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ON GROUPS WITH A STRONGLY EMBEDDED UNITARY SUBGROUP

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Abstract. A proper subgroup B of a group G is called *strongly embedded*, if $2 \in \pi(B)$ and $2 \notin \pi(B \cap B^g)$ for every element $g \in G \setminus B$, and therefore $N_G(X) \leq B$ for every 2-subgroup $X \leq B$. An element a of a group G is called *finite*, if for every $g \in G$ the subgroup $\langle a, a^g \rangle$ is finite.

In the paper, it is proved that a group with a finite element of order 4 and a strongly embedded subgroup isomorphic to the Borel subgroup of $U_3(Q)$ over a locally finite field Q of characteristic 2 is locally finite and isomorphic to the group $U_3(Q)$.

Keywords: A strongly embedded subgroup of a unitary type, Borel subgroup, Cartan subgroup, involution, finite element.

Introduction

A proper subgroup B of a group G is called *strongly embedded*, if $2 \in \pi(B)$ and $2 \notin \pi(B \cap B^g)$ for every element $g \in G \setminus B$, and therefore $N_G(X) \leq B$ for every 2-subgroup $X \leq B$ [1, Definition 4.20]. As D. Gorenstein wrote [1, pp. 26-27, 196-202], the notion of a strongly embedded subgroup constitutes one of the most important instruments of the theory of finite simple groups. According to known results by M. Suzuki and H. Bender [2, 3], a finite simple group with a strongly embedded subgroup is isomorphic to one of the groups $L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$ [1, Theorem 4.24].

An element a of a group G is called *finite*, if for every $g \in G$ the subgroup $\langle a, a^g \rangle$ is finite. Thus, for example, in a periodic group, each involution is a finite element. In [4, 5], it was proved that a group with a finite involution isomorphic to a Borel subgroup of the group $L_2(Q)$ or $Sz(Q)$ over a locally finite field Q of characteristic 2, is locally finite and isomorphic to the group $L_2(Q)$ or $Sz(Q)$ respectively. These results were instrumental for the study of groups with saturation conditions [6] – [10], and among them

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the characterization of the unitary groups $U_3(Q)$ from Suzuki-Bender's list was found. In the present paper, the characterization of groups $U_3(Q)$ without saturation conditions was obtained:

Theorem. *A group with a finite element of order 4 and a strongly embedded subgroup isomorphic to a Borel subgroup of the group $U_3(Q)$ over a locally finite field Q of characteristic 2 is locally finite and isomorphic to the group $U_3(Q)$.*

1. Definitions and preliminary results

In the proof, we use a special case of Theorem 4.18 from [11] (see also [7, Proposition 2]):

Proposition 1. *Let a locally finite group G be a union of an ascending series of subgroups, each of which is isomorphic to $U_3(q)$ for some even number q. Then $G \simeq U_3(Q)$, where Q is a locally finite field of characteristic 2.*

According to this proposition, the group G under investigation will be represented as a union $G = \bigcup_{i=2}^{\infty} G_i$ of an ascending chain of finite subgroups determined in Lemma 11:

$$G_2 < G_3 < \dots, \quad \text{where } G_i = U_3(Q_i), \ Q_i \text{ is a finite field of characteristic 2.} \tag{1}$$

For the infinite locally finite group $G = U_3(Q)$, a set of chains of the type (1) has the cardinality of the continuum, however, the structure of G and its Borel subgroup $B = U \leftthreetimes H$ is uniquely defined by the structure of the field P with an automorphism σ of order 2 and its subfield Q of points fixed by σ. In this paper, we use the designations ${}^2A_2(2^{2n}) = PSU_3(2^n) = U_3(2^n)$ from [13, Table 6.4.1], [1, pp. 166-168], and the field Q is used in the wording of statements, while the field P "stays off the screen". We hope it does not cause misunderstanding, especially because similar designations were used in the results from [1] employed in our paper.

The locally finite field Q is countable and can be viewed as a union $Q = \cup_{i=2}^{\infty} Q_i$ of finite subfields Q_i forming a chain $Q_2 \subset Q_3 \subset \dots$. Each of such chains prompts a corresponding ascending chain $B_2 < B_3 < \dots$ of Borel subgroups $B_i = U_i \leftthreetimes H_i$ of the groups $G_i = U_3(Q_i)$, whose union $B = \cup_{i=2}^{\infty} B_i$ coincides with a strongly embedded subgroup given by the conditions of the Theorem. Sylow 2-subgroups U_i of groups B_i also form a chain $U_2 < U_3 < \dots$, $U = \cup_{i=2}^{\infty} U_i$; a chain of Cartan subgroups is set in a similar way: $H_2 < H_3 < \dots$, $H = \cup_{i=2}^{\infty} H_i$. According to item 2 of Proposition 3, $H = H_0 \times H_1$ is a locally cyclic group, H_0 is isomorphic to a multiplicative group of the field Q and can be viewed as a union $H_0 = \cup_{i=2}^{\infty} H_{0i}$ of a chain of subgroups $H_{0i} \leq H_i$ isomorphic to multiplicative groups of the fields Q_i:

$$H_{02} < H_{03} < \dots, \quad \text{where } H_{0i} = H_0 \cap H_i. \tag{2}$$

The chain (2) uniquely defines the chain $Q_2 \subset Q_3 \subset \dots$ of subfields Q_i. We should note this detail, because in the theorem proof, the presentation goes in the direction opposite to the one outlined above, that is, from the chain (2) to the chain (1).

In the proof, we use Theorem 4.1 from [4]:

Proposition 2. [4, Theorem 4.1] *Let the group G contain a finite involution and an elementary abelian 2-subgroup Z, whose normalizer $N_G(Z)$ is strongly embedded into G and is a Frobenius group with the kernel Z and the locally cyclic complement T. Then G is locally cyclic and isomorphic to a group $L_2(Q)$, where Q is a locally finite field of characteristic 2.*

We will employ the properties of Borel subgroups of the groups $U_3(Q)$, [1, CЃC,CЂ. 166-168]:

Proposition 3. *Suppose that Q is a (locally) finite field of characteristic 2, $B = U \leftthreetimes H$ is a Borel subgroup of the group $G = U_3(Q)$, H is its Cartan subgroup, U is a Sylow 2-subgroup of the groups B and G, and $Z = Z(U)$. Then the following holds:*

(1) *$B = U \rightthreetimes H$, H is a (locally) cyclic group, U is a Sylow 2-subgroup in G of nilpotency class 2 and period 4, and that $Z = Z(U) = U' = \Phi(U) = \Omega_1(U)$;*
(2) *$H = H_0 \times H_1$, where $H_1 = C_H(Z)$, H_0 and H_1 are (locally) cyclic subgroups, the intersection $\pi(H_0) \cap \pi(H_1)$ is empty;*
(3) *if $|Q|$ is a finite field containing $q = 2^n$ elements, then $|G| = q^3(q^2-1)(q^3+1)/d$, $|U| = q^3$, $|Z| = q$, $|H| = (q^2-1)/d$, $|H_1| = (q+1)/d$, $|H_0| = q-1$, where $d = (3, q+1)$;*
(4) *$U \rightthreetimes H_0$ is a Frobenius group with a noninvariant quotient H_0 acting transitively on the set $Z^{\#}$ of all involutions of the group U;*
(5) *B/Z is a Frobenius group with the kernel U/Z and the complement HZ/Z, and the quotient group HZ/Z is either transitive on the set of nonidentity elements of the quotient group U/Z, when there is an element of order 3 in H_0, or has exactly 3 orbits.*

Elementary properties of an arbitrary group G with a finite involution and a strongly embedded subgroup B can be found in [4, Lemmas 1.7, 2.1, 2.2] and in [5, 15]:

Lemma 1. *Suppose that G is a group with a finite involution and a strongly embedded subgroup B, J is a set of involutions in G, $i \in J \cap B$, $k \in J \setminus B$ and $K = B \cap B^k$. Then the following holds:*

(1) *every involution in the group G is conjugate and finite, $J \cap B = i^B$, the order of the product ik is odd and the set J of involutions in G is evenly distributed among the cosets of G/B: $|J \cap Bg| = |J \cap B|$ for every element $g \in G$;*
(2) *if a Sylow 2-subgroup S of the group G intersects nontrivially with B, then it lies in B;*
(3) *if $L < G$, $L \nleq B$ and $2 \in \pi(L \cap B)$, then the subgroup $L \cap B$ is strongly embedded in L;*
(4) *if b is a nonidentity element of odd order of G and $b^j = b^{-1}$ for some involution $j \in G$, then there are no involutions in the centralizer $C_G(b)$;*
(5) *in every coset $C_B(i) \cdot b$, where $b \in B$, there exists a unique element invertible by the involution k, and its order is finite and odd;*
(6) *$B = K \cdot C_B(i) = T \cdot C_B(i)$ and $i^B = i^T$, where T is a subgroup in $K = B \cap B^k$, generated by the elements of K invertible by the involution k.*

2. Lemmas on a group with a finite involution

In this section, G denotes a group with a finite involution and a strongly embedded subgroup B isomorphic to a Borel subgroup of a group $U_3(Q)$ over a locally finite field Q of characteristic 2. Assume from here on that $B = U \rightthreetimes H$, U, H, H_0, H_1 and Z are subgroups of the group G defined in Proposition 3.

Lemma 2. *Suppose that k is an involution from $G \setminus B$ and T is a subgroup generated by all elements of $B \cap B^k$ invertible by the involution k. Then $T = H_0^u$ for a suitable element $u \in U$, in particular, H_0 is inverted by some involution $v \in G \setminus B$.*

Доказательство. There are no involutions in the subgroup $K = B \cap B^k$ under given conditions; by item 1 of Proposition 3, $K \simeq KU/U$ is a locally cyclic group; and so the set of elements of K invertible by the involution k is a subgroup $T = \{t \in K \mid t^k = t^{-1}\}$. From item 1 of Lemma 1, we have that $C_U(t) = 1$ for every $t \in T^{\#}$. So T acts freely on U by conjugation. The group B is locally finite, the Frobenius theorem holds for its subgroups, and therefore the subgroups $Z \rightthreetimes T$, $U \rightthreetimes T$ are Frobenius groups with the complement T. According to item 1 of Proposition 3, in the locally finite group B, all finite p-subgroups for odd p are cyclic. By the Sylow theorem, all subgroups of B of the same prime odd order p are conjugate. By item 2 of Proposition 3, $H_1 \leq C_B(Z)$, therefore, $\pi(T) \subseteq \pi(H_0)$, and T lies in the Frobenius group $U \rightthreetimes H_0$ (item 3 of Proposition 3). Due to the properties of (locally) finite Frobenius groups [12, Proposition 1.14], T is contained in one of the complements of the group $U \rtimes H_0$, and up to conjugacy in $U \rtimes H_0$, we can assume that $T \leq H_0$. According to items 5 and 6 of Lemma 1, T acts transitively on $Z^{\#}$, and from item 3 of Proposition 3, it follows that the equality $T = H_0$ is true. As explained above, the involution $v = k$ inverts T. The lemma is proved. □

We fix notation v for the involution from Lemma 2 for the rest of the paper.

Lemma 3. *The subgroup H coincides with the centralizer $C_G(t)$ of every nonidentity element $t \in H_0$. The following equalities are true: $C_B(v) = H_1$, $B \cap B^v = H$, and $N_G(H) = H \rtimes \langle v \rangle$.*

Доказательство. Suppose that $1 \neq t \in H_0$, and assume that $C_G(t) \neq H$. Due to items 1 and 4 of Proposition 3, $C_B(t) = H$, therefore, there exists an element g in $C_G(t) \setminus B$. From item 1 of Lemma 1, we have $g = bk$, where $b \in B$, k is an involution from $G \setminus B$, and $t \in K = B \cap B^k$. By Lemma 2, the elements of K inverted by the involution k form a subgroup $T = H_0^u$, and item 4 of Proposition 3 implies that $U \rtimes T = U \rtimes H_0$ is a Frobenius group with the kernel U and the complements T and H_0. Since K does not contain involutions, $K \cap U = 1$, $t \in T \cap H_0$, and $T = H_0$ being the complements of the same Frobenius group. By Lemma 2, $t^k = t^{-1}$, and therefore $t^b = t^{-1}$, which contradicts with the structure of the subgroup B (Proposition 3). Hence, $C_G(t) = H$ for every nonidentity element $t \in H_0$. From Lemma 2, we have $t^v = t^{-1}$, and therefore $H^v = H$ and $H \leq B \cap B^v$. Since every subgroup of B containing H as a proper subgroup contains involutions (item 1 of Proposition 3), while the subgroup $K = B \cap B^v$ does not contain any involutions, we then have $B \cap B^v = H$. By Proposition 3, $H = H_0 \times H_1$ is a locally cyclic group, $\pi(H_0) \cap \pi(H_1)$ is empty, and $H_1 \leq C_B(Z)$. We conclude that $H_1^v = H_1$ and $H_1 = C_H(v)$.

From items 4 and 5 of Proposition 3, it follows that $N_B(H) = H$. Suppose that $g \in N_G(H) \setminus B$. By item 1 of Lemma 1, $g = bk$, where $b \in B$, and k is an involution of $G \setminus B$, and $H = B \cap B^g = B \cap B^k$. Hence, $H^k = H$, $H^b = H$, and $b \in H$. From Lemma 2, we have $H_0 = \{h \in H \mid h^k = h^{-1}\}$ and $kv \in C_G(H_0) = H$. Therefore, $k \in Hv$, $N_G(H) = H \rtimes \langle v \rangle$, and the lemma is proved. □

Lemma 4. *$C_G(H_1) = H_1 \times L$, where $L = H_0 Z \langle v \rangle Z$ is a subgroup isomorphic to $L_2(Q)$. For every element $t \in H_1^{\#}$, the equality $N_G(\langle t \rangle) = C_G(H_1)$ holds.*

Доказательство. Consider the quotient group $\bar{G}_0 = C_G(H_1)/H_1$ and its subgroup $\bar{B}_0 = ZH/H_1$. Suppose that $\bar{x} \in \bar{G}_0 \setminus \bar{B}_0$ is an arbitrary element and g is one of its preimages in G. It is obvious that $g \notin B$. Since the intersection $ZH \cap (ZH)^g \leq B \cap B^g$ does not contain involutions and H_1 is a periodic group, $\bar{B}_0 \cap \bar{B}_0^{\bar{x}}$ does not contain involutions as well. Hence, $\bar{B}_0$ is strongly embedded in $\bar{G}_0$. According to items 2 and 4 of Proposition 3, $C_G(H_1) \cap B = H_1 \times (Z \rtimes H_0)$, and therefore $\bar{B}_0 \simeq \bar{Z} \rtimes \bar{H}_0$ is a Frobenius group with an elementary abelian kernel $ZH_1/H_1 \simeq Z$ and a locally cyclic complement $H/H_1 \simeq H_0$. Due to item 1 of Lemma 2, every involution in the group $\bar{G}_0$ is finite. By Proposition 2, $\bar{G}_0 \simeq L_2(R)$ for a suitable locally finite field R of characteristic 2. Since the multiplicative groups of the locally finite fields R and Q are isomorphic to H_0, the fields R and Q are isomorphic, and $\bar{G}_0 \simeq L_2(Q)$. By Lemma 3, $\bar{v} \in \bar{G}_0$ and $\bar{G}_0 = \bar{B}_0 \langle \bar{v} \rangle \bar{B}_0$.

By Schmidt theorem [14], the group $C_G(H_1)$ is locally finite. The field Q is a union $Q = \cup Q_i$ of an ascending chain (2) of finite fields of characteristic 2, which corresponds in G_0 to an ascending chain

$$\overline{K}_2 < \overline{K}_3 < \dots \tag{3}$$

of finite subgroups $\overline{K}_i \simeq L_2(Q_i) = L_2(2^{n_i})$, and moreover, $\cup \overline{K}_i = \bar{G}_0$. Since the Schur multiplier equals Z_2 in $L_2(2^n)$ ([1], Table 4.1), and the subgroup H_1 does not contain involutions, the full preimage K_i of the subgroup $\overline{K}_i$ can be decomposed into a direct product $K_i = H_1 \times L_i$, where $L_i \simeq L_2(Q_i) = L_2(2^{n_i})$. Hence, we conclude that the subgroups L_i form a chain

$$L_2 < L_3 < \dots, \tag{4}$$

whose union L is isomorphic to $L_2(Q)$ by Proposition 1. Thus, $C_G(H_1) = H_1 \times L$, $L = ZH_0\langle v\rangle Z$, and the first statement of the lemma is proved.

Suppose that $t \in H_1^{\#}$, $g \in N_G(\langle t\rangle)$, and $g \notin C_G(H_1)$. By Lemma 1, $g = bk$, where $b \in B$, $k \in J \setminus B$, and based on Lemmas 2 and 3, we have $\langle t^k\rangle = \langle t\rangle$ and $k \in C_G(H_1)$. Therefore, $b \notin N_B(\langle t\rangle) = H_1 \times ZH_0$, and due to Proposition 3, we can assume $b = u$ to be an element of order 4. But then $\langle t^{uk}\rangle = \langle t\rangle$ and $\langle t^u\rangle = \langle t\rangle$, $u^t = u$, which contradicts with the action of H_1 on the quotient group U/Z (item 5 of Proposition 3). Hence, $N_G(\langle t\rangle) = C_G(H_1)$, and the lemma is proved. □

Lemma 5. *The subgroup UH_0 acts (by conjugation) transitively on the set of involutions $J \setminus B$, the group G acts doubly transitively on the set $\Omega = B^G$ of subgroups conjugate to B, and the subgroup U acts regularly on $\Omega \setminus \{B\}$. In particular, $G = B\langle v\rangle B = B \cup BvU$, U is a Sylow 2-subgroup of the group G, and Sylow 2-subgroups in G are conjugate and pairwise coprime.*

Доказательство. By Proposition 3, $H = C_B(H_0) = N_B(H_0)$, and according to Lemma 3, $N = N_G(H_0) = H \leftthreetimes \langle v\rangle$. Based on Proposition 3, we have that UH_0 is a Frobenius group with a kernel U and a complement H_0, and therefore every complement K of the subgroup U in B has a form $K = H^u$, where $u \in U$. Suppose that k is an arbitrary involution from $J \setminus B$ and T is a subgroup generated by all elements of $B \cap B^k$ inverted by the involution k. By Lemma 2, for a suitable element $u \in U$ we have that $T = H_0^{u^{-1}}$, and according to Lemma 3, $B \cap B^k = H^{u^{-1}}$ and $k^u \in N$. Lemma 3 implies that $J \cap N = v^{H_0}$, hence, $k^{uh} = v$ for a suitable $h \in H_0$. Since $C_G(v) \cap UH_0 = 1$, it follows that UH_0 acts regularly by conjugation on the set $J \setminus B$, that is, transitively and fixed-point-freely.

If $V \in Syl_2(G)$ and $V \cap U \neq 1$, then by item 2 of Lemma 1, $V \leq B$, and from Proposition 3 it follows that $V = U$. Therefore, $U \cap V = 1$, if $V \neq U$. Suppose that $V = U^v$. Due to the transitivity of the action of the group UH_0 on the set $J \setminus B$, it follows that $J \setminus U \subseteq \cup_{x \in UH_0} V^x$ and $Syl_2(G) \setminus \{U\} = V^{UH_0}$. So, $Syl_2(G) = U^G$, and G acts doubly transitively on the set $\Omega = B^G$. Since $V^{H_0} = \{V\}$ and $N_U(V) = 1$, $Syl_2(G) \setminus \{U\} = V^U$, and U acts regularly by conjugation on the set $\Omega \setminus \{B\}$. It easily follows from the argument above that $G = B\langle v\rangle B = B \cup BvU$, U is a Sylow 2-subgroup of the group G, and Sylow 2-subgroups in G are conjugate and pairwise coprime. The lemma is proved. □

Lemma 6. *Every element $g \in G \setminus B$ has a unique canonical representation of the form $g = hu_1vu_2$, where $u_1, u_2 \in U$, $h \in H$. Every involution t of $G \setminus B$ can be canonically expressed in the form $t = u_t^{-1}h_tvu_t = h_tu_t^{-h_t}vu_t$, where $u_t \in U$, $h_t \in H_0$.*

Доказательство. By Lemma 5, $G = B \cup BvU$, and every element $g \in G \setminus B$ can be expressed in the form $g = hu_1vu_2$, where $u_1, u_2 \in U$, $h \in H$. According to [1][p. 160], such representation is called *canonical*. It is unique since from $g = hu_1vu_2 = h_1u_3vu_4$ it first follows that $u_3^{-1}h_1^{-1}hu_1 = vu_4u_2^{-1}v$, and later that $u_4 = u_2$ due to the fact that B is strongly embedded in G, and finally $h = h_1$ and $u_3 = u_1$, based on the equality $B = U \leftthreetimes H$ (Proposition 3).

By Lemmas 5 and 3, $J \setminus B = v^{H_0U} = (H_0v)^U$, and the involution t of $G \setminus B$ can be expressed canonically in the form $t = u_t^{-1}h_tvu_t = h_tu_t^{-h_t}vu_t$, where $u_t \in U$, $h_t \in H_0$. The lemma is proved. □

We use the following designations from Section 1: Q_k, U_k, H_k, B_k, here $k \geq 2$; we also designate $Z_k = Z(U_k)$, $H_{1k} = H_1 \cap H_k$, and $H_{0k} = H_0 \cap H_k$, where H_1, H_0 are subgroups of H (Proposition 3), so that $H_k = H_{1k} \times H_{0k}$. We introduce finite subsets

$$M_k = B_k\langle v\rangle B_k = B_k \cup B_k v U_k. \tag{5}$$

Since H is a locally cyclic group (item 1 of Proposition 3) and $v \in N_G(H)$ (Lemma 3), all subgroups H_k and subgroups H_{1k} and H_{0k} are admissible with respect to the involution v (Lemma 2). Therefore, in (5), the equality $B_k v B_k = B_k v U_k$ holds.

In the group $Z = Z(U)$, there is a uniquely defined involution u_0 [1][p. 163], which satisfies the following *Suzuki's structural equation for groups L and G*:

$$vu_0v = u_0vu_0, \text{ or } (vu_0)^3 = 1. \tag{6}$$

Remark 1. *If $u_1 \in Z$, $u_1 \neq u_0$, and $vu_1v = u_1vu_1$, then the subgroup $\langle u_0, v, u_1\rangle$ is isomorphic to the symmetric group S_4* [16, §6.2]. *However, there are no strongly embedded subgroups in S_4, which contradicts with item 3 of Lemma 1. Hence, the involution u_0 of (6) is unique in Z.*

We can assume that $u_0 \in U_k$ for all k. Along with the multiplication formulas in the subgroups $Z \leftthreetimes H_0$ and $H_0 \leftthreetimes \langle v\rangle$, the equation (6) uniquely defines multiplication in the group $L = ZH_0\langle v\rangle Z$. Since $H_0v = vH_0$ $(hv = vh^{-1})$, for any $z_1, z_2, z_3, z_4 \in Z$, $h_1, h_2 \in H_0$ we have $(z_1h_1vz_2)\cdot(z_3h_2vz_4) = (z_1h_1)\cdot(vz_2z_3v)\cdot(h_2^{-1}z_4)$, and using (6) we find an expression for the element vtv in the form $z_5h_3vz_6$, where $t = z_2z_3 \in Z$. Since the involution $t \in Z$ can be expressed in the form $t = h_t^{-1}u_0h_t$ for a suitable and unique $h_t \in H_0$ (item 3 of Proposition 3), then by applying (6), we obtain

$$vtv = vh_t^{-1}u_0h_tv = h_tvu_0vh_t^{-1} = h_tu_0vu_0h_t^{-1} = u_0^{h_t{}^{-1}}h_t^2vu_0^{h_t^{-1}}, \tag{7}$$

where $z_5 = u_0^{h_t{}^{-1}}$, $h_3 = h_t^2$, $z_6 = u_0^{h_t^{-1}}$. Hence, multiplication in L is uniquely defined and the isomorphism $L \simeq L_2(Q)$ is established. The same arguments work when $L = L_k$ is a finite set. It follows that L_k is a subgroup and $L_k \simeq L_2(Q_k)$. So, the following lemma is true:

Lemma 7. *The set M_k contains the subgroups $L_k = Z_kH_{0k}\langle v\rangle Z_k$ and $H_{1k} \times L_k$.*

3. Proof of the theorem

Lemma 8. *Suppose that u is an element of order 4 in U and the subgroup $M = \langle u, v\rangle$ is finite. Then either $M = A \leftthreetimes \langle u\rangle$ is a Frobenius group with a complement $\langle u\rangle$ and an abelian kernel A, or M is isomorphic to the group $U_3(Q_m)$ over some finite subfield Q_m of the field Q.*

Доказательство. By Lemma 1, the subgroup $B_m = B \cap M$ is strongly embedded in M, and $U_m = B_m \cap U$ is a normal in B_m Sylow 2-subgroup of the group M. From Lemma 5, it follows that Sylow 2-subgroups in M are pairwise coprime, in particular, $O_2(M) = 1$.

If the involution u^2 in U_m is unique, then $M = A\cdot C_M(u^2)$, where $A = O(M)$, Moreover, $v \in A \leftthreetimes \langle u^2\rangle$, and therefore $M = A \leftthreetimes \langle u\rangle$. Hence, $U_m = \langle u\rangle$, according to Proposition 3, $A \cap B = 1$, $B_m = U_m = \langle u\rangle$, $C_A(u^2) = 1$, and $M = A \leftthreetimes \langle u\rangle$ is a Frobenium group with a complement $\langle u\rangle$ and abelian kernel A, and $a^v = a^{u^2} = a^{-1}$ for any $a \in A$.

Let $Z_m = U_m \cap Z$ be a noncyclic group. Then due to Proposition 3 and Lemma 1, B_m contains a subgroup $Z_m \leftthreetimes H_{0m}$, where $H_{0m} \leq H_0$, and by items 2 and 3 of Proposition 3, H_{0m} acts regularly by conjugation on $Z_m^{\#}$. According to Lemma 7, M contains a subgroup $L_m = Z_mH_{0m}\langle v\rangle Z_m$ isomorphic to $L_2(Q_m)$, where $|Q_m| = |H_{0m}| + 1 = q_m$ (the subfield Q_m in Q obviously coincides with the set of solutions Q of the equation $x^{q_m} - x = 0$). Since U_m is normal in B_m and contains an element of order 4, due to the structure of the subgroup B (Proposition 3), we have that $O(B_m) = 1$. The noncyclic group Z_m cannot act on a finite group fixed-point-freely, hence, $O(B_m) = 1$ implies that $O(M) = 1$. So, $O_2(M) = O(M) = 1$, and by Lemma 1, all involutions in M are conjugate. Therefore, the minimal normal subgroup K in the group M is generated by the set $J \cap M$ and is a nonabelian simple group. Since $v, u^2 \in K$, it follows that $[M:K] \leq 2$. On the other hand, $Z_m \leq K$, $|U_m/Z_m| \geq |H_{0m}| > 2$, and by item 4 of Proposition 3, the quotient group B_m/Z_m is a Frobenius group with a complement of odd order, and it does not contain any subgroups of index 2. Hence, $B_m \leq K$ and $K = M$. Since M contains a subgroup isomorphic to $L_2(Q_m)$ (Lemma 7), it follows that || (??) is divisible by 3. Taking the above mentioned properties into account, we conclude that according to [1, Theorem 4.24], M is isomorphic to $U_3(Q_m)$. The lemma is proved. □

Lemma 9. *A set of subgroups $M_z = \langle uz, v\rangle$, where u is a fixed element of order 4 of U and z ranges over Z, contains at most one finite nonsimple group.*

Доказательство. Assume that $M = \langle u, v\rangle$ and $M_z = \langle uz, v\rangle$ are two distinct finite nonsimple groups. By Lemma 8, $M = A \leftthreetimes \langle u\rangle$ and $M_z = A_z \leftthreetimes \langle uz\rangle$ are Forbenius groups with abelian kernels A and A_z respectively and with different complements $\langle u\rangle$, $\langle uz\rangle$, such that $1 \neq z \neq u^2$. Set $t = u^2 = (uz)^2$, $c = vt$, and $d = c^u$. We have $cd = dc$, $c, d^z \in A_z$, $cd^z = d^z c$, and $d, d^z \in C_G(c)$. Since $c^v = c^{-1}$, by Lemma 1, there are no involutions in the subgroup $C_G(c)$, and $D = \langle d, d^z\rangle$ is a subgroup without involutions. Because $d^t = d^{-1}$, $(d^z)^t = (d^z)^{-1}$, a four-subgroup $T = \langle t\rangle \times \langle z\rangle$ lies in $N_G(D)$. Hence, $K = \langle D, T\rangle = D \leftthreetimes T$, and the involutions t, z in K are not conjugate. Due to Proposition 3, we have that $d \notin B$, $K \not\leq B$, by item 3 of Lemma 1, the subgroup $K \cap B$ is strongly embedded in K, and by item 1 of Lemma 1, all involutions in K are conjugate. The obtained contradiction proves the lemma. □

Lemma 10. *Every set M_k of the form (5) lies in some finite subgroup V of G isomorphic to $U_3(R)$, where R is a finite subfield of Q and $Q_k \subseteq R$.*

Доказательство. By the assumption of the theorem, U contains an element u of order 4, finite in G. Due to item 4 of Proposition 3, up to conjugacy in B we can assume that $u \in U_k$ for every k, and due to transitivity of H_0 on $Z^{\#}$, we assume that $u^2 = u_0$. We exclude trivial cases of small orders and suppose that $|U_2| > 2^4$. Let $H_{0k} = \langle h_k\rangle$ and $x = u^{h_k}$. Under the assumptions made for the group U_k, we have $|x^{U_k}| > 4$, and hence there exists $z \in Z_k^{\#}$, $u_0 \neq z \neq u_0^{h_k}$ such that $zx \in x^{U_k}$. Under conditions of the theorem, the subgroups $\langle x, v\rangle$ and $\langle xz, v\rangle$ are finite; by Lemma 9, at least one of them, which we designate as V, is simple and isomorphic to the group $U_3(R)$, where $R \subset Q$. It is clear that $B_m = B \cap V$ is a Borel subgroup of the group M, $B_m = U_m \leftthreetimes H_m$, $U_m = U \cap V$, $H_m = H \cap V$, and $H_m = H_{1m} \times H_{0m}$, where $H_{0m} = H_m \cap H_0$, $H_{1m} = H_m \cap H_1$. Moreover, according to Lemmas 2 – 4, $v \in N_V(H_m)$ and $V = B_m \cup B_m v U_m$.

The group U_m contains an involution $u_0^{h_k} = x^2 = (xz)^2$ and the only involution u_0 in Z (Remark 1) that satisfies Suzuki's structural equation (6). According to item 3 of Proposition 3, the involutions u_0 Pё $u_0^{h_k}$ are conjugate in $Z_m H_{0m}$ and ZH_0 by a single element h_k of H_0, therefore, $h_k \in H_{0m}$, $H_{0k} \leq H_{0m}$, and $Z_k \leq Z_m$ ($Z_k^{\#} = u_0^{H_{0k}}$). Given $q_k = |Q_k| = |H_{0m}| + 1$, the set of solutions of the equation $x^{q_k} = x$ forms a subfield isomorphic to Q_k in the field R. Hence, H_{1m} contains the unique subgroup of order $\frac{q_k+1}{(3,q_k+1)}$ in H_1 that coincides with the subgroup H_{1k}. Therefore, $H_{1k} \leq H_m$, $H_k \leq H_m$, and since H_k is irreducible on U_k/Z_k, it follows that $U_k = \langle x^{H_k}\rangle Z_k \leq U_m$. So, $B_k \leq B_m$, $M_k = B_k\langle v\rangle U_k \subseteq V$, and the lemma is proved. □

Lemma 11. *The group G is locally finite and isomorphic to the group $U_3(Q)$.*

Доказательство. Finite sets M_k, defined in (5), form a chain, and by Lemma 6 their union $\cup_{k=2}^{\infty} M_k$ coincides with G:

$$M_2 \subset M_3 \subset \dots, \quad G = \bigcup_{k=2}^{\infty} M_k. \tag{8}$$

From Lemma 10, we have $M_k \leq V_k < G$, $V_k \simeq U_3(R_k)$, where R_k is a finite field and $Q_k \subseteq R_k \subset Q$. For some of the subgroups V_k, the equalities $V_k = M_{k'}$ $(k' \geq k)$ can hold, and if there are infinitely many of such coincidings for the elements $M_{k'}$ of the chain (8) (for $k' = k_1, k_2, ...$), then the corresponding subgroups $V_{k_i} = M_{k_i} \simeq U_3(R_{k_i})$ form a chain $V_{k_1} \leq V_{k_2} \leq ...$ with a union $\cup_{i=1}^{\infty} = G$ satisfying the conditions of Proposition 1. The chain of finite fields $R_{k_1} \subseteq R_{k_2} \subseteq ...$ is a subchain of the chain $Q_2 \subset Q_3 \subset ...$, and $\cup_{i=1}^{\infty} R_{k_i} = Q$. By Proposition 1, G is locally finite and isomorphic to $U_3(Q)$, and the lemma is proved.

If there is a finite number of coincidings $V_k = M_{k'}$ $(k' \geq k)$ for the chain (8), then for uniformity we exclude such sets $M_{k'}$ from (8) and then $V_k \neq M_{k'}$ for all k, k'.

We indicate the chain $G_2 < G_3 < ...$ of the form (1) of finite subgroups $G_m \simeq U_3(P_m)$ with a corresponding chain $P_2 \subset P_3 \subset ...$ of finite subfields of the field Q. For $m = 2$, we set $G_2 = V_2 \simeq U_3(R_2)$ and $P_2 = R_2$. For $m = 3$ we find a set M_{k_2} in the chain (8) containing the subgroup V_2, and set $G_3 = V_{k_2}$, $P_3 = R_{k_2}$ (since $V_{k_2} \simeq U_3(R_{k_2})$ by the definition of the groups V_k). Similarly, for $m = n+1$ we find a set $M_{k_{n+1}}$ containing the subgroup G_n, and set $G_{n+1} = V_{k_{n+1}}$, $P_{n+1} = R_{k_{n+1}}$, thus providing the isomorphism $G_{n+1} \simeq U_3(P_{n+1})$ and the required embeddings $G_n < G_{n+1}$, $P_n \subset P_{n+1}$.

For the chain $G_2 < G_3 < ...$ of finite subgroups $G_n \simeq U_3(P_n)$ we have $\cup_{n=1}^{\infty} G_n = G$, $P_2 \subset P_3 \subset ...$, $\cup_{n=1}^{\infty} P_n = Q$, and according to Proposition 1, G is locally finite and isomorphic to $U_3(Q)$. The lemma and hence the theorem are completely proved. □

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