S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 17, стр. 1217–1226 (2020) УДК 517.544
DOI 10.33048/semi.2020.17.090 MSC 47A68

TRUNCATED WIENER-HOPF EQUATION AND MATRIX FUNCTION FACTORIZATION

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Abstract. We will study relationship between a convolution equation of second kind on a finite interval and the Riemann —Hilbert boundary value problems. In addition, as a consequence of the results obtained in the work, Theorem 2 of the following article will be supplemented [3].

Keywords: Riemann boundary value problems, factorization of matrix functions, partial indices, stability, unique, convolution equation, truncated Wiener —Hopf equation.

1. Introduction

This paper studies relationship between a convolution equation of second kind on a finite interval (which is also called a truncated Wiener–Hopf equation) and Riemann boundary value problems (also referred to as Riemann–Hilbert boundary value problems).

The following convolution equation of second kind on a finite interval $(0,\tau)$ is considered:

$$u(t) - \int_0^\tau k(t-s)u(s)\,ds = f(t), \quad t \in (0,\tau), \tag{0.1}$$

where

$$k \in L_1(-\tau,\tau), \quad f \in L_1(0,\tau),\ \tau > 0. \tag{0.2}$$

It is easy to see that the values of the function $k(t)$ outside of the interval $(-\tau,\tau)$ have no affect on the solutions of equations (0.1). For convenience, we assume that $k(t)$ is a defined function as $t \in (-\tau,\tau)$ and arbitrary enough as $t \notin (-\tau,\tau)$:

$$k \in L_1\big(e^{a|x|};R\big)$$

Voronin, A.F., Truncated Wiener-Hopf equation and matrix function factorization.

Received September, 17, 2019, published September, 1, 2020.

is a space with a norm

$$||k|| = \int\limits_{-\infty}^{\infty} |k(t)|e^{a|t|}\, dt,\ a > 0.$$

It is well-known that there are theories developed for convolution equation of second kind on a semi-infinite interval (Wiener–Hopf equations) and the scalar Riemann boundary value problem (for example, see [1–2]). The interrelation between Wiener–Hopf equation and Riemann boundary value problem was found in the middle of the previous century (see historical information in [2]). For truncated Wiener–Hopf equations (for the problem (0.1)–(0.2)), a general theory has not been developed by the present time. Only more recently a generalization of the Wiener–Hopf method for the problem (0.1)–(0.2) was obtained. In papers [3],[4, Lemma 1.1],[5, Theorem 1], the relationship (and equivalence conditions) between convolution equations on a finite interval with an arbitrary kernel from $L_1(0,\tau)$ and Riemann boundary problem with a matrix coefficient admitting a standard factorization in the Wiener algebra has been found. The authors of paper [3] have obtained for the first time generalization of the Wiener–Hopf method for problem (0.1)–(0.2) under condition of existence of a unique solution of the latter (and also for the case of truncated Wiener–Hopf equation system). In the second half of the previous century, the works of M.P. Ganin, B.V. Pal'sev, Y.I. Novokshenov, I.M. Spitkovsky, Y.I. Karlovich and others gave first results on generalization of the Wiener–Hopf method for the case of truncated Wiener–Hopf equation for special kernels and (or) matrix functions which do not lie in the Wiener algebra and hence do not admit standard factorization in the Wiener algebra (for history of the question, see, for example, [6]).

In this paper, the connection is found between problem (0.1)–(0.2) and the Riemann boundary problem in the Wiener algebra whose matrix coefficient (we denote it by $G_\beta(x)$) parametrically depends on two functions $\beta^{\pm}(x)$ and admits standard factorization. Moreover, $G_\beta = G$ as $\beta^{\pm} = 0$, where $G(x)$ is a matrix coefficient of the Rienmann problem, considered in papers [4],[5]. Given

$$\beta^{\pm}(x) = 1 + \mathcal{F}\{k(t)\theta(\pm t)\}(x),$$

where θ is the Heaviside function, and $\mathcal{F}$ is the Fourier transform, the matrix $G_\beta(x)$ will coincide with the matrix $A(x)$, whose factorization is studied in [3]. Taking into account the fact that the results in [4],[5] are obtained due to the considerations distinct from such in [3], and the matrices G and A are (fundamentally) different, relationship between the matrices G and A, established in our work, constitutes an important complement to the results of papers [3] and [4]–[5]. Apart from that, in the present work we have obtained more general results for problem (0.1)–(0.2) compared to the ones in [3]—[5], and have conducted comparative analysis of papers [3] and [4]–[5].

Before proceeding directly to the Riemann boundary problems and relationship between those and the truncated Wiener–Hopf equation, we introduce the following designations. For $1 \le n, m \le 2$, assume that $L_{n\times m}$ is a space $n \times m$ of matrix functions with elements from $L_1(R)$, and $\mathcal{F}f$ is a Fourier image of the matrix function $f \in L_{n\times m}$:

$$\mathcal{F}f(x) = \int\limits_{-\infty}^{\infty} f(t)e^{ixt}\, dt,\ x \in \mathbb{R},$$

where $\mathbb{R}$ is the extended real line (R is the real line); $W^{n\times n}$ is the Wiener algebra of continuous matrix functions of the form $C + \mathcal{F}f$, where C is a constant matrix of order n and $f \in L_{n\times n}$; $W_+^{n\times n}$ ($W_-^{n\times n}$) is a subalgebra in $W^{n\times n}$, consisting of matrix functions of the form $C + \mathcal{F}f$, such that $f(t) = 0$ as $t < 0$ (as $t > 0$); given C=0, we assign a subscript 0 to the corresponding algebras and subalgebras ($W_0^{n\times n}$, $W_{0\pm}^{n\times n}$). As $n = 1$, we drop the superscript $n \times n$ of W. If B is some algebra, we denote by $\mathcal{G}B$ the group of invertible elements in B.

Consider the following Riemann boundary value problem for vector functions $\Phi^{\pm} \in W^{2\times 1}_{0\pm}$ on the line $\mathbb{R}$:

$$\Phi^{+}(x) = G(x)\,\Phi^{-}(x) + g(x),\ x \in \mathbb{R}, \tag{0.3}$$

where

$$G \in \mathcal{G}W^{2\times 2},\ g \in W^{2\times 1}_{0}. \tag{0.4}$$

Here the classes of functions $W^{2\times 1}_{0}$, $W^{2\times 1}_{0\pm}$ are defined similarly to the classes $W^{2\times 2}_{0}$, $W^{2\times 2}_{0\pm}$, respectively. For example, the condition $\Psi^{+} \in W^{2\times 1}_{0+}$ means that

$$\Psi^{+} = (\Psi^{+}_{1}, \Psi^{+}_{2})^{T},\ \Psi^{+}_{j} \in W_{0+},\ j = 1, 2,$$

where A^{T} denotes the transpose of a matrix A.

Note also the well-known results from the theory of Riemann boundary problem and factorization of matrix functions (see, for example, [7, Chapter 6],[8, §7], [9, Chapter I]).

We say that the matrix $G \in \mathcal{G}W^{2\times 2}$ admits standard (left) factorization if it is represented in the form of the following matrix product:

$$G(x) = G_{+}(x)D(x)G_{-}(x),\ x \in \mathbb{R},$$

where $G_{\pm} \in \mathcal{G}W^{2\times 2}_{\pm}$ ($G_{\pm}$ are called the factors), $D(x)$ is a diagonal matrix function,

$$D(x) = \Big\{\Big(\frac{x-i}{x+i}\Big)^{\kappa_1}, \Big(\frac{x-i}{x+i}\Big)^{\kappa_2}\Big\},$$

$\kappa_1 \geq \kappa_2$ are partial indices of the matrix G (integers), and

$$\kappa := \text{Ind det } G(t) \equiv \frac{1}{2\pi}\Delta_{\mathbb{R}} \arg\det G(x) = \sum_{j=1}^{2} \kappa_j$$

is the total index of the matrix G.

The Riemann boundary value problem is well-defined depending on the partial indices of its matrix coefficient. In particular, we have the following theorem:

Theorem 1. *Let the total index of the matrix $G(t)$ equal zero. For the numbers p and l (where l is a number of linearly independent solutions, and p is a number of conditions for solvability of the Riemann problem (0.3)–(0.4)) to be stable with respect to elements of the matrix $G(x)$ it is necessary and sufficient that the partial indices of the matrix $G(x)$ equal zero. Moreover, if the partial indices of the matrix $G(x)$ equal zero, then for the homogeneous Riemann problem (0.3)–(0.4) only a trivial solution exists, and the non-homogeneous problem is correctly solvable (a solution exists, is unique and stable with respect to the coefficients G and g of the problem).*

§**1. Preliminary statements. The basis for research.** We define on the algebra W_0 projectors P^{+}_{0} and P^{-}_{0} complementary to each other by formulas

$$P^{\pm}_{0}\ :\ W_0\ \to\ W_{0\pm},\ P^{\pm}_{0}\mathcal{F}g(x) = \int_{-\infty}^{\infty} e^{ixt} g(t)\theta(\pm t)\,dt,\ x \in \mathbb{R}.$$

Note the following properties of the linear operators $P^{\pm}_{0}$:

$$P^{+}_{0} + P^{-}_{0} = I,\ \mathcal{F}^{-1}\{P^{\pm}_{0}\mathcal{F}g(x)\}(t) = g(t)\theta(\pm t),\ t \in R,$$

where I is the identity operator, and $\mathcal{F}^{-1}$ is the reverse Fourier transform.

Set

$$k_{\pm}(t) := \theta(\pm t)k(t),\ \mathcal{F}k_{\pm}(x) := \int_{-\infty}^{\infty} e^{ixt} k_{\pm}(t)\,dt,\ f(t) := 0,\ t\overline{\in}(0,\tau). \tag{1.1}$$

Consider the problem of finding the function $u \in L_1(0,\tau),\ u(t) = 0,\ t \notin (0,\tau)$ from the functional equation

$$\mathcal{F}u(x) - P_0^+\{\mathcal{F}k_-(x)\mathcal{F}u(x)\} - e^{ix\tau}P_0^-\{e^{-ix\tau}\mathcal{F}k_+(x)\mathcal{F}u(x)\} = \\ = \mathcal{F}f(x),\ x \in \mathbb{R}. \tag{1.2}$$

The following proposition holds [4, Statement 1.2] (see also [5, Statement 2]):

Proposition 1. *Equation (0.1) with condition (0.2) is equivalent to equation (1.2) with condition* $u \in L_1(0,\tau),\ u(t) = 0,\ t \notin (0,\tau)$.

Set

$$G(x) := -\frac{1}{\Lambda^-(x)} \begin{pmatrix} 1 & -e^{ix\tau}\mathcal{F}k_-(x) \\ e^{-ix\tau}\mathcal{F}k_+(x) & 1 - \mathcal{F}k_-(x) - \mathcal{F}k_+(x) \end{pmatrix}, \tag{1.3}$$

where

$$\Lambda^\pm(x) = 1 - \mathcal{F}k_\pm(x);$$

$$g(x) := \frac{\mathcal{F}f(x)}{\Lambda^-(x)}\big(\mathcal{F}k_-(x),\ e^{-ix\tau}\mathcal{F}k_+(x)\big)^T. \tag{1.4}$$

We may assume in advance [4, Remark 1.1] that

$$\Lambda^\pm(x) \neq 0,\ x \in R. \tag{1.5}$$

Indeed, if the inequalities (1.5) do not hold, then instead of equation (0.1) we consider a similar equation with the kernel $k_1(t) = e^{-th}\,k(t)$ and the right side $f_1(t) = e^{-th}\,f(t)$, where for the parameter h the inequality $-a < h < a$ is valid. For the newly obtained convolution equation we can always choose a parameter h such that the inequalities in (1.5) will have been satisfied, because only for a finite number of different values of the parameter h these inequalities might not hold since

$$\Lambda_1^\pm(x) = \Lambda^\pm(x + ih),\ x \in R,$$

and since by construction the functions $\Lambda^\pm(z)$ can have only a finite number of zeros on the strip $-a <\mathrm{Im}\, z < a$. Note that the restriction on the behavior of the kernel $k(t)$ on infinity was adopted only to justify (as mentioned above) the fact that inequalities (1.5) hold.

Consider the Riemann boundary problem (0.3) with conditions (1.3)–(1.5). The following theorem holds (see [4, Lemma 1.1], [5, Theorem 1]):

Theorem 2. *The problem (0.1)–(0.2) is equivalent to the Riemann boundary problem (0.3) under conditions (1.3)–(1.5) with the restriction*

$$\hat{u}(x) := \Phi_1^+(x) + e^{ix\tau}\Phi_2^-(x) + \mathcal{F}f(x) \in W_{0+},\ e^{-ix\tau}\hat{u}(x) \in W_{0-}. \tag{1.6}$$

Solutions of the problem (0.1)–(0.2) and the boundary problem (0.3),(1.3)–(1.6) are connected by equations

$$\Phi_1(x) = \mathcal{F}k_-(x)\mathcal{F}u(x),\ \Phi_2(x) = e^{-ix\tau}\mathcal{F}k_+(x)\mathcal{F}u(x), \tag{1.7}$$

$$\mathcal{F}u(x) = \Phi_1^+(x) + e^{ix\tau}\Phi_2^-(x) + \mathcal{F}f(x)\ \big(\hat{u}(x) = \mathcal{F}u(x)\big), \tag{1.8}$$

where

$$\Phi = \Phi^+ + \Phi^-,\ \Phi^\pm(x) = P_0^\pm\Phi(x).$$

Remark 1. It is easy to see that the restriction (1.6) is satisfied if a solution of the problem (0.1)–(0.2) exists (see formulas (1.7)–(1.8)). Moreover, the restriction (1.6) is satisfied in advance if

$$||k_\pm|| < 1. \tag{1.9}$$

Indeed, it follows from condition (1.9) that $|\Lambda^\pm(z)| > 0,\ \pm\,\mathrm{Im}\, z \geq 0$. Then from the first equality of system (0.3) and the second equality of the following system

$$G^{-1}(x)\,\Phi^+(x) = \Phi^-(x) + g^0(x),\ x \in \mathbb{R},$$

where

$$G^{-1}(x) = -\frac{1}{\Lambda^+(x)} \begin{pmatrix} 1 - \mathcal{F}k_-(x) - \mathcal{F}k_+(x) & e^{ix\tau}\mathcal{F}k_-(x) \\ -e^{-ix\tau}\mathcal{F}k_+(x) & 1 \end{pmatrix},$$

$$g^0(x) = \frac{\mathcal{F}f(x)}{\Lambda^+(x)} \big(\mathcal{F}k_-(x),\ e^{-ix\tau}\mathcal{F}k_+(x)\big)^T,$$

we have respectively that

$$e^{-ix\tau}\Phi_1^+(x) \in W_{0-},\ e^{ix\tau}\Phi_2^-(x) \in W_{0+}.$$

The obtained conclusion guarantee that restriction (1.6) is satisfied.

We can see that the condition (1.9) is weaker compared to the condition

$$||k|| < 1.$$

Elementary considerations show that

$$\det G = \frac{\Lambda^+}{\Lambda^-},\ G \in \mathcal{G}W^{2\times 2},\ g \in W_0^{2\times 1}.$$

It follows from the above mentioned inclusion for G that a standard factorization for the matrix G exists.

It is easy to see that Theorems 1–2 imply

Corollary 1. *If the partial indices of the matrix $G(x)$ equal zero, then the homogeneous truncated Wiener–Hopf equality (0.1) has a unique (trivial) solution in $L_1(0,\tau)$ and*

(i) the problem (0.1)–(0.2) is correctly solvable in $L_1(0,\tau)$ (a solution exists, is unique and stable with respect to the coefficients k, f of the equality in the norm L_1),

(ii) the Riemann boundary value problem (0.3) with respect to conditions (1.3)–(1.4) is also correctly solvable in the Wiener algebra,

(iii) solutions of equality (0.1) and the Riemann boundary problem (0.3) are connected by equalities (1.7)–(1.8).

If the restriction (1.9) is satisfied and the homogeneous problem (0.1)–(0.2) has a unique (trivial) solution in $L_1(0,\tau)$, then the partial indices of the matrix $G(x)$ equal zero.

Set

$$A(x) = \begin{pmatrix} e^{-ix\tau}\mathcal{F}k(x) & -1 - \mathcal{F}k(x) \\ 1 - \mathcal{F}k(x) & e^{ix\tau}\mathcal{F}k(x) \end{pmatrix},\ x \in R. \tag{1.10}$$

We have

$$\det A = 1,\ A \in \mathcal{G}W^{2\times 2}.$$

We mention the following theorem from [3] in the case of equation (0.1) to compare it to Corollary 1.

Theorem 3. *The partial indices of the matrix $A(x)$ equal zero if and only if the homogeneous truncated Wiener–Hopf equation (0.1) has a unique (trivial) solution in $L_1(0,\tau)$.*

In [3], it was also established that in the case when a canonical factorization of the matrix $A(t)$ exists, the solution of the problem (0.1)–(0.2) is written in explicit formulas in terms of that factorization. Moreover, it is shown that the above mentioned results of paper [3] remain valid if the kernel $k \in L_1(R)$ constitutes an arbitrary extension of the kernel k from the interval $(-\tau,\tau)$ to the real line.

Note that the results in [3] are obtained only for the case when the partial indices of the matrix $A(x)$ equal zero, in contrast with [4]–[5] (see Theorem 2 in our paper).

Next we obtain the analogue of Theorem 2 for the matrix A, thus complementing paper [3] for the case of non-zero partial indices of the matrix A. We will also find a direct relationship between the matrices G and A. In order to do that, we will solve a problem that is more general: we construct a matrix $G_\beta(x)$ parametrically dependent on two functions $\beta^\pm \in W_{0\pm}$ and possessing the following property:

$$G_\beta = G \text{ when } \beta^\pm = 0;\ G_\beta(x) = \{1, -1\}A(x)I_1 \text{ when } \beta^\pm = \mathcal{F}k_\pm(x),$$

where $\{1, -1\}$ is a diagonal matrix with elements 1 and -1, and I_1 is an anti-diagonal matrix with identity elements. Thus the relationship between the matrices G and A is established. Using the same reasoning as in [4]–[5], we obtain an analogue of Theorem 2 for the matrix $G_\beta(x)$.

§2. Constructing the matrix $G_\beta(x)$. The main results of the work. By the property of the operators $P_0^{\pm}$, we write the functional equation (1.2) in the equivalent form

$$\big(1-\mathcal{F}k(x)\big)\mathcal{F}u(x)+P_0^-\{\mathcal{F}k_-(x)\mathcal{F}u(x)\}+e^{ix\tau}P_0^+\{e^{-ix\tau}\mathcal{F}k_+(x)\mathcal{F}u(x)\}= \\ =\ \mathcal{F}f(x),\ x\in\mathbb{R}. \tag{2.1}$$

Set

$$b_1(x):=\mathcal{F}k_-(x)+\beta^+(x),\ b_2(x):=e^{-ix\tau}\big(\mathcal{F}k_+(x)+\beta^-(x)\big), \tag{2.2}$$

where

$$\beta^{\pm}\in W_{0\pm},\ \mathcal{F}^{-1}\beta^{\pm}\in L_1\big(e^{a|t|};R\big).$$

From (2.1) and taking into account the evident chain of equalities

$$P_0^-\{\beta^+(x)\mathcal{F}u(x)\}=P_0^+\{e^{-ix\tau}\beta^-(x)\mathcal{F}u(x)\}=0,$$

we get

$$\big(1-\mathcal{F}k(x)\big)\mathcal{F}u(x)+P_0^-\{b_1(x)\mathcal{F}u(x)\}+e^{ix\tau}P_0^+\{b_2(x)\mathcal{F}u(x)\}=\mathcal{F}f(x), \tag{2.3}$$

Hence, from Proposition 1 it follows that the following statement is true.

Proposition 2. *Equation (0.1) with condition (0.2) is equivalent to equation (2.3) with the condition $u\in L_1(0,\tau),\ u(t)=0,\ t\notin(0,\tau)$.*

Using the standard method (as in [4, the proof of Lemma 1.1], [5, the proof of Theorem 1]), from Proposition 2 we obtain a Riemann boundary value problem on determining the functions $\Phi^{\pm}\in W_{0\pm}^{2\times1}$ from the boundary condition

$$\Phi^+(x)=G_\beta(x)\,\Phi^-(x)+g_\beta(x),\ x\in\mathbb{R}, \tag{2.4}$$

where

$$G_\beta(x)=-\frac{1}{\Lambda_\beta^-(x)}\begin{pmatrix}1+\beta^+(x)+\beta^-(x) & -e^{ix\tau}\big(\mathcal{F}k_-(x)+\beta^+(x)\big)\\ e^{-ix\tau}\big(\mathcal{F}k_+(x)+\beta^-(x)\big) & 1-\mathcal{F}k_-(x)-\mathcal{F}k_+(x)\end{pmatrix}, \tag{2.5}$$

$$\Lambda_\beta^{\pm}(x)=1-\mathcal{F}k_{\pm}(x)+\beta^{\pm}(x), \tag{2.6}$$

$$g_\beta(x)=\frac{\mathcal{F}f(x)}{\Lambda_\beta^-(x)}\big(\mathcal{F}k_-(x)+\beta^+(x),\ e^{-ix\tau}\big(\mathcal{F}k_+(x)+\beta^-(x)\big)\big)^T. \tag{2.7}$$

For the ease of presentation, we assume that the inequalities similar to (1.5) hold:

$$\Lambda_\beta^{\pm}(x)\neq0,\ x\in\mathbb{R}. \tag{2.8}$$

Here the remark made after condition (1.5) remains true.

The following theorem holds:

Theorem 4. *The problem (0.1)–(0.2) is equivalent to the Riemann boundary value problem (2.4)–(2.8) with the restriction*

$$\hat{u}(x):=\frac{\Phi_1^+(x)+e^{ix\tau}\Phi_2^-(x)+\mathcal{F}f(x)}{1+\beta^+(x)+\beta^-(x)}\in W_{0+},\ e^{-ix\tau}\hat{u}(x)\in W_{0-}. \tag{2.9}$$

Solutions of the problem (0.1)–(0.2) and the Riemann boundary value problem (2.4)–(2.9) are interrelated by equalities

$$\Phi_1(x)=b_1(x)\mathcal{F}u(x),\ \Phi_2(x)=b_2(x)\mathcal{F}u(x), \tag{2.10}$$

$$\mathcal{F}u(x)=\frac{\Phi_1^+(x)+e^{ix\tau}\Phi_2^-(x)+\mathcal{F}f(x)}{1+\beta^+(x)+\beta^-(x)}\ \big(\hat{u}(x)=\mathcal{F}u(x)\big), \tag{2.11}$$

where

$$\Phi=\Phi^++\Phi^-,\ \Phi^{\pm}(x)=P_0^{\pm}\Phi(x)\in W_{0\pm}^{2\times1}.$$

The proof of Theorem 4, in fact, repeats those of Lemma 1.2 in [4] and Theorem 1 in [5]. Hence, we mention only the scheme of the proof of Theorem 4 here without going into details.

First, suppose that a solution to the problem (0.1)–(0.2) exists. Then by Proposition 2 the equality (2.3) is true. From the equality (2.3) we directly get

$$\big(1-\mathcal{F}k(x)\big)\mathcal{F}u(x)+\Phi_1^-(x)+e^{ix\tau}\Phi_2^+(x)=\mathcal{F}f(x),\ x\in\mathbb{R}, \tag{2.12}$$

having defined in advance the functions Φ_1 and Φ_2 by formulas in (2.10). Multiplying the left-hand and right-hand sides of equality (2.12) by the function $b_j(x)$, $j=1,2$, we correspondingly get the following equalities:

$$\big(1-\mathcal{F}k(x)\big)\big(\Phi_1^+(x)+\Phi_1^-(x)\big)+b_1(x)\Phi_1^-(x)+b_1(x)e^{ix\tau}\Phi_2^+(x)=b_1(x)\mathcal{F}f(x),\ x\in\mathbb{R}, \tag{2.13}$$

$$\begin{aligned}&\big(1-\mathcal{F}k(x)\big)\big(\Phi_2^+(x)+\Phi_2^-(x)\big)+b_2(x)\Phi_1^-(x)+\\&+b_2(x)e^{ix\tau}\Phi_2^+(x)=b_2(x)\mathcal{F}f(x),\ x\in\mathbb{R}.\end{aligned} \tag{2.14}$$

We write the system of equalities (2.13)–(2.14) in vector form:

$$B_1(x)\Phi^+(x)+B_2(x)\Phi^-(x)=\mathcal{F}f(x)\big(b_1(x),\,b_2(x)\big)^T, \tag{2.15}$$

where

$$\begin{aligned}B_1(x)&=\begin{pmatrix}1-\mathcal{F}k(x) & e^{ix\tau}b_1(x)\\ 0 & 1-\mathcal{F}k(x)+e^{ix\tau}b_2(x)\end{pmatrix},\\ B_2(x)&=\begin{pmatrix}1-\mathcal{F}k(x)+b_1(x) & 0\\ b_2(x) & 1-\mathcal{F}k(x)\end{pmatrix}.\end{aligned} \tag{2.16}$$

For $x\in R$ such that $1-\mathcal{F}k(x)\neq 0$, we multiply the left-hand and right-hand sides of the equality (2.15) on the left by the matrix

$$B_1^{-1}(x)=\frac{1}{d_b(x)}\begin{pmatrix}1-\mathcal{F}k(x)+e^{ix\tau}b_2(x) & -e^{ix\tau}b_1(x)\\ 0 & 1-\mathcal{F}k(x)\end{pmatrix},$$

where $d_b(x)=det\,B_1(x)=\big(1-\mathcal{F}k(x)\big)\Lambda_\beta^-(x)$. After trivial transformations and reduction in the left-hand and right-hand sides of the newly obtained equality by the factor $1-\mathcal{F}k(x)$, we get the required boundary condition (2.4). The latter is now true for every $x\in\mathbb{R}$ due to the fact that the analytical function $1-\mathcal{F}k(z)$ on the strip $-a<\mathrm{Im}\,z<a$ can have no more than a finite number of zeros on this strip (by analogy with the evident statement 1.1 in [4]).

From (2.3) and the following property of the projectors $P_0^++P_0^-=I$, it follows that

$$\big(1+\beta^+(x)+\beta^-(x)\big)\mathcal{F}u(x)=\Phi_1^+(x)+e^{ix\tau}\Phi_2^-(x)+\mathcal{F}f(x),\ x\in\mathbb{R}.$$

In the above mentioned equality, putting $\hat{u}(x):=\mathcal{F}u(x)$, we obtain that restriction (2.9) is satisfied.

Equalities in (2.10)-(2.11) hold by construction.

We now prove the second half of the statement of Theorem 4. Let the Riemann boundary value problem (2.4)–(2.9) have a solution. From the first equality of system (2.4), we get

$$\big(\Phi_1^+(x)+\Phi_1^-(x)\big)\big(1+\beta^+(x)+\beta^-(x)\big)=\big(\Phi_1^+(x)+e^{ix\tau}\Phi_2^-(x)+\mathcal{F}f(x)\big)b_1(x).$$

Then from restriction (2.9), we have that

$$\Phi_1(x)=\hat{u}(x)b_1(x). \tag{2.17}$$

We will show that

$$\Phi_2(x)=\hat{u}(x)b_2(x). \tag{2.18}$$

In order to do so, first we multiply the boundary condition (2.4) on the left by the vector-row $\big(b_2(x),\,-b_1(x)\big)$, taking into account the equality

$$\big(b_2(x),\,-b_1(x)\big)G_\beta(x)=-\big(b_2(x),\,-b_1(x)\big),$$

which can be directly obtained, and we have that

$$b_2\Phi_1 = b_1\Phi_2. \tag{2.19}$$

From (2.17) and (2.19), we get

$$b_1(x)\big(\Phi_2(x) - \hat{u}(x)b_2(x)\big) = 0,\ x \in R. \tag{2.20}$$

The function $b_1(z)$ is analytical on the strip $-a <\mathrm{Im}\, z < a$ by construction, hence, it can have only a finite number of zeros on every interval of R. Then equality (2.18) follows from equality (2.20) due to continuity of the functions $\Phi_2, \hat{u}, b_2$ in (2.20) and the following equality

$$\Phi_2(\pm\infty) - \hat{u}(\pm\infty)b_2(\pm\infty) = 0.$$

It only remains to show that $\hat{u} = \mathcal{F}u$. From the second equality of the system (2.4), we have that

$$\Phi_2(x)\big(1 - \mathcal{F}k(x)\big) + b_2(x)\Phi_1^-(x) + b_2(x)e^{ix\tau}\Phi_2^+(x) = b_2(x)\mathcal{F}f(x). \tag{2.21}$$

Substituting in (2.21) the expression for Φ_2 from (2.18) and reducing in the newly obtained equality its left-hand and right-hand sides by the factor b_2, we get

$$\hat{u}(x)\big(1 - \mathcal{F}k(x)\big) + \Phi_1^-(x) + e^{ix\tau}\Phi_2^+(x) = \mathcal{F}f(x). \tag{2.22}$$

Reduction of the left-hand and right-hand sides of equality (2.21) by the factor b_2 is a well-defined operation due to the reasoning similar to that for the reduction by the factor b_1 in equality (2.20). Set $\mathcal{F}u(x) := \hat{u}(x)$. Then from (2.22), we get (2.3). Then by Proposition 2 the problem (0.1)-(0.2) has a solution. Theorem 4 is proved.

§3. Some corollaries of Theorem 4. Direct considerations show that

$$det\, G_\beta = \frac{\Lambda_\beta^+}{\Lambda_\beta^-},\ G_\beta \in \mathcal{G}W^{2\times 2},\ g_\beta \in W_0^{2\times 1}.$$

From the above mentioned inclusion for G_β it follows that there exists a standard factorization for the matrix G_β.

It is easy to see that Theorem 4 implies

Theorem 5. *Let the inequalities in (2.8) hold. Then if the partial indices of the matrix $G_\beta(x)$ in (2.5) equal zero, then the truncated Wiener–Hopf equation (0.1) under condition (0.2) has a unique solution in $L_1(0,\tau)$.*

Converse statement: in the homogeneous truncated Wiener–Hopf equation (0.1) with condition (0.2) has a unique (trivial) solution in $L_1(0,\tau)$ and every solution of the Riemann boundary value problem (2.4)-(2.7) complies with the restriction (2.9), then the partial indices of the matrix $G_\beta(x)$ in (2.5) equal zero. Moreover, the solutions of the problem (0.1)–(0.2) and the Riemann boundary value problem (2.4)–(2.8) are connected by equalities (2.10)–(2.11).

Consider the case when

$$\beta^\pm = \mathcal{F}k_\pm. \tag{2.23}$$

By definition of the functions $b_1, b_2, \Lambda_\beta^\pm$, we have that

$$b_1 = \mathcal{F}k,\ b_2 = e^{-ix\tau}\mathcal{F}k,\ \Lambda_\beta^\pm = 1,\ 1 + \beta^+ + \beta^- = 1 + \mathcal{F}k.$$

Then in this case the matrix $G_\beta(x)$ will be interrelated with the matrix $A(x)$ in (1.10) by a simple equation

$$G_\beta(x) = \{1,\, -1\}A(x)I_1. \tag{2.24}$$

Now we expand Theorem 3. We consider a Riemann boundary value problem with a matrix coefficient $A(x)$ and get the relationship between this boundary value problem and the problem (0.1)–(0.2). In order to do so, we multiply the left-hand and right-hand sides of the boundary condition (2.4) by a diagonal matrix $\{1, -1\}$ and set

$$\Psi^+(x) := \{1, -1\}\Phi^+(x),\ \Psi^-(x) := I_1\Phi^-(x),$$

$$q(x) := \{1, -1\} g_\beta(x) = \mathcal{F}k(x)\mathcal{F}f(x)(1,\ -e^{-ix\tau})^T.$$

Taking (2.24) into account, we obtain a Riemann boundary value problem on defining functions

$$\Psi^{\pm} \in W_{0\pm}^{2\times 1}$$

by the boundary condition:

$$\Psi^+(x) = A(x)\,\Psi^-(x) + q(x),\ \ x \in \mathbb{R}. \tag{2.25}$$

It is easy to see that Theorem 4 and [3, Theorem 2] imply

Theorem 6. *Let the condition (2.23) be satisfied. Then the problem (0.1)–(0.2) is equivalent to the Riemann boundary value problem (2.25) with a restriction*

$$\hat{u}(x) := \frac{\Psi_1^+(x) + e^{ix\tau}\Psi_1^-(x) + \mathcal{F}f(x)}{1 + \mathcal{F}k(x)} \in W_{0+},\ \ e^{-ix\tau}\hat{u}(x) \in W_{0-}. \tag{2.26}$$

Solutions of the problem (0.1)–(0.2) and the Riemann boundary value problem (2.25)–(2.26) are connected by the equalities

$$\Psi_1^+(x) = P_0^+\{\mathcal{F}k(x)\mathcal{F}u(x)\},\ \ \Psi_2^+(x) = -P_0^+\{e^{-ix\tau}\mathcal{F}k(x)\mathcal{F}u(x)\},$$

$$\Psi_1^-(x) = P_0^-\{e^{-ix\tau}\mathcal{F}k(x)\mathcal{F}u(x)\},\ \ \Psi_2^-(x) = P_0^-\{\mathcal{F}k(x)\mathcal{F}u(x)\},$$

where

$$\mathcal{F}u = \hat{u}.$$

The Riemann boundary value problem (2.25)–(2.26) has a unique solution if and only if the homogeneous problem (0.1)–(0.2) has a unique (trivial) solution in $L_1(0,\tau)$.

Note that the task of calculating partial indices of matrix functions in general form from the algebra $W^{2\times 2}$, specifically, matrices G, A and G_β, is an open problem (see, for example, [10, Introduction],[11]). However, the approach developed in papers [4]–[5] provided a possibility to study (thoroughly enough) the problem (0.1)–(0.2) with a periodic kernel [12], in fact, in this case the partial indices of the matrix $G(x)$ from (1.3) were found. Moreover, Theorem 2 allowed us to relate the ancient problem of complex analysis – the problem of $\mathbb{R}$-linear conjugation (known also as Markuschewisch problem or generalised Riemann boundary value problem) – to the problem (0.1)–(0.2) and to find new effective conditions for correct solvability of these two problems [13].

We would like on this occasion to make a remark to the work [10], in which a method of defining partial indices of matrix functions with a partial symmetry is proposed. Theorem 3 in [10] is not true (the proof of the theorem contains a mistake). In other words, the proposed method in [10] does not apply to matrices with the following symmetry:

$$G(x) = G^{-1}(-x),\ \ x \in \mathbb{R}.$$

The corollaries of Theorem 3 are not valid as well: Proposition 1, Corollary 2, Lemma 1, and Theorem 4. We are thankful to I.M. Spitkovsky for his message (sent to us in due time) regarding the mistake in [10, Theorem 3].

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