alpha_{Nj}^2 x^{2N} \otimes x^j \otimes x^j$ from the first sum. Similarly, we may consider a maximal M such that $\alpha_{0M} \neq 0$. So, the only possible solution is a tensor $q(1 \otimes 1)$. It is easy to see that either $q = \lambda$, or $q = 0$. □

Corollary 1. *The only nonzero solution of AYBE of weight $\lambda \neq 0$ on $\Bbbk[X]$ is $\lambda(1 \otimes 1)$.*

Given a nonempty set X, we denote by $\Bbbk_0[X]$ the free commutative algebra generated by X. Using the formula (5), we obtain only trivial RB-operators on $\Bbbk[X]$ and $\Bbbk_0[X]$.

3. Monomial RB-operators of weight zero on $\Bbbk_0[x]$

A linear operator R defined on $\Bbbk[x]$ ($\Bbbk_0[x]$) is called monomial if for all n we have $R(x^n) = \alpha_n x^{t_n}$ for some $\alpha_n \in \Bbbk$ and $t_n \in \mathbb{N}$ ($t_n \in \mathbb{N}_{>0}$).

Theorem 1. *Given a nonzero monomial RB-operator R on $\Bbbk_0[x]$, there exist positive $m \in \mathbb{N}$, nonnegative $p_0, \ldots, p_{m-1} \in \mathbb{N}$, and some $q_0, \ldots, q_{m-1} \in \Bbbk$ such that $p_i = 0$ if and only if $q_i = 0$, and R is defined by the formula*

$$R(x^{ma+b}) = q_b \frac{x^{m(a+p_b)}}{m(a+p_b)}, \tag{7}$$

where $a \in \mathbb{N}$ and $0 < b \leq m$.

Proof. Since R is nonzero, at least one monomial lies in $\mathrm{Im}(R)$. So, $\mathrm{Im}(R)$ is an infinite-dimensional subalgebra of $\Bbbk[x]$. Set $m = \gcd(t \mid x^t \in \mathrm{Im}(R))$. We have $\mathrm{Im}(R) \supset \mathrm{Span}\{x^{mk} \mid k \geq N\}$ for some natural N.

Lemma 4. *Let $0 < b \leq m$. Suppose that $x^{ma+b} \in \ker(R)$ for some $a \geq 0$. Then $x^{mc+b} \in \ker(R)$ for all $c \geq 0$.*

Proof. By (1), $\ker(R)$ is an $\mathrm{Im}(R)$-module, therefore $x^{mc+b} \in \ker(R)$ for all $c \geq a + N$. Assume that there exists a c such that $R(x^{mc+b}) = \alpha x^{mt}$ and $\alpha \neq 0$. Then by (3), we get

$$\alpha^k x^{mtk} = (R(x^{mc+b}))^k = kR(x^{mc+b}(R(x^{mc+b}))^{k-1}) = k\alpha^{k-1}R(x^{m(c+t(k-1))+b}).$$

We obtain a contradiction when $k \geq a + N + 1$. □

Lemma 5. *Let $0 < b \leq m$. Suppose that $R(x^b) = \alpha_0 x^{mp_b}$ with $\alpha_0 \neq 0$. Then $R(x^{ma+b}) = \alpha_a x^{m(a+p_b)}$ for all $a \geq 0$. Here α_a are some nonzero elements from $\Bbbk$.*

Proof. First, we prove Lemma 5 for all $a \geq N$. Since $x^{ma} \in \mathrm{Im}(R)$, we can find a $k \geq 0$ such that $R(x^k) = \gamma x^{ma}$ for some $\gamma \neq 0$. Then we have

$$\alpha_0 \gamma x^{m(a+p_b)} = R(x^b)R(x^k) = \alpha_0 R(x^{mp_b+k}) + \gamma R(x^{ma+b}).$$

By Lemma 4, $R(x^{ma+b}) \neq 0$. Since R is monomial, we have $R(x^{ma+b}) = \alpha_a x^{m(a+p_b)}$ for some $\alpha_a \in \Bbbk$.

Now, consider the case $0 < a < N$. Suppose that $R(x^{ma+b}) = \beta x^{mt}$ for some $t > 0$. By Lemma 4, $\beta \neq 0$. Then

$$\alpha_0 \beta x^{m(t+p_b)} = R(x^b)R(x^{ma+b}) = \alpha_0 R(x^{m(a+p_b)+b}) + \beta R(x^{mt+b}).$$

If $t = a + p_b$, we are done. If $t \neq a + p_b$, then the monomials $x^{m(a+p_b)+b}$ and x^{mt+b} have proportional images under R, which means that there exist $k < l$ and $\delta \neq 0$ such that $x^{mk+b} + \delta x^{ml+b} \in \ker(R)$. Multiplying, if necessary, this element by $x^{mc} \in \mathrm{Im}(R)$ for $c \geq N$, we may assume that $k \geq N$. But for such degrees we have already proved that $R(x^{mk+b} + \delta x^{ml+b}) = \alpha_k x^{m(k+p_b)} + \alpha_l \delta x^{m(l+p_b)} \neq 0$, so there is a contradiction. □

We can rewrite each positive natural number n as $n = ma + b$ for $m \geq 0$ and $0 < b \leq m$. We define $\alpha_{a,b} \in \Bbbk$ such that $R(x^{ma+b}) = \alpha_{a,b} x^{m(a+p_b)}$. If $x^b \in \ker(R)$, then $\alpha_{0,b} = p_b = 0$. Otherwise, $\alpha_{0,b}, p_b \neq 0$. Thus, the identity (1)

$$\begin{aligned}\alpha_{a,b}\alpha_{c,b'} x^{m(a+c+p_b+p_{b'})} &= R(x^{ma+b})R(x^{mc+b'}) \\ &= \alpha_{a,b} R(x^{m(a+c+p_b)+b'}) + \alpha_{c,b'} R(x^{m(a+c+p_{b'})+b}) \\ &= (\alpha_{a,b}\alpha_{a+c+p_b,p_{b'}} + \alpha_{c,b'}\alpha_{a+c+p_{b'},p_b}) x^{m(a+c+p_b+p_{b'})}\end{aligned}$$

is equivalent to the following one:

$$\alpha_{a,b}\alpha_{c,b'} = \alpha_{a,b}\alpha_{a+c+p_b,p_{b'}} + \alpha_{c,b'}\alpha_{a+c+p_{b'},p_b}. \tag{8}$$

By Lemma 4, it is reasonable to study only the cases when $x^b, x^{b'} \notin \ker(R)$. We put $b = b'$ and $a = c$ in (8), then $\alpha_{a,b} = 2\alpha_{2a+pb,b}$. This relation can be rewritten as $\gamma_{a,b} = \gamma_{2a+p_b,b}$, where $\gamma_{a,b} = (a + p_b)\alpha_{a,b}$. For $a = 0$, we get $\gamma_{0,b} = \gamma_{p_b,b}$. We prove that $\gamma_{np_b,b} = \gamma_{0,b}$ by induction on n. The base case $n = 1$ is trivial. Suppose that this equality holds for all natural numbers less then or equal to n. The relation (8), for $a = 0$, $c = np_b$, and $b' = b$, becomes

$$\alpha_{0,b}\alpha_{np_b,b} = \alpha_{(n+1)p_b,b}(\alpha_{0,b} + \alpha_{np_b,b}).$$

By the induction hypothesis, we obtain

$$\begin{aligned}\gamma_{(n+1)p_b,b} &= (n+2)p_b\alpha_{(n+1)p_b,b} = (n+2)p_b\frac{\alpha_{0,b}\alpha_{np_b,b}}{\alpha_{0,b} + \alpha_{np_b,b}} \\ &= (n+2)p_b\frac{\gamma_{0,b}\gamma_{np_b,b}}{(n+1)p_b^2} \cdot \frac{1}{\gamma_{0,b}(1/p_b + 1/((n+1)p_b)} = \gamma_{np_b,b} = \gamma_{0,b}.\end{aligned}$$

Next we show that $\gamma_{a,b} = \gamma_{0,b}$ for all a. Indeed, the relation (3) applied to x^{ma+b} leads to

$$\alpha_{a,b}^k = k\alpha_{a,b}^{k-1}\alpha_{ka+(k-1)p_b,b}$$

or $\gamma_{a,b} = \gamma_{ka+(k-1)p_b,b}$. Setting $k = p_b$, by the above property, we conclude that $\gamma_{a,b} = \gamma_{0,b}$.

Finally, set $q_b = m\gamma_{0,b}$ for all $0 < b \leq m$. We see by the definition that $q_b = 0$ if and only if $p_b = 0$. It is easy to check that the obtained linear operator R is an RB-operator of weight 0 on $\Bbbk_0[x]$. The theorem is proved. □

Example 3. The linear map R on $\Bbbk_0[x]$ defined by $R(x^n) = \frac{x^n}{n}$ is an RB-operator of weight zero. It is easy to see that R is invertible and thus $d = R^{-1}$ is a derivation on $\Bbbk_0[x]$ with $d(x^n) = nx^n$. Hence, the restriction of the RB-operator R from Example 1 to $\Bbbk_0[x]$ equals $d^{-1} \circ x$.

Corollary 2. [4] *For a nonzero monomial RB-operator R on $\Bbbk[x]$, there exist positive $m \in \mathbb{N}$, nonnegative $p_1, \ldots, p_m \in \mathbb{N}$, and some $q_1, \ldots, q_m \in \Bbbk$ such that $p_i = 0$ if and only if $q_i = 0$, and R is defined by*

$$R(x^{ma+b}) = q_b\frac{x^{m(a+p_b)}}{m(a+p_b)},$$

where $a \in \mathbb{N}$ and $0 \leq b < m$.

Proof. By Lemma 3b, $1 \notin \mathrm{Im}(R)$, so $\langle x \rangle$ is R-invariant. Now we can apply Theorem 1. It remains to extend R from $\langle x \rangle$ to $\Bbbk[x]$. It is enough to take $R(x^0) = \alpha_{0,0}x^{mp_0}$ instead of $R(x^m) = \alpha_{0,m}x^{mp_m}$, and we similarly prove that $\gamma_{a,m} = \gamma_{0,m}$ for all $a \geq 0$. □

Let us make a few remarks on the classification of monomial RB-operators of weight zero on $\Bbbk[x]$.

Remark 1. When $x^m \in \ker(R)$, the RB-operator R is defined by Lemma 5.1b from [3] with the decomposition $\Bbbk[x] = B \oplus C$ (of vector spaces) for $B = \mathrm{Span}\{x^{am+b} \mid p_b = 0\}$ and $C = \mathrm{Span}\{x^{am+b} \mid p_b \neq 0\}$.

Remark 2. If $x^m \notin \ker(R)$, then we may consider the restriction of R to the subalgebra $\mathrm{Span}\{x^{mk} \mid k \geq 0\}$. Then, up to a scalar multiple, R is defined on $\Bbbk[y]$ for $y = x^m$ by the formula $R(y^a) = \frac{y^{a+p_0}}{a+p_0}$. In other words, $R = J_0 \circ l_{y^{p_0-1}}$. So, R is an analytically modelled RB-operator on $\Bbbk[x]$ using the terminology of [5].

4. Monomial RB-operators of nonzero weight on $\Bbbk_0[x]$

Let R be an RB-operator of nonzero weight λ on an algebra A. By Lemma 1a, we may assume that $\lambda = 1$.

Theorem 2. *Let R be a monomial RB-operator of weight 1 on $\Bbbk_0[x]$. Then there exists $\alpha \in \Bbbk$ such that $(\alpha+1)^n \neq \alpha^n$ for all $n \geq 1$ and*

$$R(x^n) = \frac{\alpha^n}{(\alpha+1)^n - \alpha^n} x^n. \tag{9}$$

Proof. Trivial RB-operators on $\Bbbk_0[x]$ are monomial and they correspond to the cases $\alpha = 0$ and $\alpha = -1$ in (9). Suppose that R is a nontrivial monomial RB-operator of weight 1 on $\Bbbk_0[x]$.

<u>Case 1: $\ker(R) = (0)$</u>. Suppose that there exists a monomial x^n such that $R(x^n) = \alpha x^m$ for $n \neq m$. Since R is injective, $\alpha \neq 0$. Then

$$\alpha^2 x^{2m} = R(x^n)R(x^n) = 2\alpha R(x^{n+m}) + R(x^{2n}).$$

Since $n + m \neq 2n$, the kernel of R is nonzero, which is a contradiction. Thus, $R(x^n) = \alpha_n x^n$, $\alpha_n \in \Bbbk$, for all $n \geq 1$.

We have $R(x) = \alpha_1 x$, set $\alpha = \alpha_1$. Let us prove the formula (9) by induction on n. The base case $n = 1$ is trivial. Suppose that (9) has been proved for all numbers less than n. Then

$$R(x)R(x^{n-1}) = \alpha x \cdot \frac{\alpha^{n-1}}{(\alpha+1)^{n-1} - \alpha^{n-1}} x^{n-1} = \frac{\alpha^n}{(\alpha+1)^{n-1} - \alpha^{n-1}} x^n,$$

$$\begin{aligned} R(x)R(x^{n-1}) &= R(R(x)x^{n-1} + xR(x^{n-1}) + x^n) \\ &= \left(\alpha + \frac{\alpha^{n-1}}{(\alpha+1)^{n-1} - \alpha^{n-1}} + 1\right) R(x^n). \end{aligned}$$

Thus,

$$R(x^n) = \frac{\alpha^n}{(\alpha+1)^{n-1} - \alpha^{n-1}} \cdot \frac{(\alpha+1)^{n-1} - \alpha^{n-1}}{\alpha^{n-1} + (\alpha+1)^n - \alpha^n - \alpha^{n-1}} x^n = \frac{\alpha^n}{(\alpha+1)^n - \alpha^n} x^n,$$

as required.

<u>Case 2: $\ker(R) \neq (0)$</u>. Since R is nontrivial, we as well have $\mathrm{Im}(R) \neq (0)$. Set $m = \gcd(t \mid x^t \in \mathrm{Im}(R))$. It is easy to get that

$$\mathrm{Im}(R) \supset \mathrm{Span}\{x^{mk} \mid k \geq N\}$$

for some positive integer N.

CASE 2A: $\ker(R)$ contains a monomial x^k. Since $\ker(R)$ and $\mathrm{Im}(R)$ are subalgebras of $\Bbbk_0[x]$, we have that $x^{mkN} \in \ker(R) \cap \mathrm{Im}(R)$. We denote $t = kN$ and consider x^s such that $R(x^s) = ax^{mt}$, $a \neq 0$. It is well-known that $\ker(R)$ is an ideal in $\mathrm{Im}(R+\mathrm{id})$, i.e., $x^{ps+qmt} \in \ker(R)$ for all $p, q \geq 1$. Then $R(x^s)R(x^s) = a^2x^{2mt} = R(x^{2s})$ and, similarly, $R(x^{smt}) = a^{mt}x^{m^2t^2}$. We have $x^{smt} \in \ker(R)$, so $a = 0$, which is a contradiction.

CASE 2B: $\ker(R)$ does not contain a monomial. Since R is monomial, $\ker(R)$ is nonzero only if there exists $p \neq s$ such that $R(x^p) = ax^{mt}$ and $R(x^s) = bx^{mt}$ for some $t \geq 1$ and $a, b \neq 0$. We have

$$a^2x^{2mt} = R(x^p)R(x^p) = 2aR(x^{p+mt}) + R(x^{2p}),$$

so $R(x^{2p}) = \alpha_{2p}x^{2mt}$. We can prove by induction that $R(x^{pk}) = \alpha_{pk}x^{mtk}$ for all $k \geq 1$. Analogously, $R(x^{sk}) = \alpha_{sk}x^{mtk}$ for all $k \geq 1$. Note that $R(x^{ps}) = \alpha_{ps}x^{mts}$ and $R(x^{ps}) = \alpha_{ps}x^{mtp}$ at the same time. We have a contradiction to $p \neq s$. □

Corollary 3. *Up to conjugation by an automorphism of $\Bbbk[x]$, each nontrivial monomial RB-operator on $\Bbbk[x]$ of nonzero weight is splitting with subalgebras $\Bbbk$ and $\langle x \rangle$.*

Proof. Suppose that R is a nontrivial monomial RB-operator of weight 1 on $\Bbbk[x]$. Let $R(1) = \alpha x^k$ and suppose that $k > 0$ and $\alpha \neq 0$. Then from (1), we have

$$2\alpha R(x^k) = 2R^2(1) = R(1)R(1) - R(1) = \alpha^2x^{2k} - \alpha x^k, \tag{10}$$

which is a contradiction to the monomiality condition.

Thus, $R(1) \in \Bbbk$, i.e., $R(1) \in \{0, -1\}$, and R is splitting by Lemma 3a. Since R is nontrivial, both $\ker(R)$ and $\mathrm{Im}(R)$ are nonzero. Hence, $\mathrm{Im}(R)$ has a basis of monomials and R acts on $\mathrm{Im}(R)$ as the operator $-\mathrm{id}$.

CASE 1. Suppose that $\langle x \rangle = \Bbbk_0[x]$ is R-invariant. By Theorem 2, $R(x^n) = \frac{\alpha^n x^n}{(\alpha+1)^n - \alpha^n}$, $n > 0$, for some α such that the denominator is nonzero for all n. If $\alpha = 0$, then we either have a trivial RB-operator (when $R(1) = 0$) or a splitting one with subalgebras $\Bbbk$ and $\langle x \rangle$ (when $R(1) = -1$).

If $\alpha \neq 0$, then, since R acts on $\mathrm{Im}(R)$ as the operator $-\mathrm{id}$, it follows that $\alpha = -1$ and $R(x^n) = -x^n$ for all $n > 0$. Again, we either have a trivial RB-operator (when $R(1) = -1$) or a splitting one with subalgebras $\Bbbk$ and $\langle x \rangle$ (when $R(1) = 0$).

CASE 2. Suppose that $\langle x \rangle = \Bbbk_0[x]$ is not R-invariant. We will show that $R(1) = -1$. Indeed, suppose that $R(1) = 0$ and there exists $k > 0$ such that $R(x^k) = \alpha 1$ for some $\alpha \in \Bbbk$. Since $\ker(R)$ is an ideal in $\mathrm{Im}(R + \mathrm{id})$, we have $x^k \in \ker(R)$, and so $\alpha = 0$. Thus, $\langle x \rangle$ is R-invariant, which is a contradiction.

Consider $R(x)$. If $R(x) = -x$, then $\ker(R+\mathrm{id}) = \Bbbk[x]$ and R is trivial. If $R(x) = 0$, then R is splitting with subalgebras $\Bbbk$ and $\langle x \rangle$. Assume that $R(x) = \alpha x^k$ for some $\alpha \neq 0$ and $k > 0$, and moreover, $R(x) \neq -x$. From the last condition we get $k > 1$. If $\ker(R)$ contains a monomial x^t, then $x^{tk} \in \ker(R) \cap \mathrm{Im}(R) = (0)$, a contradiction. Since $R(x^k) = -x^k$, we may repeat the arguments of the proof of Theorem 2 to get a contradiction.

It remains to study the case when $R(x) = \alpha 1$ for $\alpha \neq 0$. We assume for convenience that the weight of the RB-operator R equals -1. Then by induction on n, it is easy to show that the equality $R(x) = \alpha 1$ implies $R(x^n) = \alpha^n 1$ for all $n \geq 1$. We define the automorphism $\varphi\colon x^n \to x^n/\alpha^n$ of $\Bbbk[x]$. Then the RB-operator $R' = \varphi^{-1}R\varphi$ (we apply Lemma 1b) acts on all monomials as $R'(x^n) = 1$ and R' is splitting with subalgebras $\mathrm{Im}(R') = \Bbbk$ and $\ker(R') = \langle x-1\rangle$. Define $\psi \in \mathrm{Aut}(\Bbbk[x])$ as $\psi\colon x \to x-1$. Note that $\psi^{-1}R'\psi$ is a splitting RB-operator with subalgebras $\Bbbk$ and $\langle x\rangle$. $\square$

5. Monomial RB-operators of nonzero weight on $\Bbbk_0[X]$

Example 4. Let R be a splitting RB-operator on $\Bbbk_0[x,y]$ with subalgebras $\Bbbk_0[x]$ and $\langle y\rangle$. Then R is monomial.

Theorem 3. *Let R be an injective monomial RB-operator of weight 1 on the algebra $\Bbbk_0[x_1,\ldots,x_n]$, then there exist nonzero $\alpha_1,\ldots,\alpha_n \in \Bbbk$ such that*

$$R(x_1^{i_1}\ldots x_n^{i_n}) = \frac{\alpha_1^{i_1}\ldots\alpha_n^{i_n}}{(\alpha_1+1)^{i_1}\ldots(\alpha_n+1)^{i_n} - \alpha_1^{i_1}\ldots\alpha_n^{i_n}} x_1^{i_1}\ldots x_n^{i_n} \tag{11}$$

for all $i_1,\ldots,i_n \geq 0$, $i_1^2+\ldots+i_n^2 > 0$. Moreover, all denominators are nonzero.

Proof. First, suppose that there exists a monomial $w = x_1^{i_1}\ldots x_n^{i_n}$ such that $R(w) = \alpha w'$ with $w' \neq w$. Then

$$\alpha^2 w'^2 = R(w)R(w) = 2\alpha R(ww') + R(w^2).$$

Since $ww' \neq w^2$, the kernel of R is nonzero, which is a contradiction.

Define scalars $\alpha_i \in \Bbbk\setminus\{0\}$ such that $R(x_i) = \alpha_i x_i$. We prove the formula (11) for a monomial w by induction on the degree $\deg(w)$. Given a monomial w from $\Bbbk_0[x_1,\ldots,x_n]$, we denote by $\alpha(w)$ the coefficient at w in the right-hand side of (11). For $\deg(w) = 1$, (11) follows from the definition of α_i.

Suppose that we have proved (11) for all monomials of degree not greater than d. Let $w = w'x_j$ be a monomial of degree $d{+}1$ for $w' = x_1^{i_1}\ldots x_n^{i_n}$, where $i_1{+}\ldots{+}i_n{=}d$. From the equality

$$\alpha(w')\alpha_j w = R(w')R(x_j) = (\alpha(w') + \alpha_j + 1)R(w),$$

we calculate the coefficient k in $R(w) = kw$ as

$$\begin{aligned}
k &= \frac{\alpha(w')\alpha_j}{\alpha(w')+\alpha_j+1}\\
&= \frac{\alpha_1^{i_1}\ldots\alpha_n^{i_n}\alpha_j}{(\alpha_1+1)^{i_1}\ldots(\alpha_n+1)^{i_n}-\alpha_1^{i_1}\ldots\alpha_n^{i_n}} \cdot \frac{1}{\frac{\alpha_1^{i_1}\ldots\alpha_n^{i_n}}{(\alpha_1+1)^{i_1}\ldots(\alpha_n+1)^{i_n}-\alpha_1^{i_1}\ldots\alpha_n^{i_n}}+\alpha_j+1}\\
&= \frac{\alpha_1^{i_1}\ldots\alpha_n^{i_n}\alpha_j}{\alpha_1^{i_1}\ldots\alpha_n^{i_n} + (1+\alpha_j)\big((\alpha_1+1)^{i_1}\ldots(\alpha_n+1)^{i_n}-\alpha_1^{i_1}\ldots\alpha_n^{i_n}\big)}\\
&= \frac{\alpha_1^{i_1}\ldots\alpha_n^{i_n}\alpha_j}{(\alpha_1+1)^{i_1}\ldots(\alpha_n+1)^{i_n}(\alpha_j+1)-\alpha_1^{i_1}\ldots\alpha_n^{i_n}\alpha_j} = \alpha(w),
\end{aligned}$$

and we are done. $\square$

Corollary 4. *Let R be an injective monomial RB-operator of weight 1 on the algebra $\Bbbk[x_1,\ldots,x_n]$. Then $R = -id$.*

Proof. Suppose that $R(1) \notin \Bbbk$, i.e., $R(1) = \alpha x_1^{i_1}\ldots x_n^{i_n}$ with $\alpha \neq 0$. Then, similarly to (10), we get that $R(x_1^{i_1}\ldots x_n^{i_n}) = (1/2)x_1^{i_1}\ldots x_n^{i_n}(\alpha x_1^{i_1}\ldots x_n^{i_n} - 1)$ is not a monomial, which is a contradiction. Thus, $R(1) = -1$ and R is splitting.

Since $\ker(R) = (0)$, the ideal $\Bbbk_0[x_1, \ldots, x_n]$ is R-invariant, and we may apply Theorem 3. Since R acts on $\mathrm{Im}(R)$ as $-\mathrm{id}$, we derive that $\alpha_1 = \ldots = \alpha_n = -1$. Thus, $R = -\mathrm{id}$. □

We should say that the direct analogue of Theorem 3 does not hold in case of RB-operators of weight zero. According to Example 2, we have a collection

$$S = \{J_a \circ l_f \mid f \in \Bbbk[x] \setminus \{0\},\, a \in \Bbbk\}$$

of injective RB-operators of weight zero on $\Bbbk[x]$. If f is a monomial and $a = 0$, then $J_0 \circ l_f$ is an injective monomial RB-operator of weight zero on $\Bbbk[x]$. Recently, in [13], the following conjecture about the form of all injective RB-operators on $\Bbbk[x]$ was confirmed over any field $\Bbbk$ of characteristic zero.

Conjecture 1. *(Guo, Rosenkranz, Zheng, 2015* [5]*). The set of all injective RB-operators of weight zero on $\Bbbk[x]$ coincides with S.*

Example 5. Let $\alpha_1, \ldots, \alpha_n \in \Bbbk$ be nonzero scalars such that all sums $\sum\limits_{i_j \in \overline{1,n}} \frac{1}{\alpha_{i_j}}$ are nonzero. We define a linear operator R on $\Bbbk_0[x_1, \ldots, x_n]$ as follows:

$$R(x_1^{i_1} \ldots x_n^{i_n}) = \frac{x_1^{i_1} \ldots x_n^{i_n}}{\frac{i_1}{\alpha_1} + \ldots + \frac{i_n}{\alpha_n}}, \tag{12}$$

where $i_1, \ldots, i_n \geq 0$ and $i_1^2 + \ldots + i_n^2 > 0$. Then R is an injective monomial RB-operator of weight 0 on $\Bbbk_0[x_1, \ldots, x_n]$.

6. Gradings by the spectrum of an RB-operator

It is well-known that for a given finite-dimensional algebra A we have a grading on A by the spectrum of its derivation or automorphism [8]. At the end of this work, we state similar results for RB-operators related to the formulas (11) and (12).

In this section, a ground field $\Bbbk$ may have a positive characteristic.

For nonzero scalars $\lambda, \mu \in \Bbbk$, we put

$$\lambda \circ \mu = \begin{cases} \frac{\lambda\mu}{\lambda+\mu+1}, & \lambda + \mu \neq -1, \\ \text{not defined}, & \text{otherwise}. \end{cases}$$

Note that the partially defined operation $\circ$ on $\Bbbk_0$ is commutative and associative in the sense that the equality

$$(\lambda \circ \mu) \circ \nu = \frac{\lambda\mu\nu}{(\lambda+1)(\mu+1)(\nu+1) - \lambda\mu\nu} = \lambda \circ (\mu \circ \nu)$$

holds when all four involved products are defined.

Example 6. The set $\mathbb{R}_{>0}$ of all positive real numbers with multiplication $\circ$ form a semigroup. Moreover, it is isomorphic to the semigroup $\langle \mathbb{R}_{>1}, \cdot \rangle$. Indeed, we define $\varphi\colon \mathbb{R}_{>0} \to \mathbb{R}_{>1}$ as $\varphi(x) = 1 + 1/x$. It is a bijection and

$$\varphi(x \circ y) = 1 + \frac{x+y+1}{xy} = 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy} = \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) = \varphi(x)\varphi(y).$$

Given a finite-dimensional algebra A over an algebraically closed field $\Bbbk$ and an RB-operator R of weight 1 on A, consider the generalized eigenvalue decomposition

$$A = \bigoplus_{\lambda \in \mathrm{Spec}(R)} A_\lambda. \tag{13}$$

Proposition 2. *Suppose that A is a finite-dimensional algebra over an algebraically closed field $\Bbbk$ and R is an RB-operator of weight 1 on A. Let $\lambda,\mu\in Spec(R)$ be such that $\lambda,\mu\neq 0$. Then* $A_\lambda A_\mu \subseteq \begin{cases}(0), & \lambda+\mu=-1 \text{ or } \lambda\circ\mu\notin Spec(R),\\ A_{\lambda\circ\mu}, & \text{otherwise.}\end{cases}$

Proof. Let $u\in A_\lambda$ and $v\in L_\mu$. Then we have $(R-\lambda\mathrm{id})^k u=0$ and $(R-\mu\mathrm{id})^l v=0$. We will prove the statement by induction on $k+l$. Suppose that $k+l=2$. Then we have

$$\lambda\mu uv = R(u)R(v) = R(R(u)v+uR(v)+uv) = (\lambda+\mu+1)R(uv).$$

If $\lambda+\mu=-1$, then $uv=0$. Otherwise, $R(uv)=(\lambda\circ\mu)uv$. From this, we get the base case.

Suppose that we have proved the statement for all numbers less than $k+l$. We set $\tilde u=(R-\lambda\mathrm{id})u$ and $\tilde v=(R-\mu\mathrm{id})v$. Then

$$\begin{aligned} R(u)R(v) &= (\lambda u+\tilde u)(\mu v+\tilde v)=\lambda\mu uv+\lambda u\tilde v+\mu\tilde u v+\tilde u\tilde v,\\ R(R(u)v+uR(v)+uv) &= (\lambda+\mu+1)R(uv)+R(\tilde u v+u\tilde v). \end{aligned}\tag{14}$$

If $\lambda+\mu=-1$, then $uv=0$, as all other products in (14) are zero by induction. Otherwise, we conclude that $(\lambda+\mu+1)(R-\lambda\circ\mu\mathrm{id})(uv)\in A_{\lambda\circ\mu}$, i.e., $uv\in A_{\lambda\circ\mu}$. □

Example 7. Let A be the quotient of $\Bbbk_0[x]$ by the ideal generated by x^{N+1}. Suppose that $\Bbbk$ is either $\mathbb{Q}$ or $\mathbb{Z}_p$ for some prime p. Then a linear operator R defined on A by

$$R(x^i)=\frac{x^i}{2^i-1},\quad i=1,\dots,N$$

is a monomial RB-operator of weight 1 on A. To avoid division by zero in the case $\Bbbk=\mathbb{Z}_p$, we require that none of the numbers 2^2-1, 2^3-1, ..., 2^N-1 is divisible by p. For this, we may put a restriction $N<\log_2(p+1)$. Multiplying the unit i times, we get $1_\circ^i=\dfrac{1}{2^i-1}$. Thus, we have a decomposition

$$A=A_1\oplus A_{1_\circ^2}\oplus\ldots\oplus A_{1_\circ^N},\quad A_{1_\circ^i}=\mathrm{Span}\{x^i\}.$$

We want to verify that Proposition 2 holds true in this example. For $i+j\le N$, if follows that $x^i\cdot x^j=x^{i+j}\in A_{1_\circ^{i+j}}$, since $1_\circ^i\circ 1_\circ^j=1_\circ^{i+j}$. When $i+j>N$, we have $x^i\cdot x^j=0$ and $1_\circ^{i+j}\notin\mathrm{Spec}(R)$.

Further, we give a particular example of this construction for $N=3$ and $p=5$. In this case, $2^2-1=3$ and $2^3-1=7$ are not divisible by 5. So, we have $R(x)=x$, $R(x^2)=2x^2$, $R(x^3)=3x^3$ and get a decomposition $A=A_1\oplus A_2\oplus A_3$ with $A_i=\mathrm{Span}\{x^i\}$. By Proposition 2, $x\cdot x^2=x^3\in A_3$, as $1\circ 2=3$. On the other hand, $1\circ 3$ is not defined, and therefore we have $x\cdot x^3=0$. Finally, $2\circ 3=1$, and $x^2\cdot x^3=0\in A_1$.

We now proceed to the case of weight zero. For nonzero scalars $\lambda,\mu\in\Bbbk$, we put

$$\lambda*\mu=\begin{cases}\frac{\lambda\mu}{\lambda+\mu}, & \lambda\neq-\mu,\\ \text{not defined}, & \text{otherwise.}\end{cases}$$

The operation $*$ on $\Bbbk_0$ is again commutative and associative, i.e.,

$$(\lambda*\mu)*\nu=\frac{\lambda\mu\nu}{\lambda\mu+\lambda\nu+\mu\nu}=\lambda*(\mu*\nu)$$

holds when all products involved are defined.

Example 8. The set $\mathbb{R}_{>0}$ of all positive real numbers with multiplication $\circ$ forms a semigroup, which is isomorphic to the semigroup $\langle \mathbb{R}_{>0}, +\rangle$. Indeed, we define $\varphi\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ by $\varphi(x) = 1/x$. It is a bijection and $\varphi(x * y) = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y} = \varphi(x) + \varphi(y)$.

Once again, we consider the generalized eigenvalue decomposition (13) of an algebra A by an RB-operator R of weight zero.

Proposition 3. *Suppose that A is a finite-dimensional algebra over an algebraically closed field $\Bbbk$ and R is an RB-operator of weight 0 on A. Let $\lambda, \mu \in Spec(R)$ be such that $\lambda, \mu \neq 0$. Then* $A_\lambda A_\mu \subseteq \begin{cases} (0), & \lambda + \mu = 0 \text{ or } \lambda * \mu \notin Spec(R), \\ A_{\lambda * \mu}, & \text{otherwise}. \end{cases}$

Proof. Similar to the proof of Proposition 2. □

Example 9. We deal with the same algebra as in Example 7. A linear operator R defined on A by

$$R(x^i) = x^i/i, \quad i = 1, \dots, N$$

is a monomial RB-operator of weight 0 on A. Here we need the restriction $N < p$. Multiplying the unit i times, we get $1_*^i = 1/i$. So, we have a decomposition

$$A = A_1 \oplus A_{1/2} \oplus \ldots \oplus A_{1/N}, \quad A_{1/i} = \mathrm{Span}\{x^i\}.$$

For $i + j \leq N$, it follows that $x^i \cdot x^j = x^{i+j} \in A_{1/(i+j)}$, since $1_*^i * 1_*^j = \frac{1}{i} * \frac{1}{j} = \frac{1}{i+j} = 1_*^{i+j}$. When $i + j > N$, we have $x^i \cdot x^j = 0$ and $1_*^{i+j} \notin \mathrm{Spec}(R)$.

Consider Example 9 with $N = 3$ and $p = 5$. In this case $R(x) = x$, $R(x^2) = 3x^2$, $R(x^3) = 2x^3$, and $A = A_1 \oplus A_2 \oplus A_3$. Here $x^2 \cdot x^3 = 0$, since $(1/2) * (1/3)$ is not defined. Further, $x \cdot x^2 = x^3 \in A_2$, as $1 * 3 = 2$. Finally, $x \cdot x^3 = 0$, because $1 * 2 = 4 \notin \mathrm{Spec}(R)$.

Remark 3. Note that Example 7 is nothing more than a quotient of the RB-algebra (9) with $\alpha = 1$ by the Rota—Baxter ideal generated by x^{N+1}. Analogously, the quotient of the RB-algebra from Example 3 gives the RB-operator from Example 9.

Remark 4. It is important that we exclude a unit in both Examples 7 and 9, which means we do not consider the quotient B of the polynomial algebra $\Bbbk[x]$ by the ideal generated by x^{N+1}. Indeed, in this case the spectrum of every RB-operator of weight λ on B is a subset of $\{0, -\lambda\}$ [3, 14].

The results of Propositions 2 and 3 can be useful in attempt to construct a universal enveloping associative Rota—Baxter algebra for a given Lie Rota—Baxter algebra, see the exact formulation of the problem in [15].

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