Abstract. This paper is dedicated to problems of perceptibility and recognizability in pre-Heyting logics, that is, in extensions of the minimal logic $J$ satisfying the axiom $\neg\neg(\bot \rightarrow p)$. These concepts were introduced in [8, 11, 10]. The logic $Od$ and its extensions were studied in [5, 14] and other papers. The semantic characterization of the logic $Od$ and its completeness were obtained in [5]. The formula $F$ and the logic $JF$ were studied in [12]. It was proved that the logic $JF$ has disjunctive and finite-model properties. The logic $JF$ has Craig's interpolation property (established in [17]). The perceptibility of the formula $F$ in well-composed logics is proved in [14]. It is unknown whether the formula $F$ is perceptible over $J$ [8]. We will prove that the formula $F$ is perceptible over the minimal pre-Heyting logic $Od$ and the logic $OdF$ is recognizable over $Od$.

Keywords: Recognizability, perceptibility, minimal logic, pre-Heyting logic, Johansson algebra, Heyting algebra, superintuitionistic logic, calculus.

In this paper, we investigate the problems of perceptibility and recognizability in pre-Heyting logics, that is, in extensions of the minimal logic $J$ [1] satisfying the axiom $\neg\neg(\bot \rightarrow p)$. The mentioned notation was introduced in [8, 10]. For example, a formula $A$ is perceptible over the logic $J$, if there exists an algorithm that, given any finite system $Ax$ of axioms schemes, checks whether $A$ can be derived in $J + Ax$.

The logic $Od$ and its extentions were investigated in [5, 14] and other papers. The semantic characterization of the logic $Od$ and its completeness were obtained in [5].
The formula $F = (\bot \to p \lor q) \to (\bot \to p) \lor (\bot \to q)$ and the logic JF were studied in [12], where it was proved that the logic JF possesses disjunctive and finite-model properties. The fact that the logic JF has Craig’s interpolation property CIP was established in [17]. In [14] we proved that the formula $F$ is perceptible in well-composed logics. It is unknown whether $F$ is perceptible over J [8].

In this paper we prove that the formula $F$ is perceptible over the minimal pre-Heyting logic $Od = J + \neg\neg(\bot \to p)$, whereas the logic $OdF = Od + F$ is recognizable over $Od$.

Section 1 provides preliminary information. In Section 2, the basic known results concerning pre-Heyting logics are listed. Section 3 is dedicated to proving of theorems on perceptibility of the formula $F$ over the logic Od, and on recognizability and perceptibility of the logic OdF over Od.

1. Preliminary data

The language of Johansson’s minimal logic J contains connectives $\&$, $\lor$, $\to$ and the propositional constant $\bot$; $\top = \bot \to \bot$, $\neg A = A \to \bot$.

The minimal logic J possesses the same axioms as the positive intuitionistic calculus, and the only rule of inference is modus ponens $R1$: $A, A \to B \vdash B$.

J-logic is defined as any set of formulas that contains all the axioms of the calculus of J and is closed under $R1$ and substitution. For J-logic $L$ we write $\Gamma \vdash L A$, if $A$ can be derived from the set $L \cup \Gamma$ using the rule $R1$.

Consider the formulas:

$Int = \bot \to p$,

$Od = \neg\neg(\bot \to p)$,

$X = (\bot \to p) \lor (p \to \bot)$,

$F = (\bot \to p \lor q) \to (\bot \to p) \lor (\bot \to q)$.

Let us introduce the designations for some of the J-logics:

$Int = J + Int$,

$Neg = J + \bot$,

$Od = J + Od$,

$JX = J + X$,

$JF = J + F$,

$For = J + p$,

$OdF = Od + F$.

A J-logic is said to be nontrivial if it does not coincide with For.

A logic is called superintuitionistic (negative, pre-Heyting, well-composed) if it contains Int, Neg, Od, JX respectively.

Given the list of variables $p$, we denote by $A(p)$ a formula all the variables of which belong to $p$, and by $F(p)$ — the set of all such formulas.

Suppose that the lists $p, q, r$ do not intersect pairwise.

Craig’s interpolation property CIP [2] is defined the following way:

CIP. If $\Gamma \vdash L A(p, q) \to B(p, r)$, then there exists a formula $C(p)$, such that $\Gamma \vdash L A(p, q) \to C(p)$ and $\Gamma \vdash L C(p) \to B(p, r)$.

Weak interpolation property WIP is defined the following way:

WIP. If $A(p, q), B(p, r) \vdash L \bot$, then there exists a formula $A'(p)$, such that $A(p, q) \vdash L A'(p)$ and $A'(p), B(p, r) \vdash L \bot$. 

An algebraic semantics of the logic J is constructed using *Johansson’s algebras* (J-algebras) $\mathbf{A} = \langle A; \&; \lor; \to; \bot; \top \rangle$, where

$\langle A; \&; \lor; \to; \bot; \top \rangle$ is a lattice with respect to $\&$, $\lor$ with the greatest element $\top$, and $\bot$ is an arbitrary element in $A$,

$z \leq x \to y \iff z \& x \leq y$.

It is well-known (see, for example, [18]) that the family of J-algebras forms a variety, and there is a one-to-one correspondence between the lattice of J-logics and the lattice of varieties of J-algebras.

If $A$ is a formula and $A$ is an algebra, then we say that the formula $A$ is valid in $A$, and write $A \models A$, if the identity $A = \top$ holds in $A$.

We denote

$$V(L) = \{A | A \models L\}.$$  

In particular, $V(\text{Int})$ is a variety of Heyting algebras, $V(\text{Neg})$ is a variety of negative algebras, and $V(\text{JX})$ is a variety of well-composed algebras.

The following result is well-known:

**Theorem 1.1** (Completeness theorem). Let $L$ be a J-logic. Then $L + A \vdash B$ if and only if $A \models A$ implies $A \models B$ for any algebra $A \in V(L)$.

Let $L$ be a logic, and $\text{Rul} - a$ set of axioms and inference rules. We write $J + \text{Rul} \geq L$, if all the formulas in $L$ are deducible in $(J + \text{Rul})$; we write $J + \text{Rul} = L$, if $L$ coincides with the set of deducible in $(J + \text{Rul})$ formulas.

The terms of perceivable formula and recognizable logic were introduced in [8, 10]. Let $L_0$ be a finitely axiomatizable J-logic, $A$ formula $A$ is perceptive over $L_0$ if there exists an algorithm verifying for any finite system of axioms $Ax$ whether the relation $L_0 + Ax \vdash A$ holds; the formula $A$ is strongly perceptive over $L_0$ if there is an algorithm verifying for any finite system $\text{Rul}$ of axioms and inference rules whether the relation $L_0 + \text{Rul} \vdash A$ holds.

Let $L, L_0$ be finitely axiomatizable J-logics, $L \supseteq L_0$.

The logic $L$ is perceptive over $L_0$ if the problem $L_0 + Ax \geq L$ is decidable; $L$ is recognizable over $L_0$ if the problem $L_0 + Ax = L$ is decidable.

The most important J-logics prove to be recognizable over J [8].

**Proposition 1.2** ([8]). Let $L_0$ be a finitely axiomatizable J-logic, $L = L_0 + A$. The logic $L$ is recognizable over $L_0$ if and only if $A$ is perceptive over $L_0$ and $L$ is decidable.

## 2. PRE-HEYTING LOGICS

Let us consider the class of pre-Heyting logics in more detail.

In [16] it is shown that Od is a J-logic for which the analogue of Glivenko’s theorem holds [15]: For any formula $A$

$$\text{Cl} \vdash \neg A \iff \text{Od} \vdash \neg A.$$  

Moreover, Od is the smallest of such J-logics [5].

Note that the formula $\text{Od}$ is J-conservative [13].

Recall that a formula $A(p)$ with one variable $p$ is called J-conservative [7] if

$$A(\bot), A(p), A(q) \vdash J A(p \& q); A(\bot), A(p), A(q) \vdash J A(p \lor q);$$
If \( L \) is a \( J \)-logic and the formula \( A \) is \( J \)-conservative, then the logic \( L + A \) is deduced to \( L \). We have

**Lemma 2.1** ([13], [7]). Suppose that a formula \( A(p) \) is \( J \)-conservative. Then for every logic \( L \) and every formula \( B(p_1, \ldots, p_k) \)

\[
L + A(p) \vdash B(p_1, \ldots, p_k) \iff L \vdash A(\bot) \& A(p_1) \& \ldots \& A(p_k) \to B(p_1, \ldots, p_k).
\]

The addition of conservative formulas as new axioms preserves the finite model property FMP and interpolation properties.

Taking into consideration the fact that the logic \( J \) possesses the properties FMP and CIP, we get

**Theorem 2.2** ([13], [7]). For any \( J \)-logic \( L \)

1. If \( L \) possesses FMP, then \( L + Od \) possesses FMP;
2. If \( L \) possesses CIP, then \( L + Od \) possesses CIP.

In particular, the logic \( Od \) has the properties CIP and FMP.

The semantic completeness of the logic \( Od \) with respect to modified semantics is established in [5]. This logic is complete with respect to a class of frames that satisfies the condition of maximality and also the condition

\( (F2) \) For every maximal in \( W - Q \) element \( x \) and every \( y \in Q, x \leq y \Rightarrow y = \infty \).

Similarly to superintuitionistic logics, all pre-Heyting logics possess the weak interpolation property WIP [19].

The characterization of pre-Heyting algebras by nonembeddability is obtained in [6], and the information on strong perceptibility and strong recognizability can be found in [14].

### 3. Perceptibility of \( F \) over \( Od \)

The majority of the most important formulas are perceptible over \( J \) and, consequently, over any \( J \)-logic. In [8], the question of the perceptibility of the formula \( F \) over \( J \) and the recognizability of the logic \( JF \) over \( J \) is left open. The decidability of this logic is proved in [12]. The perceptibility of the formula \( F \) in the class of well-compounded logics is obtained in [14]. In this section, we will prove the perceptibility of the formula \( F \) over the logic \( Od \).

Our proof uses algebraic semantics. Every variety is generated by its subdirectly irreducible algebras. Recall [3] that a \( J \)-algebra is said to be subdirectly irreducible (s.i.) if and only if it has an *opremum*, that is, the greatest of the elements distinct from \( \top \).

The following lemma is known:

**Lemma 3.1** ([3]). If \( a \nleq b \) in a \( J \)-algebra \( A \), then there exists a homomorphism \( h \) from \( A \) on a subdirectly irreducible \( J \)-algebra \( B \) such that \( h(a) = \top, h(b) = \Omega_B \).

We will prove the following

**Theorem 3.2.** Suppose \( Od + A \vDash F \). Then there exists a s.i. pre-Heyting algebra with not more than 14 elements, satisfying \( A \) and refuting \( F \).
Proof. Suppose $Od + A \not\vdash F$. By Lemma 3.1, there exist s.i. pre-Heyting algebra and elements $a_1, b_1$ such that
\[ \bot \leq a_1 \lor b_1, (\bot \rightarrow a_1) \lor (\bot \rightarrow b_1) = \Omega. \]
Let $a = \bot \rightarrow a_1$, $b = \bot \rightarrow b_1$. Then
\[ \bot \rightarrow a = a, \bot \rightarrow b = b, a \lor b = \Omega. \]
Let $B$ be a subalgebra generated by elements $a, b$.
Then it is s.i. pre-Heyting, and the elements $a, b$ satisfy the conditions:
\[ \bot \leq a \lor b, (\bot \rightarrow a) \lor (\bot \rightarrow b) = \Omega; \]
\[ \bot \rightarrow a = a, \bot \rightarrow b = b; \]
\[ a \rightarrow \bot = (\bot \rightarrow a) \rightarrow \bot = \bot, b \rightarrow \bot = \bot. \]
In particular, $a, b, \bot$ are pairwise incomparable.
Also $a \& b$ is incomparable to $\bot$, $\bot \& a$ is incomparable to $b$, $\bot \& b$. So, the algebra has to include the following elements of the set $FM$:
\[ 1) \ a \& b \& \bot, 2) \ a \& b, 3) \ a \& \bot, 4) \ b \& \bot, 5) \ a, 6) \ b, 7) \ \bot, 8) \ a \lor \bot, 9) \ b \lor \bot, 10) \ \Omega = a \lor b, 11) \ (a \& b) \lor \bot, 12) \ a \& (b \lor \bot), 13) \ b \& (a \lor \bot), 14) \ \top. \]
Here $1$ is the least, $\top$ is the greatest element, $10$ is an opremum. Next, $2, 3, 4$ are pairwise incomparable,
\[ 3 < 5 < 8; 4 < 6 < 9; 3, 4 < 7 < 8, 9; 11 \leq 10; \]
\[ 8, 9 \leq 10; 12 \leq 5; 13 \leq 6. \]
The number of elements of the algebra can be estimated with the help of Proposition 3.3.
\[ \square \]

**Proposition 3.3.** Suppose $Od + Ax \not\vdash F$, then there exists a pre-Heyting s.i. algebra $B$ with underlying set $FM = \{1 - 14\}$, such that $B \models Ax$:
\[ 1) \ a \& b \& \bot, 2) \ a \& b, 3) \ a \& \bot, 4) \ b \& \bot, 5) \ a, 6) \ b, 7) \ \bot, 8) \ a \lor \bot, 9) \ b \lor \bot, 10) \ \Omega = a \lor b, 11) \ (a \& b) \lor \bot, 12) \ a \& (b \lor \bot), 13) \ b \& (a \lor \bot), 14) \ \top. \]

Proof. The set $FM$ contains $\bot$. We want to prove that the set $FM$ is closed under all operations. First we will check whether $FM$ is closed under binary operations. Let us prove that the set $FM$ is closed under $\&$.
It is obvious that $x \& y = x \iff x \leq y$ and $x \& \Omega = x$ for each $x \neq \top$; $\top \& \bot = \bot$. Note that $2 \& 3 = 2 \& 4 = 2 \& 7 = 3 \& 4 = 3 \& 6 = 1$. The element $4$ is comparable to all the elements except $2, 3, 5, 12$. The following equalities hold:
\[ 4 \& 5 = 5 \& 9 = a \& (b \lor \bot) = 12, 6 \& 8 = 13, 11 \& 12 = ((a \& b) \lor \bot) \& (a \& (b \lor \bot)) = (a \lor \bot) \& (b \lor \bot) \& (a \& (b \lor \bot)) = a \& (b \lor \bot) = 12, 11 \& 13 = 12 \& 13 = a \& b = 2. \]

The results of the operation $\&$ for the elements $1-14$ belong to the set $FM$ and are listed in the table below.
So, $FM$ is closed under $\&$.

Now, we prove that the set $FM$ is closed under $\lor$.

Note that $(a \& \bot) \lor b = (a \lor b) \& (b \lor \bot) = \Omega \& (b \lor \bot) = b \lor \bot$,

$8 \lor 12 = a \lor \bot \lor a \& (b \lor \bot) = a \lor \bot$,

$11 \lor 12 = (a \& b) \lor \bot \lor (a \& (b \lor \bot)) = (a \lor \bot) \& (a \lor \bot \lor b) = (a \lor \bot) \& \Omega = a \lor \bot$,

$11 \lor 13 = (a \& b) \lor \bot \lor b \& (a \lor \bot) = (a \lor \bot) \lor \bot \lor b \& \bot = (a \& b) \lor \bot = 11$,

$12 \lor 13 = (a \& (b \lor \bot)) \lor (b \& (a \lor \bot)) = (a \& b) \lor (a \& \bot) \lor (a \& b) \lor (b \& \bot) = (a \& b) \lor \bot = 11$.

The results of the operation $\lor$ are listed in the table:

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Thus, we have proved that $FM$ is closed under $\lor$.

Now, let us prove that the set $FM$ is closed under $\rightarrow$.

We have
\[ x \leq y \iff x \to y = \top, \]
\[ \top \to x = x, \]
\[ \Omega \to x = x \text{ if } x < \Omega, \; \Omega \to x = \top \text{ if } x \geq \Omega. \]

\[ 2 \to 3 = a \& b \to a \& \bot = a \& b \to \bot = a \to \bot = \bot = 7. \]
\[ 2 \to 4 = \bot = 7. \]
\[ \text{Since } 2 \leq 5, 2 \leq 6, \text{ then } 2 \to 5 = 2 \to 6 = \top. \]
\[ 2 \to 7 = a \& b \to \bot = \bot. \]
\[ \text{As } 2 \leq 8, 9, 10, 11, 12, 13, \text{ then } 2 \to 8 = 2 \to 9 = 2 \to 10 = 2 \to 11 = 2 \to 12 = 2 \to 13 = \top. \]

Next, \[ 3 \to 2 = a \& \bot \to a \& b = a \& \bot \to b = b, \]
\[ 3 \to 4 = a \& \bot \to b \& \bot = a \& \bot \to b = a \to b = b, 3 \leq 5, 7, 8, 9, 11. \]
\[ 3 \to 6 = a \& \bot \to b = b, \]
\[ 3 \to 12 = a \& \bot \to (a \& (b \lor \bot)) = \top. \]
\[ 3 \to 13 = a \& \bot \to b \& (a \lor \bot) = a \& \bot \to b = b. \]

For \( 4 \to k \) the proof is similar.

We note that \[ 5 \to 3 = a \to a \& \bot = a \to \bot = \bot. \]

The results of \[ 5 \to 1, 5 \to 2, 5 \to 4 \] belong to the set \( FM \) because 1-14 are closed under conjunction, as \( x \to y \& z = (x \to y) \& (x \to z). \)

We have \[ 5 \to 6 = a \to b = b, \]
\[ 5 \to 7 = \bot. \]

Since \( 5 \leq 8 \), then \[ 5 \to 8 = \top. \]

Next, \[ 5 \to 9 = a \to b \lor \bot = \Omega \to b \lor \bot \text{ holds.} \]

It is easy to show that \[ 5 \to 10, 5 \to 11, 5 \to 12, 5 \to 13 \in 1-14. \]

For \( 6 \to k \) the proof is similar.

We note that \[ 7 \to 5 = \bot \to a = a, 7 \to 6 = \bot \to b = b, \text{also } 7 \leq 8, 9, 10, 11. \]

The results of \[ 7 \to 12, 7 \to 14 \] belong to the set \( FM \) because \( FM \) is closed under \&, as \( x \to y \& z = (x \to y) \& (x \to z). \)

Similarly, \[ 8 \to k, 9 \to k, 11 \to k \] belong to the set \( FM \) due to the fact that \( FM \) is closed under \&.

Since \[ 12 \to k = a \& (b \lor \bot) \to k = a \to (b \lor \bot \to k), \text{ then } 12 \to k \text{ belongs to the set } FM \text{ due to the already proved fact that the elements } (b \lor \bot) \to k, a \to k \text{ belong to the set } FM. \]

Similarly, it can be shown that \[ 13 \to k \in FM. \]

So, we proved that the set \( FM \) is closed under \( \to. \)

\[ \square \]
Theorem 3.4. The formula $F$ is perceptible over $Od$, and the logic $OdF$ is recognizable over $Od$.

Proof. Let $Ax$ be an arbitrary finite set of axioms. If $Od + Ax \not\vdash F$, then we apply Theorem 3.2. Conversely, if $B \models Ax$ and $B \not\models F$ for some pre-Heyting algebra $B$, then $Od + Ax \not\vdash F$. Therefore, $F$ is perceptible over $Od$.

The decidability of the logic $OdF$ follows from the finite model property of the logic $JF$ [12] and the fact that the formula $Od$ is $J$-conservative. Thus, the logic $OdF$ is perceptible over $Od$.

□

Note that the 14 elements in the obtained algebra do not have to be distinct. Let us consider, for example, the following cases:

(1) $a \lor \bot = b \lor \bot = \Omega$.

The elements 8, 9, 11, 12, 13 stick together with the others, and 9 elements remain.

(2) $a \lor \bot = \Omega$, $b \lor \bot < \Omega$.

There are 11 elements remaining in the algebra.

(3) $b \lor \bot = \Omega$, $a \lor \bot < \Omega$.

There are also 11 elements that remain in the algebra.

References


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