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PERIODIC LOCALLY NILPOTENT GROUPS OF FINITE c-DIMENSION

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Abstract. According to Bryant's theorem a periodic locally nilpotent group satisfying minimal condition on centralizers is virtually nilpotent. The c-dimension of a group is the supremum of lengths of chains of centralizers. We bound the index of the nilpotent radical of a locally nilpotent p-group of finite c-dimension k in terms of k and p.

Keywords: c-dimension, periodic locally nilpotent group, locally nilpotent p-group.

1. Introduction

Let G be a group and A be a subset of G. The set of all elements of G that commute with the elements of A is called the centralizer of A in G and is denoted by $C_G(A)$.

A group G satisfies the minimal condition on centralizers if every strictly descending chain of centralizers stabilizes in a finite number of steps. The class of all such groups is denoted by $\mathfrak{M}_c$.

There are several important classes of $\mathfrak{M}_c$-groups: finite groups, abelian groups, free groups, and linear groups. The class $\mathfrak{M}_c$ is closed under taking subgroups, finite direct product, and finite extensions, but it is not closed under taking homomorphic images and arbitrary extensions [1, Example 1].

A group is called periodic if every element of the group has finite order. Periodic $\mathfrak{M}_c$-groups were studied in [1, 2]. In particular, in [1] R. Bryant showed that periodic locally nilpotent $\mathfrak{M}_c$-group has a normal nilpotent subgroup of finite index. Recall

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that a group is called locally nilpotent if every finitely generated subgroup of the group is nilpotent.

Following [5], the supremum of lengths of strictly descending chains of centralizers is called the c-dimension of a group G and is denoted by $\operatorname{cdim}(G)$. Observe that all of the groups listed above are groups of finite c-dimension. An example of an $\mathfrak{M}_c$-group of infinite c-dimension was given in [1, Example 2].

It follows from Bryant's result that periodic locally nilpotent $\mathfrak{M}_c$-group has the nilpotent radical, i.e. the largest normal nilpotent subgroup. One can take instead of $\mathfrak{M}_c$-group a group of finite c-dimension. In this case, two natural questions arise: whether we can bound the nilpotency class or the index of the nilpotent radical of a group in terms of c-dimension.

The answer to both questions is negative. In [4] Khukhro constructs the following example: for any prime p there is a group of c-dimension 3 such that every its nilpotent subgroup of finite index has nilpotency class $p-1$. The counterexample to the second question is much simpler: for a prime p take the wreath product $\mathbb{Z}_p^{\infty} \wr \mathbb{Z}_p$. It has c-dimension 2, but the index of its nilpotent radical is p.

Nevertheless, we can prove the following:

Theorem 1. *Let G be a locally nilpotent p-group of c-dimension k. Then the index of its nilpotent radical is bounded in terms of p and k.*

Corollary 1. *Let G be a periodic locally nilpotent group of c-dimension k. Then the index of its nilpotent radical is bounded in terms of p and k, where p is the largest prime such that the Sylow p-subgroup of G is non-abelian.*

Remark. Note that such p always exists, as will be seen in the proof of the corollary.

Observe that the derived length of G itself is bounded by k according to [4, Lemma 1]. It is unknown whether one can bound the nilpotency class of the nilpotent radical of p-group of finite c-dimension k in terms of p and k.

2. Preliminaries

Let us recall some basic properties of c-dimension.

Lemma 1. *Let G and K be groups of finite c-dimension and let A be a subset of G. Then*

1. *If $H \leqslant G$, then $\operatorname{cdim}(H) \leqslant \operatorname{cdim}(G)$;*
2. *If $C_G(A) < G$, then $\operatorname{cdim}(C_G(A)) < \operatorname{cdim}(G)$;*
3. $\operatorname{cdim}(G \times K) = \operatorname{cdim}(G) + \operatorname{cdim}(K)$.

The proof of Theorem 1 is essentially based on the proof of the original Bryant theorem and requires some lemmas from [1].

Let us denote by $Z_k(G)$ the k-th member of the upper central series of a group G.

Lemma 2 ([1], Corollary 2.2)**.** *Suppose that G is a locally nilpotent $\mathfrak{M}_c$-group such that $Z_{k-1}(G) < G$, where k is a positive integer. Then $Z_{k-1}(G) < Z_k(G)$.*

Lemma 3 ([1], Lemma 2.6)**.** *Let G be a periodic nilpotent $\mathfrak{M}_c$-group. Then $G/Z_1(G)$ has finite exponent.*

Recall that a periodic group G has finite exponent if there exists a positive integer e such that $x^e = 1$ for every x in G. The smallest such integer e is called the exponent of G.

Lemma 4 ([1], Lemma 2.7)**.** *Let G be a locally nilpotent $\mathfrak{M}_c$-group such that $G/Z_k(G)$ has finite exponent for some integer $k \geqslant 0$. Then G is nilpotent.*

Lemma 5 ([1], Lemma 2.8)**.** *Let D be an elementary abelian group of order p^2, where p is a prime. let $n = \frac{1}{2}(p^2 + p)$. Then there exist non-trivial elements $x_1, x_2, \ldots, x_n$ of D such that*

$$(x_1 - 1)(x_2 - 1)\ldots(x_n - 1) = 0$$

in the group ring of D over the integers.

We write $G = A.B$ if G has a normal subgroup N isomorphic to A such that G/N is isomorphic to B.

3. Proofs

For every prime p, define the functions ε_p, $\psi_p(h, k)$, $h = 0, \ldots, k$; $k \in \mathbb{N}$, and $\varphi_p(k)$, $k \in \mathbb{N}$ as follows:

$$\varepsilon_p = \begin{cases} p, & \text{if } p \neq 2; \\ 4 & \text{if } p = 2. \end{cases}$$
$$\psi_p(h, k) = 1,\ k \in \{0, 1\},$$
$$\psi_p(k, k) = \varepsilon_p,\ k > 1,$$
$$\psi_p(h, k) = \varphi_p^{h+1}(k-1)\psi_p(h+1, k)^{\frac{p(p+1)}{2}},$$
$$\varphi_p(k) = \psi_p(0, k)!.$$

We are going to prove a more precise version of Theorem 1.

Theorem 2. *Let G be a locally nilpotent p-group of c-dimension k. Then the index of its nilpotent radical is at most $\varphi_p(k)$.*

Proof. We proceed by induction on the c-dimension of G. We may assume that G is nonabelian, i.e. $\mathrm{cdim}(G) > 0$.

It follows from Lemma 2 that $Z_1(G) < Z_2(G)$, so there exists $u \in Z_2(G) \setminus Z_1(G)$ such that its image in the quotient group $Z_2(G)/Z_1(G)$ has order p. The map $g \mapsto [g, u]$ is a homomorphism from G to $Z_1(G)$ with the kernel $C = C_G(u)$. Let us denote by E the image of G under this map. Observe that E is an elementary abelian group as $[g, u]^p = [g, u^p] = 1$.

Lemma 1 implies that $\mathrm{cdim}(C) < \mathrm{cdim}(G)$, so by inductive hypothesis $C = N.F$, where N is the nilpotent radical of C and $|F| \leqslant \varphi_p(k-1)$. Note that N is normal in G and $G/N \cong F.E$.

To complete the proof it is sufficient to show that G contains a nilpotent subgroup G_0 whose index does not exceed $\psi_p(0, k)$. Indeed, if such a subgroup exists, then the index of its core is at most $\psi_p(0, k)! = \varphi_p(k)$ as stated.

First, let us construct a rooted tree Γ such that every vertex γ of the tree is labeled by some centralizer M_γ from $Z_1(N)$ (note that different vertices can have the same label). The depth of the vertex is the length of the path from the tree root to this vertex.

We label the root of the tree by $Z_1(N)$. Now, let γ be the vertex of depth h. We attach children to γ in the following way:

Set $H = N_G(M_\gamma)$, $K = C_G(M_\gamma)$. The map $\overline{}: H \to H/K$ is the natural homomorphism.

If $\overline{H}$ has no non-cyclic elementary abelian subgroups, then γ is a leaf.

Now let $\overline{D} \leqslant \overline{H}$ be an elementary abelian subgroup of order p^2 and let $D \leqslant H$ be its preimage. By Lemma 5 there exist elements $x_1, \cdots, x_n \in D \setminus K$, $n = \frac{p(p+1)}{2}$, such that

$$(\overline{x}_1 - 1)(\overline{x}_2 - 1)\cdots(\overline{x}_n - 1) = 0$$

in the group ring of D over the integers. Now we attach n new vertices γ_i to γ with labels $M_{\gamma_i} = C_{M_\gamma}(x_i) < M_\gamma$.

Observe that every path from the root to a leaf corresponds to some chain of strictly descending centralizers in $Z_1(N)$, so Γ is a finite tree and the depth of every vertex is at most $\operatorname{cdim}(G) = k$.

Lemma 6. *If γ is a vertex of Γ of depth h, then $|G : N_G(M_\gamma)| \leqslant \varphi_p^h(k-1)$.*

Proof. We use induction on h. If $h = 0$, then $M_\gamma = Z_1(N)$ and $|G : N_G(M_\gamma)| = 1 = \varphi_p^0(k-1)$.

If $h > 0$ then γ has the parent vertex δ. Let $H = N_G(M_\delta)$, $K = C_G(M_\delta)$, $\overline{L} = C_{\overline{H}}(\overline{D})$ and let L be the preimage of $\overline{L}$.

Since L normalizes M_γ, we have

$$|G : N_G(M_\gamma)| \leqslant |G : L| = |G : H||\overline{H} : \overline{L}|.$$

Observe that the index of $\overline{L}$ in $\overline{H}$ is at most the order of the commutator subgroup $\overline{H}'$ of $\overline{H}$. Indeed, let $\overline{d_1}, \dots, \overline{d_s}$ be some elements of distinct conjugacy classes of $\overline{L}$ in $\overline{H}$. Then for every $i > 1$ there exists $\overline{d}$ in $\overline{D}$ such that $\left[\overline{d}, \overline{d_i}\right] \neq \left[\overline{d}, \overline{d_1}\right]$. Hence, the order of $\overline{H}'$ is not less than the index of $\overline{L}$.

Since M_δ is a central subgroup of N, $\overline{H}$ is a homomorphic image of a subgroup of $F.E$, so $|\overline{H}'| \leqslant |F| \leqslant \varphi_p(k-1)$. By induction hypothesis, $|G : H| \leqslant \varphi_p^h(k-1)$ and we have

$$|G : N_G(M_\gamma)| \leqslant \varphi_p^{h+1}(k-1).$$

□

Lemma 7. *For every vertex γ of Γ of depth h, there exists a subgroup $G_0 \leqslant G$ such that $|G : G_0| \leqslant \psi_p(h,k)$ and $M_\gamma \leqslant Z_m(G_0)$ for some positive integer m.*

Proof. We use the decreasing induction on h.

Let γ be a leaf of depth h. Then $\overline{H}$ has no non-cyclic elementary abelian subgroups, which means that it is finite. Indeed, assume that $\overline{H}$ is infinite. It has finite exponent, so there exists an element $\overline{v}$ of the largest order. Since the commutator subgroup $\overline{H}'$ is finite, the centralizer $C_{\overline{H}}(\overline{v})$ has finite index, so it is infinite. Then there exists an element $\overline{w} \in C_{\overline{H}}(\overline{v}) \setminus \langle \overline{v} \rangle$. The group $\langle \overline{v}, \overline{w} \rangle$ is abelian and non-cyclic; hence, it contains non-cyclic elementary abelian subgroup, so we obtain a contradiction.

Therefore, $\overline{H}$ is finite, so either it is cyclic or $p = 2$ and $\overline{H}$ is a generalized quaternion group (see, for example, [3, Theorem 6.11]). We have $|\overline{H} : \Phi(\overline{H})| \leqslant \varepsilon_p$, where $\Phi(\overline{H})$ is a Frattini subgroup of $\overline{H}$ and ε_p is defined in the beginning of this section.

Set $G_0 = K$. We have $M_\gamma \leqslant C_G(G_0) \cap G_0 = Z_1(G_0)$ and

$$|G : G_0| \leqslant |G : H||H : K| \leqslant \varphi_p^h(k-1)|\overline{H}|,$$

$$|\overline{H}| = |\overline{H} : \Phi(\overline{H})||\Phi(\overline{H})|,$$
$$|\Phi(\overline{H})| = |\Phi(H/K)| \leqslant |\Phi(H/N)| \leqslant |F|.$$

Therefore,

$$|G : G_0| \leqslant \varepsilon_p \varphi_p^h(k-1)\varphi_p(k-1) \leqslant \varphi_p^{h+1}(k-1)\psi_p(h+1,k) = \psi_p(h,k).$$

Now let γ be a branch (i.e. γ is not a leaf) of depth h, γ_i are its children for $i = 1, \dots, n$. Then by induction hypothesis there exist subgroups G_i of G such that $|G : G_i| \leqslant \psi_p(h+1,k)$ and $M_{\gamma_i} \leqslant Z_{m_i}(G_i)$.

If we set $G_0 = L \cap G_1 \cap \dots \cap G_n$, then

$$|G : G_0| \leqslant |G : L| \prod_{i=1}^{n} |G : G_i| \leqslant \varphi_p^{h+1}(k-1)(\psi_p(h+1,k))^n = \psi_p(h,k).$$

Now we show that $M_\gamma \leqslant Z_{nm}(G_0)$, where $m = \max\{m_1, \dots, m_n\}$.

$\overline{L}$ acts on M_γ by conjugation, so we may think of M_γ as of $\overline{L}$-module. Observe that $M_{\gamma_i} \leqslant Z_m(G_0)$.

We have

$$M_\gamma(\overline{x}_1 - 1)(\overline{x}_2 - 1)\cdots(\overline{x}_n - 1) = 0,$$

consequently,

$$M_\gamma(\overline{x}_1 - 1)(\overline{x}_2 - 1)\cdots(\overline{x}_{n-1} - 1) \leqslant C_{M_\gamma}(x_n) = M_{\gamma_n} \leqslant Z_m(G_0).$$

Then

$$M_\gamma(\overline{x}_1 - 1)(\overline{x}_2 - 1)\cdots(\overline{x}_{n-1} - 1)(\overline{G_0} - 1)^m = 0.$$

Since all $\overline{x_i}$ are central elements of $\overline{L}$, we obtain

$$M_\gamma(\overline{G_0} - 1)^m(\overline{x}_1 - 1)(\overline{x}_2 - 1)\cdots(\overline{x}_{n-1} - 1) = 0.$$

By continuing in a similar way, we obtain

$$M_\gamma(\overline{G_0} - 1)^{nm} = 0.$$

It is equivalent to the inclusion $M_\gamma \leqslant Z_{nm}(G_0)$. $\square$

Now we finish the proof of Theorem 2 by applying Lemma 7 to the root γ. We have $M_\gamma = Z_1(N) \leqslant Z_m(G_0)$. It follows from Lemma 3 that the quotient group $N/Z_1(N)$ has finite exponent; therefore, so does $G_0/Z_m(G_0)$. Therefore, G_0 is nilpotent by Lemma 4, and the index of G_0 in G is no more than $\psi_p(0,k)$. $\square$

One can easily derive Corollary 1 from Theorem 2.

Proof of Corollary 1. A periodic locally nilpotent group can be represented as a direct product of its Sylow subgroups $G = \prod O_p(G)$. Since $\operatorname{cdim}(G \times H) = \operatorname{cdim} G + \operatorname{cdim} H$, there are only finitely many primes p such that $O_p(G)$ is nonabelian (i.e. such that $\operatorname{cdim}(O_p(G)) > 0$). We apply Theorem 2 to $O_p(G)$ for every such p. Then $O_p(G) = N_p.F_p$, where N_p is the nilpotent radical of $O_p(G)$ and $|F_p| \leqslant \varphi_p(k)$. N_p is a normal nilpotent subgroup of G for every p; thus, their direct product N is also a nilpotent normal subgroup of G and

$$|G : N| \leqslant \prod \varphi_p(k),$$

where $\varphi_p(k) = 1$ when $O_p(G)$ is abelian.

Observe that the right-hand side of the inequality can be bounded in terms of k and the largest p such that $O_p(G)$ is nonabelian. $\square$

In the end, we prove some small refinement of Bryant's result about the structure of locally nilpotent groups satisfying the minimal condition on centralizers.

Theorem 3. *Let G be a periodic locally nilpotent group satisfying the minimal condition on centralizers and let N be its nilpotent radical. Then $Z_1(N) = C_G(N)$.*

Proof. Let us denote by C the centralizer of $Z_1(N)$ in G. We start by proving that $C = N$. The inclusion $C \geqslant N$ is obvious. To prove the converse, we show that C is nilpotent.

Since $Z_1(C) \geqslant Z_1(N)$, the quotient group $C/Z_1(C)$ is a homomorphic image of $C/Z_1(N)$, so $C/Z_1(C)$ has a finite exponent by Lemma 3. Thus, C is nilpotent by Lemma 4, and normal in G as a centralizer of a normal subgroup; hence, $C \leqslant N$.

We have $N = C_G(Z_1(N)) \geqslant C_G(N)$; therefore $C_G(N) = Z_1(N)$. $\square$

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