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A CASE STUDY IN UNIVERSAL GEOMETRY: EXTENDING APOLLONIAN CIRCLES TO RELATIVISTIC GEOMETRY AND FINITE FIELDS

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Abstract. We investigate and extend classical results for the Apollonian circles of a triangle to include relativistic geometries and to hold over general fields, in particular also to finite fields, using the framework of rational trigonometry. Our new results include curvature relations between the three Apollonian circles, criteria for the existence of Isodynamic points, more general formulations of the Lemoine and Brocard axes involving radical axes. Over finite fields the number theoretical aspects of the subject become important.

Keywords: Apollonian circles, chromogeometry, rational trigonometry, curvature, finite fields.

1. Introduction

In the last 15 years, a new direction allows us to rethink the very foundational premises upon which geometry is based. *Rational trigonometry*, initiated in [8] and [9], holds that we aim to elucidate a very general metrical geometry over an arbitrary field, so that we focus our attention primarily to concepts which are entirely algebraic, and so hold without further assumptions on the field. This allows a much wider applicability, including notably the possibility of metrical geometry over finite fields. It also elevates relativistic geometry, which has taken a very back seat compared to Euclidean geometry in classical work, to an equal footing, which is commensurate with its undeniable role in describing the large-scale physical

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structure of the world following Einstein and Minkowski. It also strengthens the role that number theory plays in the subject.

In this paper we have a new look at a classical phenomenon going back to Apollonius; the characterization of a circle as the locus of a point whose distances to two fixed points maintains a constant ratio, bring it forward into this modern view, and then have a new and more general study of the three Apollonian circles associated to a triangle. This topic has been well studied and reported on over the centuries, see for example the summary accounts of Casey ([3, p.146]), Altshiller ([1], [2, p.260-262]) and Yiu ([12, p.187]), and we will see that much of the classical theory extends to general fields and arbitrary symmetric bilinear forms. These include the centers of all three Apollonian circles lie on the Lemoine axis, all three circles are orthogonal to the circumcircle at the triangle vertices, and the common radical axis coincides with the Brocard axis. However, some new distinctions also appear when we investigate the intersections of circles; most notably, the existence of isodynamic points. The subtle number theoretic issues that underlie many such situations in geometry now come to the fore. We will also exhibit an elegant relation among the quadratic curvatures of the Apollonian circles of a triangle, for which the classical version only appeared in a special case as an exercise in [2, p.266].

Throughout we will have a chance to see rational trigonometry in action; how relativistic analogs of this theory are just as rich as the Euclidean story; and how especially the finite field situation yields new and subtle additional phenomenon and questions. Along the way we want to firmly establish some novel tools for effective algebraic investigation in affine metrical geometry: these include new interactions between the affine plane and the associated vector space and the extension of barycentric (or areal) coordinates to facilitate computations with both points and vectors consistently, and the key role of the planar cross product borrowed from Algebraic Calculus ([11]) to clarify the distinction between affine and metric forms for the equations of lines.

The only preliminaries for this paper are an undergraduate understanding of linear algebra and symmetric bilinear forms.

2. Circles and Triangles in Universal Geometry

We consider the general framework of universal planar geometry, first in a purely affine setting. Some novel notation for relations between points and vectors is introduced, including the cross product of planar vectors, and corresponding affine notions of lines, as well as affine (or barycentric) coordinates of points expressed as 1-combinations of points, and correspondingly 0-combinations of points to express vectors.

Then we add a metrical setting coming from a symmetric bilinear form on the associated vector space, with the aim of introducing circles both in standard form and from the Apollonian perspective. We will also examine pencils of circles, the radical axis of two circles, and the connections with circles in Apollonian form. We also introduce some general facts about triangles in this metrical set up, including somewhat novel circumcenter formulas.

2.1. **Affine space.** We will be working in the **affine plane** $\mathbb{A}^2$ over a field $\mathbb{F}$, consisting of **points** written as $\mathbf{A} = [a, b]$. The associated **vector space** $\mathbb{V}^2$ will be introduced as displacements, or separations between pairs of points, and consists of row vectors written as $\mathbf{v} = (r, s)$. This allows somewhat novel notation; if $\mathbf{A} = [a, b]$ and $\mathbf{B} = [c, d]$ are points then the vector $\mathbf{v} = \overrightarrow{\mathbf{AB}}$ will be denoted

$$\mathbf{v} = \overrightarrow{\mathbf{AB}} \equiv \mathbf{B} - \mathbf{A} \equiv (c - a, d - b)\,.$$

Even though there is not in general an addition operation on points themselves, we will agree that we can add a point and a vector in that order, so that following from the above

$$\mathbf{B} = \mathbf{A} + \mathbf{v}.$$

Note that $\mathbf{v} + \mathbf{A}$ has no meaning.

We may further define, for $\lambda \in \mathbb{F}$, the **affine combination**

$$\mathbf{C} = \mathbf{A} + \lambda\overrightarrow{\mathbf{AB}} = \mathbf{A} + \lambda\,(\mathbf{B} - \mathbf{A}) \equiv (1 - \lambda)\mathbf{A} + \lambda\mathbf{B}$$

which describes a general point on the line $\mathbf{AB}$. More generally if $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ are points, then we may interpret a **0-combination** of points

$$\mathbf{v} = \sum_{k=1}^{n} a_k \mathbf{A}_k \qquad \text{where} \qquad \sum_{k=1}^{n} a_k = 0$$

as a vector, while a **1-combination** of points

$$\mathbf{B} = \sum_{k=1}^{n} a_k \mathbf{A}_k \qquad \text{where} \qquad \sum_{k=1}^{n} a_k = 1$$

represents a point.

During some calculations and proofs, it may be convenient to temporarily convert from points to vectors uniformly. To enable this, we say that a point $\mathbf{A} = [a, b]$ has **position vector** $\mathbf{a} = \overrightarrow{O\mathbf{A}} = (a, b)$, where $O = [0, 0]$ is the origin. So if $\mathbf{A}$ and $\mathbf{B}$ are two points with position vectors $\mathbf{a}$ and $\mathbf{b}$, then $\overrightarrow{\mathbf{AB}} = \mathbf{B} - \mathbf{A} = \mathbf{b} - \mathbf{a}$. Nevertheless, we will aim for statements of theorems that do not involve position vectors, as they implicitly assume a distinguished point in the affine space, which is against the spirit of affine geometry.

2.2. **Cross product.** It turns out to be useful to have a two-dimensional version of the familiar three-dimensional operation. The utility of this concept has been emphasized in the *Algebraic Calculus One* course [11], where it is used to underpin the notion of affine area in the plane.

If $\mathbf{v} \equiv (a, b)$ and $\mathbf{w} \equiv (c, d)$ are vectors, then their **cross product** is the number

$$\mathbf{v} \times \mathbf{w} \equiv ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The cross product is a skew-symmetric bilinear form, that is for vectors $\mathbf{v}$ and $\mathbf{w}$, we have $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$ and so also $\mathbf{v} \times \mathbf{v} = 0$. The cross product also encodes parallelism, since $\mathbf{v}$ and $\mathbf{w}$ are parallel, written as $\mathbf{v} \parallel \mathbf{w}$, precisely when $\mathbf{v} \times \mathbf{w} = 0$. Using matrix algebra, we can write

$$\mathbf{v} \times \mathbf{w} = \mathbf{v} J \mathbf{w}^T$$

where

$$J \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a basis element for the one-dimensional space of 2×2 skew-symmetric matrices. Note that the cross product here is a purely affine concept, and does not rely on any prior metrical structure to the affine plane.

2.3. **Lines in affine geometry.** For us a line l will be an algebraic equation of the form $\ell : r + sx + ty = 0$ where s and t are not both zero. We agree that the line is unchanged if all coefficients are multiplied by the same non-zero scalar. The point $\mathbf{A} = [a, b]$ **lies on** the line ℓ precisely when it satisfies the equation, that is $r + sa + tb = 0$. Equivalently we say that ℓ **passes through** $\mathbf{A}$, or that $\mathbf{A}$ and ℓ are **incident**.

We can rewrite equations of lines using cross products. If $\mathbf{V} = [x, y]$ represents a variable point, then a line in the affine space can be expressed in a parametric form

$$l : \mathbf{V} = \mathbf{P} + \lambda \mathbf{w}$$

for some fixed point $\mathbf{P}$ and direction vector $\mathbf{w}$. This line l passes through $\mathbf{P}$ and is parallel to $\mathbf{w}$.

Alternatively we may write the above equation as

$$l : \mathbf{w} \times (\mathbf{V} - \mathbf{P}) = 0.$$

This is equivalent to, in terms of the corresponding position vectors $\mathbf{v}$ and $\mathbf{p}$, to the equation

$$g + \mathbf{w} \times \mathbf{v} = 0 \qquad \text{where} \qquad g = -\mathbf{w} \times \mathbf{p}.$$

Sometimes it may be more convenient to use a point $\mathbf{P}'$ not on the line; in such cases, we end up with a different constant term as in

$$g_1 + \mathbf{w} \times (\mathbf{V} - \mathbf{P}') = 0.$$

2.4. **Dot products.** A metrical structure on the affine plane $\mathbb{A}^2$ arises from a symmetric bilinear form on the associated vector space $\mathbb{V}^2$, which we write in the form of a **dot product** $\mathbf{v} \cdot \mathbf{w}$. In terms of linear algebra with row vectors $\mathbf{v} \equiv (a, b)$ and $\mathbf{w} \equiv (c, d)$, such a dot product may be represented as

$$\mathbf{v} \cdot \mathbf{w} \equiv \mathbf{v} B \mathbf{w}^T$$

for a 2×2 symmetric matrix B. We will assume in addition that this is non-degenerate, meaning that $\det B \neq 0$. It will be convenient for us to adopt the convention that the dot product is also represented just as a product, so that for vectors $\mathbf{v}$ and $\mathbf{w}$ we write $\mathbf{vw} \equiv \mathbf{v} \cdot \mathbf{w}$. Two vectors $\mathbf{v}$ and $\mathbf{w}$ are then **perpendicular** precisely when $\mathbf{vw} = 0$, written as $\mathbf{v} \perp \mathbf{w}$.

The main settings of our investigation, the **blue**, **red** and **green** geometries on the affine plane, are defined by the respective symmetric matrices:

$$B_b \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B_r \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B_g \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These three matrices form a natural basis for the space of 2×2 symmetric matrices. Together with J mentioned previously, we have then a distinguished basis of the vector space of 2×2 matrices. Of course the blue geometry is just the usual Euclidean geometry, while the red and green geometries are relativistic geometries usually associated with the special relativity of Einstein, as interpreted geometrically by Minkowski.

We may also use a dot product to describe lines in the affine plane, as the line through $\mathbf{P}$ perpendicular to $\mathbf{w}$, has equation

$$m : \mathbf{w} \cdot (\mathbf{V} - \mathbf{P}) = 0.$$

This is equivalent to, in terms of the corresponding position vectors $\mathbf{v}$ and $\mathbf{p}$, the equation

$$h + \mathbf{w} \cdot \mathbf{v} = 0 \qquad \text{where } h = -\mathbf{w} \cdot \mathbf{p}.$$

With respect to a point $\mathbf{P}'$ not on the line, we may also express a line perpendicular to $\mathbf{w}$ by

$$h_1 + \mathbf{w} \cdot (\mathbf{V} - \mathbf{P}') = 0.$$

Proposition 1. *The lines $\ell : g + (a, b) \times (x, y) = 0$ and $m : h + (a, b) \cdot (x, y) = 0$ are perpendicular.*

Proof. This follows directly from the relationship between these lines and the vector (a, b). □

2.5. **Quadrances.** Having fixed a dot product, we introduce the **quadrance** of a vector $\mathbf{v}$ to be the number

$$Q(\mathbf{v}) \equiv \mathbf{v}^2 \equiv \mathbf{v} \cdot \mathbf{v}.$$

We are here just giving a name to the quantity output by the associated quadratic form. Clearly $Q(\mathbf{v}) = Q(-\mathbf{v})$. The **quadrance** $Q(\mathbf{A}, \mathbf{B})$ between affine points $\mathbf{A}$ and $\mathbf{B}$ is, by definition, the quadrance of the vector between them, so that

$$Q(\mathbf{A}, \mathbf{B}) \equiv Q\left(\overrightarrow{\mathbf{AB}}\right).$$

From the above property of quadrances, this is a symmetric expression, that is $Q(\mathbf{A}, \mathbf{B}) = Q(\mathbf{B}, \mathbf{A})$. In terms of positional vectors,

$$Q(\mathbf{A}, \mathbf{B}) = (\mathbf{b} - \mathbf{a})^2 = \mathbf{a}^2 - 2\mathbf{ab} + \mathbf{b}^2.$$

The fundamental theorem in mathematics has a pleasant and simple manifestation in this general context, assuming the characteristic of our field is not two.

Theorem 1 (Pythagoras)**.** *For three points $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$,*

$$\overrightarrow{\mathbf{A}_1\mathbf{A}_2} \perp \overrightarrow{\mathbf{A}_1\mathbf{A}_3}$$

precisely when

$$Q(\mathbf{A}_1, \mathbf{A}_2) + Q(\mathbf{A}_1, \mathbf{A}_3) = Q(\mathbf{A}_2, \mathbf{A}_3).$$

Proof. If $\mathbf{v}_3 = \overrightarrow{\mathbf{A}_1\mathbf{A}_2}$ and $\mathbf{v}_2 = \overrightarrow{\mathbf{A}_1\mathbf{A}_3}$ then

$$\begin{aligned} Q(\mathbf{A}_2, \mathbf{A}_3) &= (\mathbf{v}_2 - \mathbf{v}_3)(\mathbf{v}_2 - \mathbf{v}_3) = \mathbf{v}_2^2 + \mathbf{v}_3^2 - 2\mathbf{v}_2\mathbf{v}_3 \\ &= Q(\mathbf{A}_1, \mathbf{A}_2) + Q(\mathbf{A}_1, \mathbf{A}_3) - 2\mathbf{v}_2\mathbf{v}_3. \end{aligned}$$

Now $\overrightarrow{\mathbf{A}_1\mathbf{A}_2} \perp \overrightarrow{\mathbf{A}_1\mathbf{A}_3}$ precisely when $\mathbf{v}_2\mathbf{v}_3 = 0$, which is precisely the condition

$$Q(\mathbf{A}_1, \mathbf{A}_2) + Q(\mathbf{A}_1, \mathbf{A}_3) = Q(\mathbf{A}_2, \mathbf{A}_3).$$

□

The sister theorem to Pythagoras' theorem is much less well-known, which is somewhat remarkable, since it is arguably the second most important result in geometry, see [8, p.70]. To state it, we introduce **Archimedes' function** following [8] by

$$\begin{aligned}\mathcal{A}(\alpha,\beta,\gamma) &\equiv (\alpha+\beta+\gamma)^2-2\left(\alpha^2+\beta^2+\gamma^2\right)\\ &= 2\alpha\beta+2\alpha\gamma+2\beta\gamma-\alpha^2-\beta^2-\gamma^2\\ &= 4\alpha\beta-(\alpha+\beta-\gamma)^2.\end{aligned}$$

Theorem 2 (Triple Quad Formula)**.** *For three points* $\mathbf{A}_1,\mathbf{A}_2$ *and* $\mathbf{A}_3$,

$$\overrightarrow{\mathbf{A}_1\mathbf{A}_2} \parallel \overrightarrow{\mathbf{A}_1\mathbf{A}_3}$$

precisely when

$$\mathcal{A}\left(Q\left(\mathbf{A}_1,\mathbf{A}_2\right),Q\left(\mathbf{A}_1,\mathbf{A}_3\right),Q\left(\mathbf{A}_2,\mathbf{A}_3\right)\right)=0.$$

Proof. If $\overrightarrow{\mathbf{A}_1\mathbf{A}_2}=\mathbf{v}$ and $\overrightarrow{\mathbf{A}_1\mathbf{A}_3}=\mathbf{w}$ then

$$\begin{aligned}&\mathcal{A}\left(Q\left(\mathbf{A}_1,\mathbf{A}_2\right),Q\left(\mathbf{A}_1,\mathbf{A}_3\right),Q\left(\mathbf{A}_2,\mathbf{A}_3\right)\right)\\ &= \mathcal{A}\left(\mathbf{v}^2,\mathbf{w}^2,(\mathbf{w}-\mathbf{v})^2\right)\\ &= 4\left(\mathbf{v}^2\mathbf{w}^2-(\mathbf{v}\cdot\mathbf{w})^2\right).\end{aligned} \tag{1}$$

With the definition of dot product $\mathbf{v}\cdot\mathbf{w}\equiv\mathbf{v}B\mathbf{w}^T$ let

$$B=\begin{pmatrix}a & b\\ b & c\end{pmatrix}$$

and let $\mathbf{v}=[v_x,v_y]$, $\mathbf{w}=[w_x,w_y]$, then

$$\mathbf{v}^2\mathbf{w}^2-(\mathbf{v}\cdot\mathbf{w})^2=(v_xw_y-v_yw_x)^2\left(ac-b^2\right).$$

If $\mathbf{v}\parallel\mathbf{w}$, then $v_x:v_y=w_x:w_y$ and so

$$\mathcal{A}\left(Q\left(\mathbf{A}_1,\mathbf{A}_2\right),Q\left(\mathbf{A}_1,\mathbf{A}_3\right),Q\left(\mathbf{A}_2,\mathbf{A}_3\right)\right)=0.$$

Conversely, if $\mathcal{A}\left(Q\left(\mathbf{A}_1,\mathbf{A}_2\right),Q\left(\mathbf{A}_1,\mathbf{A}_3\right),Q\left(\mathbf{A}_2,\mathbf{A}_3\right)\right)=0$, then since $ac-b^2=\det B\neq 0$ we have

$$\mathbf{v}\times\mathbf{w}=v_xw_y-v_yw_x=0$$

and thus $\mathbf{v}\parallel\mathbf{w}$. □

So the quantity $\mathcal{A}\left(Q\left(\mathbf{A}_1,\mathbf{A}_2\right),Q\left(\mathbf{A}_1,\mathbf{A}_3\right),Q\left(\mathbf{A}_2,\mathbf{A}_3\right)\right)$ is a measure of the non-collinearity of the points $\mathbf{A}_1,\mathbf{A}_2$ and $\mathbf{A}_3$ and naturally there is a connection with areas of triangles. The relation between this result and Heron's formula (the rational analog is called Archimedes' formula in [8]) is connected to the following identity.

Lemma 1 (Archimedes' linear relation)**.** *Suppose* $A,B,C\in\mathbb{F}$ *are squares, that is,* $A=a^2,B=b^2$, *and* $C=c^2$, *then* $\mathcal{A}(A,B,C)=0$ *precisely when one of the relations* $a\pm b\pm c=0$ *holds.*

Proof. By substitution, we get

$$\begin{aligned}\mathcal{A}(A,B,C) &= \mathcal{A}(a^2,b^2,c^2)\\ &= \left(a^2+b^2+c^2\right)^2-2\left(a^4+b^4+c^4\right)\\ &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c)=0.\end{aligned}$$

□

We can restate the conclusion as the sum of two of a, b, c equals either the third or its negative.

Corollary 1. *If a, b are known, then $\mathcal{A}(a^2, b^2, c^2) = 0$ implies either $c = \pm(a+b)$ or $c = \pm(a-b)$.*

Let us also note that the exact same conclusion holds if each of A, B, C is the negative of a square.

Lemma 2 (Archimedes' linear relation II)**.** *Suppose $A, B, C \in \mathbb{F}$ are negatives of squares, that is, $A = -a^2, B = -b^2$, and $C = -c^2$, then $\mathcal{A}(A, B, C) = 0$ precisely when one of the relations $a \pm b \pm c = 0$ holds.*

2.6. **Circles.** We continue with a fixed dot product on the vector space $\mathbb{V}^2$. If $\mathbf{V}$ is a general affine point and $\mathbf{P}$ is a fixed affine point, then a **circle** $\mathcal{C}$ is an equation of the form

$$\mathcal{C} : (\mathbf{V} - \mathbf{P})^2 = Q$$

where $Q \in \mathbb{F}$. In this case we say that $\mathbf{P}$ is a **center** of the circle, and Q is a **quadrance** of the circle $\mathcal{C}$.

Theorem 3. *The center and quadrance of a circle $\mathcal{C}$ are unique.*

Proof. Suppose $\mathcal{C}$ has two separate centers $\mathbf{P}_1, \mathbf{P}_2$ and respective quadrances Q_1, Q_2, then with $\mathbf{V}$ as the variable point, by definition,

$$(\mathbf{V} - \mathbf{P}_1)^2 - Q_1 = 0 \qquad \text{and} \qquad (\mathbf{V} - \mathbf{P}_2)^2 - Q_2 = 0.$$

By subtracting the second equation from the first one, we get

$$2(\mathbf{P}_1 - \mathbf{P}_2)\mathbf{V} + \left(\mathbf{P}_1^2 - \mathbf{P}_2^2 - Q_1 + Q_2\right) = 0.$$

Since the right-hand side is the zero polynomial, all coefficients on the left-hand side must equal zero. Hence $\mathbf{P}_1 = \mathbf{P}_2$ and $Q_1 = Q_2$. □

The quadrance Q of a circle need not be a square, but if $Q = r^2$ then we say that r, or $-r$, is a **radius** of the circle. Note that the radius is then not unique.

A point $\mathbf{A}$ **lies on** the circle $\mathcal{C}$ precisely when it satisfies the equation of that circle, that is $(\mathbf{A} - \mathbf{P})^2 = Q$. In such a case we also say that the circle $\mathcal{C}$ **passes through** $\mathbf{A}$, or that $\mathbf{A}$ and $\mathcal{C}$ are **incident**. A circle need not have any points lying on it. For example, in the blue geometry the circle defined by $(\mathbf{V} - [1, 5])^2 = 3$ can be written as the equation $(x-1)^2 + (y-5)^2 = 3$, and it has no points lying on it over $\mathbb{Q}$, but several points lying on it over $\mathbb{F}_7$, for example $[2, 1]$. We shall call a circle **empty** if it is incident with no points.

A circle is **null** precisely when its quadrance Q is zero. Null circles generally play a special, more degenerate, role. In the blue geometry over $\mathbb{Q}$, a null circle is a point, while in the red and green geometries null circles are geometrically pairs of lines. However over a finite field, a circle in the blue geometry may be null and not a point.

For a non-null circle, we define the reciprocal of the quadrance Q to be the **quadratic curvature** κ of the circle, so that

$$\kappa \equiv \frac{1}{Q}.$$

If this quadratic curvature happens to be a square, say $\kappa = k^2$, then we could define k, or $-k$ as a **curvature** of the circle, but over a general field this is not unique.

A circle may be defined in less direct ways as well, as the next section shows.

2.7. **Apollonian version of circles.**

Theorem 4 (Apollonian circle)**.** *Suppose that* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are fixed points and* D_1 *and* D_2 *are distinct numbers. The equation of the variable point* $\mathbf{V}$ *satisfying the relation*

$$Q(\mathbf{V}, \mathbf{P}_1) : Q(\mathbf{V}, \mathbf{P}_2) = D_1 : D_2 \tag{2}$$

gives a circle $\mathcal{C}$ *with center*

$$\mathbf{P} = \frac{D_2}{D_2 - D_1}\mathbf{P}_1 + \frac{D_1}{D_1 - D_2}\mathbf{P}_2$$

and quadrance

$$Q = \frac{D_1 D_2}{(D_1 - D_2)^2}(\mathbf{P}_1 - \mathbf{P}_2)^2.$$

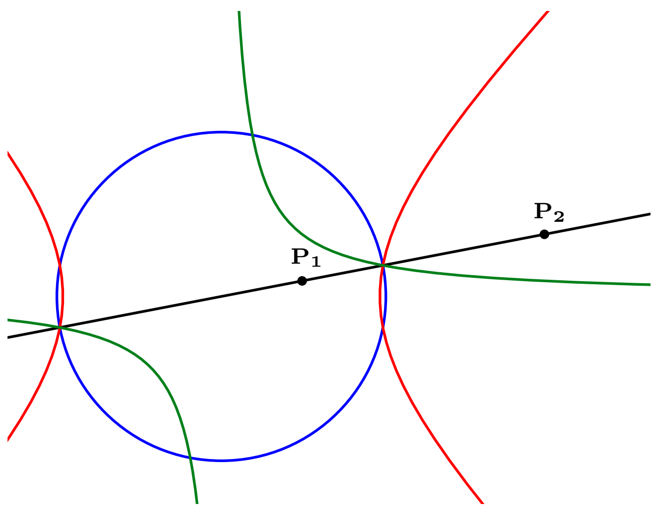


FIG. 1. An Apollonian circle with respect to points $\mathbf{P}_1$ and $\mathbf{P}_2$ in all three colors of geometry

Proof. Let $\mathbf{v}$, $\mathbf{p}_1$ and $\mathbf{p}_2$ be the position vectors of $\mathbf{V}$, $\mathbf{P}_1$ and $\mathbf{P}_2$ respectively. Then the equation $Q(\mathbf{V}, \mathbf{P}_1) : Q(\mathbf{V}, \mathbf{P}_2) = D_1 : D_2$ can be rewritten as

$$D_2(\mathbf{v} - \mathbf{p}_1)^2 = D_1(\mathbf{v} - \mathbf{p}_2)^2$$

or

$$(D_2 - D_1)\mathbf{v}^2 - 2(D_2\mathbf{p}_1 - D_1\mathbf{p}_2)\mathbf{v} + D_2\mathbf{p}_1^2 - D_1\mathbf{p}_2^2 = 0.$$

After completing the square, this can be expressed and simplified as

$$\begin{aligned}\left(\mathbf{v} - \frac{D_2\mathbf{p}_1 - D_1\mathbf{p}_2}{D_2 - D_1}\right)^2 &= \left(\frac{D_2\mathbf{p}_1 - D_1\mathbf{p}_2}{D_2 - D_1}\right)^2 - \frac{D_2\mathbf{p}_1^2 - D_1\mathbf{p}_2^2}{D_2 - D_1} \\ &= \frac{D_1 D_2}{(D_1 - D_2)^2}(\mathbf{p}_1 - \mathbf{p}_2)^2.\end{aligned}$$

In terms of just points, this is

$$\left(\mathbf{V}-\left(\frac{D_2}{D_2-D_1}\mathbf{P}_1+\frac{D_1}{D_1-D_2}\mathbf{P}_2\right)\right)^2=\frac{D_1D_2}{\left(D_1-D_2\right)^2}\left(\mathbf{P}_1-\mathbf{P}_2\right)^2.$$

which we recognize as a circle $\mathcal{C}$ with center $\mathbf{P}$ and quadrance Q as stated. □

The term **Apollonian circle** refers to such a circle determined by two points $\mathbf{P}_1$ and $\mathbf{P}_2$, and the non-equal proportion $D_1 : D_2$. We will say that $\{\mathbf{P}_1,\mathbf{P}_2\}$ is an **Apollonian focal pair** for the circle $\mathcal{C}$, but such a pair is in general not unique. Note that if $D_1 = 0$ then we get the null circle $(\mathbf{V}-\mathbf{P}_1)^2 = 0$, while if $D_2 = 0$ then we get the null circle $(\mathbf{V}-\mathbf{P}_2)^2 = 0$.

We may also consider the equation (2) in case of an equal proportion $D_1 : D_2$ when $D_1 = D_2$. Then we do not get a circle, but rather the equation

$$2\left(\mathbf{P}_1-\mathbf{P}_2\right)\mathbf{v}-\left(\mathbf{p}_1^2-\mathbf{p}_2^2\right)=0$$

which can be rewritten purely in terms of points as

$$\left(\mathbf{P}_1-\mathbf{P}_2\right)\left(\mathbf{V}-\left(\frac{1}{2}\mathbf{P}_1+\frac{1}{2}\mathbf{P}_2\right)\right)=0.$$

This is a line, which is the perpendicular bisector of $\overline{\mathbf{P}_1\mathbf{P}_2}$, meaning that it passes through the mid-point $\frac{1}{2}\mathbf{P}_1+\frac{1}{2}\mathbf{P}_2$ and is perpendicular to $\mathbf{P}_1-\mathbf{P}_2$.

We will say that $D_1 : D_2$ is a **square proportion** precisely when D_1D_2 is a square in the field. This is clearly well-defined.

Theorem 5 (Dividing points)**.** *The Apollonian circle $\mathcal{C}$ determined by two points $\mathbf{P}_1$ and $\mathbf{P}_2$ and the non-equal proportion $D_1 : D_2$ meets the line $\mathbf{P}_1\mathbf{P}_2$ precisely when $D_1 : D_2$ is a square proportion. In this case if $D_1D_2 \neq 0$ then there are exactly two meets, and they form a diameter of $\mathcal{C}$.*

Proof. The center $\mathbf{P}$ of $\mathcal{C}$ lies on $\mathbf{P}_1\mathbf{P}_2$, as it is an affine combination of the two points. So a meet of $\mathcal{C}$ with $\mathbf{P}_1\mathbf{P}_2$, if such a point exists, must be of the form $\mathbf{A} = \mathbf{P} + k\left(\mathbf{P}_1-\mathbf{P}_2\right)$ for some $k \in \mathbb{F}$. In this case we require

$$\left(\mathbf{A}-\mathbf{P}\right)^2=k^2\left(\mathbf{P}_1-\mathbf{P}_2\right)^2=Q=\frac{D_1D_2}{\left(D_1-D_2\right)^2}\left(\mathbf{P}_1-\mathbf{P}_2\right)^2$$

or

$$k^2=\frac{D_1D_2}{\left(D_1-D_2\right)^2}.$$

This has a solution precisely when D_1D_2 is a square, which occurs precisely when $D_1 : D_2$ is a square proportion. When $D_1D_2 \neq 0$ we have $k^2 \neq 0$ so there are two solutions, say $\mathbf{A}_1 = \mathbf{P}+k\left(\mathbf{P}_1-\mathbf{P}_2\right)$ and $\mathbf{A}_2 = \mathbf{P}-k\left(\mathbf{P}_1-\mathbf{P}_2\right)$ in terms of a fixed k, and clearly $\mathbf{P}$ is the midpoint of $\overline{\mathbf{A}_1\mathbf{A}_2}$. □

2.8. **Pencils of circles and the radical axis.** Given any two distinct curves, there is a canonical family, or **pencil**, of curves determined by taking linear combinations of their defining equations. The result of taking a linear combination of equations of two given circles is a pencil described as follows.

Theorem 6 (Affine combinations of circles)**.** *For any two distinct circles $\mathcal{C}_1 : \left(\mathbf{V}-\mathbf{P}_1\right)^2 - Q_1 = 0$ and $\mathcal{C}_2 : \left(\mathbf{V}-\mathbf{P}_2\right)^2 - Q_2 = 0$, the affine linear combination*

$$(3)\qquad \left(1-\alpha\right)\left(\left(\mathbf{V}-\mathbf{P}_1\right)^2-Q_1\right)+\alpha\left(\left(\mathbf{V}-\mathbf{P}_2\right)^2-Q_2\right)=0$$

defines a circle with center

$$\mathbf{P} \equiv (1-\alpha)\,\mathbf{P}_1 + \alpha\mathbf{P}_2$$

and quadrance

$$Q \equiv (1-\alpha)\,Q_1 + \alpha Q_2 - \alpha\,(1-\alpha)\,(\mathbf{P}_1 - \mathbf{P}_2)^2\,.$$

Proof. If we replace points with position vectors, then the equation

$$(1-\alpha)\left((\mathbf{V}-\mathbf{P}_1)^2 - Q_1\right) + \alpha\left((\mathbf{V}-\mathbf{P}_2)^2 - Q_2\right) = 0$$

becomes

$$\mathbf{v}^2 - 2\left((1-\alpha)\,\mathbf{p}_1 + \alpha\mathbf{p}_2\right)\mathbf{v} - \left(Q_1 - \alpha Q_1 + \alpha Q_2 - \mathbf{p}_1^2 + \alpha\mathbf{p}_1^2 - \alpha\mathbf{p}_2^2\right) = 0.$$

Completing the square we get

$$\left(\mathbf{v} - ((1-\alpha)\,\mathbf{p}_1 + \alpha\mathbf{p}_2)\right)^2 - \left((1-\alpha)\,Q_1 + \alpha Q_2 - \alpha\,(1-\alpha)\,(\mathbf{p}_1-\mathbf{p}_2)^2\right) = 0$$

which we recognize as a circle with center $\mathbf{P}$ and quadrance Q as stated. □

There is one special linear combination where the resulting equation is actually that of a line; this occurs when we take the difference of the equations of $\mathcal{C}_1$ and $\mathcal{C}_2$

$$\left((\mathbf{V}-\mathbf{P}_1)^2 - Q_1\right) - \left((\mathbf{V}-\mathbf{P}_2)^2 - Q_2\right) = 0$$

in which case we get a linear equation in $\mathbf{V}$. This is then the **radical axis** of the two circles $\mathcal{C}_1$ and $\mathcal{C}_2$, or indeed of any two circles in the pencil, as the pencil is defined by any two equations in it.

Theorem 7 (Radical axis formula)**.** *If* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are distinct points, then the radical axis of the circles*

$$(\mathbf{V}-\mathbf{P}_1)^2 - Q_1 = 0 \qquad \text{and} \qquad (\mathbf{V}-\mathbf{P}_2)^2 - Q_2 = 0$$

is the line

$$(Q_2 - Q_1) + 2\,(\mathbf{P}_2 - \mathbf{P}_1)\left(\mathbf{V} - \left(\frac{1}{2}\mathbf{P}_1 + \frac{1}{2}\mathbf{P}_2\right)\right) = 0$$

which is perpendicular to $\mathbf{P}_1\mathbf{P}_2$*. This can also be written in the form*

$$2\,(\mathbf{P}_2-\mathbf{P}_1)\left(\mathbf{V} - \left(\left(\frac{1}{2} + \frac{Q_2-Q_1}{2\,(P_2-P_1)^2}\right)\mathbf{P}_1 + \left(\frac{1}{2} + \frac{Q_1-Q_2}{2\,(P_2-P_1)^2}\right)\mathbf{P}_2\right)\right) = 0.$$

Proof. The equation for the radical axis has the form

$$(\mathbf{V}-\mathbf{P}_1)^2 - Q_1 = (\mathbf{V}-\mathbf{P}_2)^2 - Q_2.$$

If we replace the points with their position vectors, we get the equation

$$\mathbf{v}^2 - 2\mathbf{p}_1\mathbf{v} + \mathbf{p}_1^2 - Q_1 = \mathbf{v}^2 - 2\mathbf{p}_2\mathbf{v} + \mathbf{p}_2^2 - Q_2$$

or

$$(Q_2 - Q_1) + 2\,(\mathbf{p}_2 - \mathbf{p}_1)\left(\mathbf{v} - \left(\frac{1}{2}\mathbf{p}_1 + \frac{1}{2}\mathbf{p}_2\right)\right) = 0$$

which can be rewritten purely in terms of points as

$$(Q_2 - Q_1) + 2\,(\mathbf{P}_2 - \mathbf{P}_1)\left(\mathbf{V} - \left(\frac{1}{2}\mathbf{P}_1 + \frac{1}{2}\mathbf{P}_2\right)\right) = 0.$$

This is clearly a line which has normal vector $\mathbf{P}_2 - \mathbf{P}_1$ and so is perpendicular to $\mathbf{P}_1\mathbf{P}_2$. Suppose we want to find where this axis meets the line $\mathbf{P}_1\mathbf{P}_2$, say at a point $\mathbf{P} = \alpha\mathbf{P}_1 + (1-\alpha)\,\mathbf{P}_2$. Substituting and solving we get

$$\alpha = \frac{1}{2} + \frac{Q_2 - Q_1}{2\,(P_2 - P_1)^2} \qquad \text{and} \qquad 1 - \alpha = \frac{1}{2} + \frac{Q_1 - Q_2}{2\,(P_2 - P_1)^2}$$

which gives the second formula for the radical axis. □

We will call the point

$$\mathbf{P} \equiv \left(\frac{1}{2} + \frac{Q_2 - Q_1}{2\,(P_2 - P_1)^2}\right)\mathbf{P}_1 + \left(\frac{1}{2} + \frac{Q_1 - Q_2}{2\,(P_2 - P_1)^2}\right)\mathbf{P}_2$$

the **radical base** of the two circles; this is the meet of their radical axis and the line $\mathbf{P}_1\mathbf{P}_2$. It is then common to any two circles in the pencil.

In rational trigonometry, we may define the **power** of a point $\mathbf{V}$, with respect to a circle $\mathcal{C}$ centerd at $\mathbf{P}$ and with a quadrance Q, to be the number

$$(\mathbf{V} - \mathbf{P})^2 - Q$$

so that a point lies on the circle precisely when its power is 0.

Two circles are **concentric** precisely when they have a common center. Then for any two non-concentric circles, it is classically known that the locus of points with equal power to them is their radical axis ([2, p.191-194]). This is now obvious in this more general setting, as the equation of the radical axis is obtained by essentially equating powers from the two circles.

The radical axis could also be viewed as a special projective case of the affine pencil formed by two non-concentric circles, by equating α with infinity in Equation (3).

2.9. Apollonian pencil of circles.

Theorem 8 (Apollonian pencil of circles). *If $\mathbf{P}_1$ and $\mathbf{P}_2$ are two distinct points, then the Apollonian circles with Apollonian foci $\mathbf{P}_1$ and $\mathbf{P}_2$ form a pencil of circles.*

Proof. The two simplest Apollonian circles with foci $\mathbf{P}_1$ and $\mathbf{P}_2$ are just the null circles $(\mathbf{V} - \mathbf{P}_1)^2 = 0$ and $(\mathbf{V} - \mathbf{P}_2)^2 = 0$. If we take an affine linear combination of these we get

$$\begin{aligned}&(1-\alpha)\,(\mathbf{V} - \mathbf{P}_1)^2 + \alpha\,(\mathbf{V} - \mathbf{P}_2)^2 \\ &\qquad = \Big((\mathbf{V} - ((1-\alpha)\,\mathbf{P}_1 + \alpha\mathbf{P}_2))^2 + (1-\alpha)\,\alpha\,(\mathbf{P}_1 - \mathbf{P}_2)^2\Big)\end{aligned}$$

which agrees with the Apollonian circle

$$Q\,(\mathbf{V}, \mathbf{P}_1) : Q\,(\mathbf{V}, \mathbf{P}_2) = D_1 : D_2$$

in case

$$\alpha = \frac{D_1}{D_1 - D_2} \qquad \text{and} \qquad (1-\alpha) = \frac{D_2}{D_2 - D_1}$$

which amounts to

$$D_1 : D_2 = \alpha : \alpha - 1.$$

□

2.10. **Triangles, quadrea and barycentric coordinates.** A **triangle**

$$\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3} = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$$

is a set of three (distinct) non-collinear points. The **lines** of such a triangle are $\mathbf{A}_1\mathbf{A}_2$, $\mathbf{A}_1\mathbf{A}_3$ and $\mathbf{A}_2\mathbf{A}_3$, and these are distinct. A **side** $\overline{\mathbf{A}_1\mathbf{A}_2} = \{\mathbf{A}_1, \mathbf{A}_2\}$ is a set of two (distinct) points. The triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ has three sides, namely $\overline{\mathbf{A}_1\mathbf{A}_2}$, $\overline{\mathbf{A}_2\mathbf{A}_3}$ and $\overline{\mathbf{A}_1\mathbf{A}_3}$.

For a triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$, any point in the affine plane can be written uniquely as a 1-combination

$$\mathbf{P} = a_1\mathbf{A}_1 + a_2\mathbf{A}_2 + a_3\mathbf{A}_3 \qquad \text{where} \qquad a_1 + a_2 + a_3 = 1$$

while any vector in the associated vector space can be written uniquely as a 0-combination

$$\mathbf{v} = \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_3 \qquad \text{where} \qquad a_1 + a_2 + a_3 = 0.$$

Suppose we have fixed a dot product as above; then a fixed triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ has quadrances

$$Q_1 \equiv Q(\mathbf{A}_2, \mathbf{A}_3) \qquad Q_2 \equiv Q(\mathbf{A}_3, \mathbf{A}_1) \qquad Q_3 \equiv Q(\mathbf{A}_1, \mathbf{A}_2).$$

We say that $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ is a **non-null** triangle precisely when each of the three quadrances Q_1, Q_2 and Q_3 is non-zero. Motivated by the Triple quad formula, we define the **quadrea** of the triangle to be

$$\mathcal{A}\left(\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}\right) \equiv \mathcal{A}\left(Q_1, Q_2, Q_3\right) = 2Q_1Q_2 + 2Q_1Q_3 + 2Q_2Q_3 - Q_1^2 - Q_2^2 - Q_3^2.$$

The following theorem was stated in [8, p.68] in the blue situation.

Theorem 9 (Quadrea)**.** *If*

$$\mathbf{A}_1 = [x_1, y_1] \qquad \mathbf{A}_2 = [x_2, y_2] \qquad \mathbf{A}_3 = [x_3, y_3]$$

and the dot product is given by $\mathbf{v} \cdot \mathbf{w} \equiv \mathbf{v}B\mathbf{w}^T$ *for a symmetric matrix* B *with* $\det B = \delta$*, then*

$$\begin{aligned} \mathcal{A}\left(\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}\right) &= 4\delta\left(x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2\right)^2 \\ &= 16\ \delta\ \text{area}^2\left(\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}\right). \end{aligned}$$

Proof. Suppose that

$$B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

with $\det B = \delta$. The quadrances of the triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ are then

$$\begin{aligned} Q_1 &= Q(\mathbf{A}_2, \mathbf{A}_3) = (x_3 - x_2)^2 a + 2(x_3 - x_2)(y_3 - y_2) b + (y_3 - y_2)^2 c \\ Q_2 &= Q(\mathbf{A}_3, \mathbf{A}_1) = (x_3 - x_1)^2 a + 2(x_3 - x_1)(y_3 - y_1) b + (y_3 - y_1)^2 c \\ Q_3 &= Q(\mathbf{A}_1, \mathbf{A}_2) = (x_2 - x_1)^2 a + 2(x_2 - x_1)(y_2 - y_1) b + (y_2 - y_1)^2 c. \end{aligned}$$

Now we may compute that

$$\begin{aligned} \mathcal{A}\left(Q_1, Q_2, Q_3\right) &= 2Q_1Q_2 + 2Q_1Q_3 + 2Q_2Q_3 - Q_1^2 - Q_2^2 - Q_3^2 \\ &= 4\left(ac - b^2\right)\left(x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2\right)^2 \\ &= 16\ \delta\ \text{area}^2\left(\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}\right). \end{aligned}$$

□

In particular we note that $\delta = +1$ for the blue geometry, and $\delta = -1$ for the red and green geometries.

We now establish general dot product and quadrance formulas for vectors written as a 0-combination with respect to the triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$.

Theorem 10 (Triangle vector dot product)**.** *Suppose that the quadrances of the triangle $\overline{A_1A_2A_3}$ are Q_1, Q_2 and Q_3 as usual, and that $\mathbf{u} = \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_3$ and $\mathbf{v} = \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \beta_3\mathbf{A}_3$ are vectors expressed as 0-combinations. Then*

$$\mathbf{u}\cdot\mathbf{v} = -\frac{1}{2}(\alpha_2\beta_3 + \alpha_3\beta_2)Q_1 - \frac{1}{2}(\alpha_3\beta_1 + \alpha_1\beta_3)Q_2 - \frac{1}{2}(\alpha_1\beta_2 + \alpha_2\beta_1)Q_3.$$

Proof. Since $\alpha_3 = -\alpha_1 - \alpha_2$, we may write

$$\mathbf{u} = \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_3 = \alpha_1(\mathbf{A}_1 - \mathbf{A}_3) + \alpha_2(\mathbf{A}_2 - \mathbf{A}_3) = -\alpha_1\mathbf{w}_2 - \alpha_2\mathbf{w}_1$$

where $\mathbf{w}_1 = \overrightarrow{\mathbf{A}_2\mathbf{A}_3}$ and $\mathbf{w}_2 = \overrightarrow{\mathbf{A}_1\mathbf{A}_3}$. Likewise, $\mathbf{v} = -\beta_1\mathbf{w}_2 - \beta_2\mathbf{w}_1$. Now clearly $\mathbf{w}_1^2 = Q(\mathbf{w}_1) = Q_1$ and $\mathbf{w}_2^2 = Q(\mathbf{w}_2) = Q_2$ while

$$\begin{aligned} Q_3 &= Q\left(\overrightarrow{\mathbf{A}_1\mathbf{A}_2}\right) = Q(\mathbf{w}_2 - \mathbf{w}_1) = (\mathbf{w}_2 - \mathbf{w}_1)\cdot(\mathbf{w}_2 - \mathbf{w}_1) \\ &= Q_2 + Q_1 - 2\mathbf{w}_1\mathbf{w}_2 \end{aligned}$$

so that

$$2\mathbf{w}_1\mathbf{w}_2 = Q_1 + Q_2 - Q_3.$$

It follows that

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= (-\alpha_1\mathbf{w}_2 - \alpha_2\mathbf{w}_1)(-\beta_1\mathbf{w}_2 - \beta_2\mathbf{w}_1) \\ &= \alpha_1\beta_1 Q_2 + \alpha_2\beta_2 Q_1 + (\alpha_1\beta_2 + \alpha_2\beta_1)\mathbf{w}_1\mathbf{w}_2 \\ &= \alpha_1\beta_1 Q_2 + \alpha_2\beta_2 Q_1 + \frac{1}{2}(\alpha_1\beta_2 + \alpha_2\beta_1)(Q_1 + Q_2 - Q_3) \\ &= \left(\frac{1}{2}\alpha_1\beta_2 + \frac{1}{2}\alpha_2\beta_1 + \alpha_2\beta_2\right)Q_1 + \left(\alpha_1\beta_1 + \frac{1}{2}\alpha_1\beta_2 + \frac{1}{2}\alpha_2\beta_1\right)Q_2 \\ &\quad + \left(-\frac{1}{2}\alpha_1\beta_2 - \frac{1}{2}\alpha_2\beta_1\right)Q_3 \\ &= \left(\frac{1}{2}(\alpha_1\beta_2 + \alpha_2\beta_2) + \frac{1}{2}(\alpha_2\beta_1 + \alpha_2\beta_2)\right)Q_1 \\ &\quad + \left(\frac{1}{2}(\alpha_1\beta_2 + \alpha_1\beta_1) + \frac{1}{2}(\alpha_1\beta_1 + \alpha_2\beta_1)\right)Q_2 - \left(\frac{1}{2}\alpha_1\beta_2 + \frac{1}{2}\alpha_2\beta_1\right)Q_3 \\ &= -\frac{1}{2}(\alpha_2\beta_3 + \alpha_3\beta_2)Q_1 - \frac{1}{2}(\alpha_3\beta_1 + \alpha_1\beta_3)Q_2 - \frac{1}{2}(\alpha_1\beta_2 + \alpha_2\beta_1)Q_3. \end{aligned}$$

□

Note that in matrix form this relation can be expressed as

$$\mathbf{u}\cdot\mathbf{v} = -\frac{1}{2}\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}0 & Q_3 & Q_2 \\ Q_3 & 0 & Q_1 \\ Q_2 & Q_1 & 0\end{pmatrix}\begin{pmatrix}\beta_1 \\ \beta_2 \\ \beta_3\end{pmatrix}.$$

This expresses the dot product in barycentric coordinates.

Corollary 2 (Triangle vector quadrance)**.** *If $\mathbf{v} = \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_3$ is a vector expressed as a 0-combination, then*

$$Q(\mathbf{v}) = -\alpha_2\alpha_3 Q_1 - \alpha_1\alpha_3 Q_2 - \alpha_1\alpha_2 Q_3.$$

Proof. This is a direct application of the previous theorem, since $Q(\mathbf{v}) = \mathbf{v}\cdot\mathbf{v}$. $\square$

Theorem 11 (Triangle normal vector)**.** *If $\mathbf{u} = \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_3$ is a vector expressed as a 0-proportion with respect to a triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ with quadrances Q_1, Q_2 and Q_3 as usual, then a perpendicular vector to it is given by $\mathbf{v} = \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \beta_3\mathbf{A}_3$ where*

$$\begin{aligned}\beta_1 &= (Q_2 - Q_3)\,\alpha_1 + Q_1\alpha_2 - Q_1\alpha_3\\ \beta_2 &= -Q_2\alpha_1 + (Q_3 - Q_1)\,\alpha_2 + Q_2\alpha_3\\ \beta_3 &= Q_3\alpha_1 - Q_3\alpha_2 + (Q_1 - Q_2)\,\alpha_3.\end{aligned}$$

Proof. By adopting the same approach as in the proof of the Triangle vector dot product theorem, we can write $\mathbf{u} = -\alpha_1\mathbf{w}_2 - \alpha_2\mathbf{w}_1$ and $\mathbf{v} = -\beta_1\mathbf{w}_2 - \beta_2\mathbf{w}_1$, where $\mathbf{w}_1 = \overrightarrow{\mathbf{A}_2\mathbf{A}_3}$ and $\mathbf{w}_2 = \overrightarrow{\mathbf{A}_1\mathbf{A}_3}$. Since $\mathbf{u}, \mathbf{v}$ are assumed perpendicular, we have

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (-\alpha_1\mathbf{w}_2 - \alpha_2\mathbf{w}_1)(-\beta_1\mathbf{w}_2 - \beta_2\mathbf{w}_1)\\ &= \alpha_1\beta_1\mathbf{w}_2^2 + \alpha_2\beta_2\mathbf{w}_1^2 + (\alpha_1\beta_2 + \alpha_2\beta_1)\,\mathbf{w}_1\mathbf{w}_2\\ &= \alpha_1\beta_1 Q_2 + \alpha_2\beta_2 Q_1 + \frac{1}{2}(\alpha_1\beta_2 + \alpha_2\beta_1)(Q_1 + Q_2 - Q_3)\\ &= \left(\alpha_1 Q_2 + \frac{1}{2}\alpha_2(Q_1 + Q_2 - Q_3)\right)\beta_1 + \left(\alpha_2 Q_1 + \frac{1}{2}\alpha_1(Q_1 + Q_2 - Q_3)\right)\beta_2\\ &= 0.\end{aligned}$$

This establishes a proportional relationship between β_1 and β_2, and we can set

$$\begin{aligned}\beta_1 &= (Q_1 + Q_2 - Q_3)\,\alpha_1 + 2Q_1\alpha_2\\ \beta_2 &= -2Q_2\alpha_1 - (Q_1 + Q_2 - Q_3)\,\alpha_2.\end{aligned}$$

and obtain $\beta_3 = -\beta_1 - \beta_2$. We can further re-arrange terms and get

$$\begin{aligned}\beta_1 &= (Q_2 - Q_3)\,\alpha_1 + Q_1\alpha_2 + (Q_1\alpha_1 + Q_1\alpha_2)\\ &= (Q_2 - Q_3)\,\alpha_1 + Q_1\alpha_2 - Q_1\alpha_3.\end{aligned}$$

Expressions of β_2, β_3 can be obtained with a similar approach. $\square$

Note that in matrix form we can write the relation as

$$\begin{pmatrix}\beta_1 & \beta_2 & \beta_3\end{pmatrix} = \begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}Q_2 - Q_3 & -Q_2 & Q_3\\ Q_1 & Q_3 - Q_1 & -Q_3\\ -Q_1 & Q_2 & Q_1 - Q_2\end{pmatrix}.$$

In the same vein, it is useful to have a formula for the quadrance between a general point expressed with respect to the triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ and a point of that triangle.

Theorem 12 (Triangle affine quadrance)**.** *Suppose that $\mathbf{P} = \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_3$ is a point expressed as a 1-proportion with respect to a triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ with quadrances Q_1, Q_2 and Q_3 as usual. Then*

$$\begin{aligned}Q(\mathbf{A}_1, \mathbf{P}) &= \alpha_3^2 Q_2 + \alpha_2^2 Q_3 + \alpha_2\alpha_3(Q_2 + Q_3 - Q_1)\\ Q(\mathbf{A}_2, \mathbf{P}) &= \alpha_1^2 Q_3 + \alpha_3^2 Q_1 + \alpha_1\alpha_3(Q_3 + Q_1 - Q_2)\\ Q(\mathbf{A}_3, \mathbf{P}) &= \alpha_2^2 Q_1 + \alpha_1^2 Q_2 + \alpha_1\alpha_2(Q_1 + Q_2 - Q_3).\end{aligned}$$

Proof. We have, using the Triangle vector quadrance theorem, that

$$\begin{aligned} Q(\mathbf{A}_1,\mathbf{P}) &= Q\left(\overrightarrow{\mathbf{A}_1\mathbf{P}}\right) = Q\left(\left(\alpha_1-1\right)\mathbf{A}_1+\alpha_2\mathbf{A}_2+\alpha_3\mathbf{A}_3\right) \\ &= -\alpha_2\alpha_3Q_1+\left(1-\alpha_1\right)\alpha_3Q_2+\left(1-\alpha_1\right)\alpha_2Q_3 \\ &= -\alpha_2\alpha_3Q_1+\left(\alpha_2+\alpha_3\right)\alpha_3Q_2+\left(\alpha_2+\alpha_3\right)\alpha_2Q_3 \\ &= \alpha_2^2Q_3+\alpha_3^2Q_2+\alpha_2\alpha_3\left(Q_2+Q_3-Q_1\right). \end{aligned}$$

The other formulas are obtained symmetrically. □

Example 13. *To illustrate these formulas in an explicit example, in the red geometry let's consider the triangle* $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ *with vertices* $\mathbf{A}_1 \equiv [0,0], \mathbf{A}_2 \equiv [1,0]$ *and* $\mathbf{A}_3 \equiv [0,1]$ *which has quadrances*

$$Q_1 = 0 \qquad Q_2 = -1 \qquad Q_3 = 1$$

and the two vectors

$$\begin{aligned} \mathbf{u} &= (2,1) = -3\mathbf{A}_1+2\mathbf{A}_2+\mathbf{A}_3 \\ \mathbf{v} &= (3,1) = -4\mathbf{A}_1+3\mathbf{A}_2+\mathbf{A}_3 \end{aligned}$$

expressed as 0*-combinations. By the Triangle vector dot product theorem,*

$$\mathbf{u}\cdot\mathbf{v} = -\frac{1}{2}\begin{pmatrix}-3 & 2 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & -1\\ 1 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}-4\\ 3\\ 1\end{pmatrix} = 5 = 2\cdot 3-1\cdot 1$$

while

$$Q(\mathbf{u}) = -2(0)-(-3)(-1)-(-6)(1) = 3 = 2^2-1^2.$$

By the Triangle normal vector theorem, a vector perpendicular to $\mathbf{u}$ *is*

$$\begin{aligned} \mathbf{w} &= \begin{pmatrix}-3 & 2 & 1\end{pmatrix}\begin{pmatrix}-2 & 1 & 1\\ 0 & 1 & -1\\ 0 & -1 & 1\end{pmatrix} = \begin{pmatrix}6 & -2 & -4\end{pmatrix} \\ &= 6\mathbf{A}_1-2\mathbf{A}_2-4\mathbf{A}_3 = (-2,-4) \end{aligned}$$

and we can check that $\mathbf{u}\cdot\mathbf{w} = (2)(-2)-(1)(-4) = 0$.

A point $P \equiv [2,1]$ *is a* 1*-combination* $-2\mathbf{A}_1+2\mathbf{A}_2+\mathbf{A}_3$*, hence*

$$\begin{aligned} Q(\mathbf{A}_1,\mathbf{P}) &= 1(-1)+2^2(1)+(2)(-1+1-0) = 3 \\ Q(\mathbf{A}_2,\mathbf{P}) &= (-2)^2(1)+1^2(0)+(-2)(1+0+1) = 0 \\ Q(\mathbf{A}_3,\mathbf{P}) &= 2^2(0)+(-2)^2(-1)+(-4)(0-1-1) = 4 \end{aligned}$$

which agrees with the direct calculations. This example demonstrates how the theorems apply in the general geometries, even when the triangle has a null side.

2.11. **The circumcircle and circumquadrance of a triangle.** The circumcenter is one of the four triangle centers known since the time of the ancient Greeks. The associated circumcircle has an important relation with the Apollonian circles of the triangle, so we first establish some classical properties of the former in this wider context of a general bilinear form over a general field.

Theorem 14 (Circumquadrance)**.** *The circumcenter of the triangle* $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ *with quadrances* $Q_1 \equiv Q(\mathbf{A}_2, \mathbf{A}_3), Q_2 \equiv Q(\mathbf{A}_3, \mathbf{A}_1)$ *and* $Q_3 \equiv Q(\mathbf{A}_1, \mathbf{A}_2)$ *and quadrea* $\mathcal{A} = \mathcal{A}(Q_1, Q_2, Q_3)$ *is the point given by the* 1*-combination*

$$\mathbf{C} \equiv \frac{Q_1(Q_2+Q_3-Q_1)}{\mathcal{A}}\mathbf{A}_1 + \frac{Q_2(Q_3+Q_1-Q_2)}{\mathcal{A}}\mathbf{A}_2 + \frac{Q_3(Q_1+Q_2-Q_3)}{\mathcal{A}}\mathbf{A}_3.$$

Furthermore the quadrance of this circumcircle is

$$K = \frac{Q_1Q_2Q_3}{\mathcal{A}}.$$

Proof. We compute, using the Triangle affine quadrance theorem and some simplification, that

$$\begin{aligned} Q(\mathbf{A}_1, \mathbf{C}) &= \frac{1}{\mathcal{A}^2}\left[Q_3^2(Q_1+Q_2-Q_3)^2 Q_2 + Q_2^2(Q_3+Q_1-Q_2)^2 Q_3\right] \\ &\quad + \frac{1}{\mathcal{A}^2}Q_2Q_3(Q_1+Q_2-Q_3)(Q_3+Q_1-Q_2)(Q_2+Q_3-Q_1) \\ &= \frac{1}{\mathcal{A}^2}Q_1Q_2Q_3\left(2Q_1Q_2+2Q_1Q_3+2Q_2Q_3-Q_1^2-Q_2^2-Q_3^2\right) \\ &= \frac{Q_1Q_2Q_2}{\mathcal{A}}. \end{aligned}$$

Since this expression is symmetric in Q_1, Q_2, Q_3, we conclude that

$$Q(\mathbf{A}_1, \mathbf{C}) = Q(\mathbf{A}_2, \mathbf{C}) = Q(\mathbf{A}_3, \mathbf{C}) = K = \frac{Q_1Q_2Q_2}{\mathcal{A}}.$$

□

We call K the **circumquadrance** of the triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$.

3. Apollonian Circles of a Triangle

We now introduce the main objects of interest, the Apollonian circles of a non-null triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ with quadrances Q_1, Q_2 and Q_3, by replacing the classical definition involving ratios of distances with corresponding ratios of quadrances. The **Apollonian circle** $\mathcal{C}_1$ **associated to the vertex** $\mathbf{A}_1$ of $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ is the locus of a variable point $\mathbf{V}$ such that

$$Q(\mathbf{V}, \mathbf{A}_2) : Q(\mathbf{V}, \mathbf{A}_3) = Q_3 : Q_2.$$

So the equation of such an Apollonian circle $\mathcal{C}_1$ is

$$Q_2(\mathbf{V}-\mathbf{A}_2)^2 = Q_3(\mathbf{V}-\mathbf{A}_3)^2.$$

This circle clearly passes through $\mathbf{A}_1$. Similarly we define $\mathcal{C}_2$ passing through $\mathbf{A}_2$ and $\mathcal{C}_3$ passing through $\mathbf{A}_3$.

The following is then an immediate consequence of the Apollonian circle theorem.

Theorem 15 (Triangle Apollonian circles). *The Apollonian circles $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ of the triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ have equations*

$$\begin{aligned}
(4) \qquad \mathcal{C}_1 &: \left(\mathbf{V}-\left(\frac{Q_2}{Q_2-Q_3}\mathbf{A}_2+\frac{Q_3}{Q_3-Q_2}\mathbf{A}_3\right)\right)^2-\frac{Q_1Q_2Q_3}{(Q_2-Q_3)^2}=0\\
\mathcal{C}_2 &: \left(\mathbf{V}-\left(\frac{Q_3}{Q_3-Q_1}\mathbf{A}_3+\frac{Q_1}{Q_1-Q_3}\mathbf{A}_1\right)\right)^2-\frac{Q_1Q_2Q_3}{(Q_3-Q_1)^2}=0\\
\mathcal{C}_3 &: \left(\mathbf{V}-\left(\frac{Q_1}{Q_1-Q_2}\mathbf{A}_1+\frac{Q_2}{Q_2-Q_1}\mathbf{A}_2\right)\right)^2-\frac{Q_1Q_2Q_3}{(Q_1-Q_2)^2}=0.
\end{aligned}$$

Their respective centers and quadrances are

$$\begin{aligned}
(5) \qquad \mathbf{C}_1 &\equiv \frac{Q_2}{Q_2-Q_3}\mathbf{A}_2+\frac{Q_3}{Q_3-Q_2}\mathbf{A}_3 \quad \text{and} \quad R_1\equiv\frac{Q_1Q_2Q_3}{(Q_2-Q_3)^2}\\
\mathbf{C}_2 &\equiv \frac{Q_3}{Q_3-Q_1}\mathbf{A}_3+\frac{Q_1}{Q_1-Q_3}\mathbf{A}_1 \quad \text{and} \quad R_2\equiv\frac{Q_1Q_2Q_3}{(Q_1-Q_3)^2}\\
\mathbf{C}_3 &\equiv \frac{Q_1}{Q_1-Q_2}\mathbf{A}_1+\frac{Q_2}{Q_2-Q_1}\mathbf{A}_2 \quad \text{and} \quad R_3\equiv\frac{Q_1Q_2Q_3}{(Q_1-Q_2)^2}.
\end{aligned}$$

Proof. We obtain $\mathcal{C}_1$ by applying the Apollonian circle theorem and noting that

$$(\mathbf{A}_2-\mathbf{A}_3)^2=Q_1.$$

The results for $\mathcal{C}_2, \mathcal{C}_3$ follow the same derivation. □

3.1. Relationship with the circumcircle and circumquadrance.

Proposition 2. *For a triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ with quadrances Q_1, Q_2 and Q_3, quadrea $\mathcal{A}$, circumcenter $\mathbf{C}$ and centers of Apollonian circles $\mathbf{C}_1, \mathbf{C}_2$ and $\mathbf{C}_3$, we have*

$$\begin{aligned}
Q(\mathbf{C},\mathbf{C}_1) &= \frac{Q_1^2Q_2Q_3(2Q_2+2Q_3-Q_1)}{(Q_2-Q_3)^2\mathcal{A}}\\
Q(\mathbf{C},\mathbf{C}_2) &= \frac{Q_1Q_2^2Q_3(2Q_1+2Q_3-Q_2)}{(Q_1-Q_3)^2\mathcal{A}}\\
Q(\mathbf{C},\mathbf{C}_3) &= \frac{Q_1Q_2Q_3^2(2Q_1+2Q_2-Q_3)}{(Q_1-Q_2)^2\mathcal{A}}.
\end{aligned}$$

Proof. From our existing formulas, we have

$$\begin{aligned}
\overrightarrow{\mathbf{CC}_1} &= -\frac{Q_1(Q_2+Q_3-Q_1)}{\mathcal{A}}\mathbf{A}_1+\left(\frac{Q_2}{Q_2-Q_3}-\frac{Q_2(Q_3+Q_1-Q_2)}{\mathcal{A}}\right)\mathbf{A}_2\\
&\quad+\left(\frac{Q_3}{Q_3-Q_2}-\frac{Q_3(Q_1+Q_2-Q_3)}{\mathcal{A}}\right)\mathbf{A}_3.
\end{aligned}$$

Now applying the Vector quadrance corollary we get

$$\begin{aligned}
Q(\mathbf{C},\mathbf{C}_1) &= -\left(\frac{Q_2}{Q_2-Q_3}-\frac{Q_2(Q_3+Q_1-Q_2)}{\mathcal{A}}\right)\left(\frac{Q_3}{Q_3-Q_2}-\frac{Q_3(Q_1+Q_2-Q_3)}{\mathcal{A}}\right)Q_1\\
&\quad+\frac{Q_1(Q_2+Q_3-Q_1)}{\mathcal{A}}\left(\frac{Q_3}{Q_3-Q_2}-\frac{Q_3(Q_1+Q_2-Q_3)}{\mathcal{A}}\right)Q_2\\
&\quad+\frac{Q_1(Q_2+Q_3-Q_1)}{\mathcal{A}}\left(\frac{Q_2}{Q_2-Q_3}-\frac{Q_2(Q_3+Q_1-Q_2)}{\mathcal{A}}\right)Q_3
\end{aligned}$$

which pleasantly simplifies to

$$Q(\mathbf{C},\mathbf{C}_1)=\frac{Q_1^2Q_2Q_3\,(2Q_2+2Q_3-Q_1)}{(Q_2-Q_3)^2\,\mathcal{A}}.$$

The other cases are symmetrically similar. □

Theorem 16 (Circumcircle)**.** *The Apollonian circles of a triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ are perpendicular to the circumcircle at the vertices (*[1, p.262]*).*

Proof. Consider the Apollonian circle $\mathcal{C}_1$ of $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ with center $\mathbf{C}_1$ passing through $\mathbf{A}_1$ and with quadrance R_1, as well as the circumcircle of $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ with center $\mathbf{C}$ and quadrance K. Then after some simplification, we have

$$\begin{aligned}R_1+K &= Q(\mathbf{C}_1,\mathbf{A}_1)+Q(\mathbf{A}_1,\mathbf{C})=\frac{Q_1Q_2Q_3}{(Q_2-Q_3)^2}+\frac{Q_1Q_2Q_3}{\mathcal{A}}\\ &= \frac{Q_1^2Q_2Q_3\,(2Q_2+2Q_3-Q_1)}{(Q_2-Q_3)^2\,\mathcal{A}}=Q(\mathbf{C},\mathbf{C}_1).\end{aligned}$$

So by Pythagoras' theorem the two circles meet perpendicularly at $\mathbf{A}_1$. □

Motivated by the relationship between quadrea and triangle's area in the blue geometry ([8, p.146]), we define a number $m\in\mathbb{F}$ to be a **signed square** precisely when $m=\pm\, n^2, n\in\mathbb{F}$. Then by the Quadrea theorem, the quadrea $\mathcal{A}$ of a general triangle is always a signed square for each of the blue, red and green geometries.

Let us then define a triangle to be **circum-radial** precisely when its circumquadrance is a signed square.

Theorem 17. *In the blue, red and green geometries, if one of the quadrances R_1,R_2,R_3 of the Apollonian circles of a triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ is a signed square, so are the other two. This happens precisely when $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ is a circum-radial triangle.*

Proof. The explicit formulas for the quadrances R_1,R_2,R_3 of the Apollonian circles in the Triangle Apollonian circles theorem show that one of these is a signed square precisely when the product $Q_1Q_2Q_3$ is a signed square, which then occurs precisely when any of the others is also a signed square. Now from the Circumquadrance theorem we have the relation $K=\frac{Q_1Q_2Q_3}{\mathcal{A}}$, and since for each of the blue, red and green geometries the quadrea is a signed square, it follows that $Q_1Q_2Q_3$ is a signed square precisely when K is a signed square, which completes the proof. □

3.2. **Quadratic curvatures of Apollonian circles.**

Here is a result not mentioned in most literatures, except hinted as an exercise in [2, p.266]: a relation about the curvatures of the Apollonian circles of a triangle, now holding in a general metrical geometry.

Theorem 18. *If κ_1,κ_2 and κ_3 are the quadratic curvatures of the Apollonian circles $\mathcal{C}_1,\mathcal{C}_2$ and $\mathcal{C}_3$ of a non-null triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$, then*

$$\mathcal{A}(\kappa_1,\kappa_2,\kappa_3)=0.$$

Proof. Let the quadrances of $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ by Q_1, Q_2 and Q_3 as before. Then from the Triangle Apollonian Circles theorem, we may set

$$\begin{aligned} M_1 &\equiv (Q_2 - Q_3)^2 = \frac{Q_1 Q_2 Q_3}{R_1} \\ M_2 &\equiv (Q_3 - Q_1)^2 = \frac{Q_1 Q_2 Q_3}{R_2} \\ M_3 &\equiv (Q_1 - Q_2)^2 = \frac{Q_1 Q_2 Q_3}{R_3}. \end{aligned}$$

Then since

$$(Q_2 - Q_3) + (Q_3 - Q_1) + (Q_1 - Q_2) = 0$$

Archimedes' linear relation shows that

$$0 = \mathcal{A}(M_1, M_2, M_3) = M_1^2 + M_2^2 + M_3^2 - 2M_1M_2 - 2M_2M_3 - 2M_3M_1$$

so we get

$$\begin{aligned} 0 &= Q_1^2Q_2^2Q_3^2\left(\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2} - \frac{2}{R_1R_2} - \frac{2}{R_2R_3} - \frac{2}{R_3R_1}\right) \\ &= Q_1^2Q_2^2Q_3^2\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 - 2\kappa_1\kappa_2 - 2\kappa_2\kappa_3 - 2\kappa_3\kappa_1\right) \\ &= Q_1^2Q_2^2Q_3^2\mathcal{A}(\kappa_1, \kappa_2, \kappa_3). \end{aligned}$$

By assumption all the quadrances Q_i are non-zero, hence it follows that $\mathcal{A}(\kappa_1, \kappa_2, \kappa_3) = 0$. □

Corollary 3. *The quadrances R_1, R_2, R_3 of the Apollonian circles of a triangle satisfy*

$$R_1^2R_2^2 + R_1^2R_3^2 + R_2^2R_3^2 - 2R_1R_2R_3(R_1 + R_2 + R_3) = 0.$$

Proof. This follows from the proof of the previous result together with the identity

$$\begin{aligned} &\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2} - \frac{2}{R_1R_2} - \frac{2}{R_2R_3} - \frac{2}{R_3R_1} \\ &= \frac{R_1^2R_2^2 + R_1^2R_3^2 + R_2^2R_3^2 - 2R_1R_2R_3(R_1 + R_2 + R_3)}{R_1^2R_2^2R_3^2}. \end{aligned}$$

□

Suppose that the Apollonian circles of a triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ have square quadrances $R_1 = r_1^2, R_2 = r_2^2$ and $R_3 = r_3^2$. Since the radii r_1, r_2 and r_3 are in general determined only up to sign, the corresponding (classical) curvatures $k_1 \equiv r_1^{-1}, k_2 \equiv r_2^{-1}$, and $k_3 \equiv r_3^{-1}$ are also determined only up to signs. Then

$$\mathcal{A}\left(k_1^2, k_2^2, k_3^2\right) = 0$$

and hence

$$k_1 \pm k_2 \pm k_3 = 0.$$

In other words, there is a linear relation between the three classical curvatures of the Apollonian circles of a circum-radial triangle; one of them is the sum of the other two. Which relation actually holds is an ambiguous question, as these curvatures are in general defined only up to sign.

Suppose now that the Apollonian circles of a triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ have quadrances which are negatives of squares, namely $R_1 = -r_1^2, R_2 = -r_2^2$ and $R_3 = -r_3^2$. Since these new radii r_1, r_2 and r_3 are in general determined only up to sign, the corresponding (classical) curvatures $k_1 \equiv r_1^{-1}, k_2 \equiv r_2^{-1}$, and $k_3 \equiv r_3^{-1}$ are also determined only up to signs. Then

$$\mathcal{A}\left(-k_1^2, -k_2^2, -k_3^2\right) = 0$$

and hence by Archimedes' linear relation II, we may conclude that

$$k_1 \pm k_2 \pm k_3 = 0.$$

This is the kind of relation that we might witness in the red and/or green geometries.

Example 19. *The triangle* $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ *with vertices* $\mathbf{A}_1 \equiv [4,2], \mathbf{A}_2 \equiv [0,0]$ *and* $\mathbf{A}_3 \equiv [5,0]$ *is circum-radial in both blue and red geometries as :*

Blue			*Red*		
$Q_1 = 25$	$Q_2 = 5$	$Q_3 = 20$	$Q_1 = 25$	$Q_2 = -3$	$Q_3 = 12$
$R_1 = \frac{100}{9}$	$R_2 = 100$	$R_3 = \frac{25}{4}$	$R_1 = -4$	$R_2 = -\frac{900}{169}$	$R_3 = -\frac{225}{196}$
$K = \frac{25}{4}$			$K = \frac{9}{4}$		
$A = 400$			$A = -400$		

In the blue geometry, since the Apollonian quadrances are squares, we can define the (blue) curvatures, each up to a sign, by the rule $k_i^2 = R_i$ *for* $i = 1,2,3$ *to get*

$$k_1 = \frac{3}{10} \qquad k_2 = \frac{1}{10} \qquad k_3 = \frac{2}{5}.$$

Note that

$$k_1 + k_2 = \frac{3}{10} + \frac{1}{10} = \frac{2}{5} = k_3.$$

In the red geometry, the Apollonian quadrances are signed squares, and so we can define the red curvatures, each again up to a sign, by the rule $k_i^2 = -R_i$ *for* $i = 1,2,3$ *to get*

$$k_1 = \frac{1}{2} \qquad k_2 = \frac{13}{30} \qquad k_3 = \frac{14}{15}.$$

Note that

$$k_1 + k_2 = \frac{1}{2} + \frac{13}{30} = \frac{14}{15} = k_3.$$

3.3. **Collinearity of centers and Lemoine axis.** We may now easily extend some other classical results associated to Apollonian circles to this wider context.

Theorem 20 (Lemoine axis)**.** *The centers of the three Apollonian circles of the triangle* $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ *with quadrances* Q_1, Q_2 *and* Q_3 *are collinear, along a line with direction vector*

$$\mathbf{z} = Q_1\left(Q_2 - Q_3\right)\mathbf{A}_1 + Q_2\left(Q_3 - Q_1\right)\mathbf{A}_2 + Q_3\left(Q_1 - Q_2\right)\mathbf{A}_3.$$

Proof. We note that $\mathbf{z}$ is indeed a 0-combination of the points of the triangle, so it represents a vector. With the centers of the Apollonian circles $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ as in Equation (5), we compute that

$$\begin{aligned}
\overrightarrow{\mathbf{C}_1\mathbf{C}_2} &= \mathbf{C}_2 - \mathbf{C}_1 \\
&= \left(\frac{Q_1}{Q_1 - Q_3}\mathbf{A}_1 + \frac{Q_3}{Q_3 - Q_1}\mathbf{A}_3\right) - \left(\frac{Q_2}{Q_2 - Q_3}\mathbf{A}_2 + \frac{Q_3}{Q_3 - Q_2}\mathbf{A}_3\right) \\
&= \frac{\left(Q_1\left(Q_2 - Q_3\right)\mathbf{A}_1 + Q_2\left(Q_3 - Q_1\right)\mathbf{A}_2 + Q_3\left(Q_1 - Q_2\right)\mathbf{A}_3\right)}{\left(Q_1 - Q_3\right)\left(Q_2 - Q_3\right)} \\
&= \frac{1}{\left(Q_1 - Q_3\right)\left(Q_2 - Q_3\right)}\mathbf{z}
\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{\mathbf{C}_1\mathbf{C}_3} &= \mathbf{C}_3 - \mathbf{C}_1 \\ &= \left(\frac{Q_1}{Q_1-Q_2}\mathbf{A}_1 + \frac{Q_2}{Q_2-Q_1}\mathbf{A}_2\right) - \left(\frac{Q_2}{Q_2-Q_3}\mathbf{A}_2 + \frac{Q_3}{Q_3-Q_2}\mathbf{A}_3\right) \\ &= \frac{\left(Q_1\left(Q_2-Q_3\right)\mathbf{A}_1 + Q_2\left(Q_3-Q_1\right)\mathbf{A}_2 + Q_3\left(Q_1-Q_2\right)\mathbf{A}_3\right)}{\left(Q_1-Q_2\right)\left(Q_2-Q_3\right)} \\ &= \frac{1}{\left(Q_1-Q_2\right)\left(Q_2-Q_3\right)}\mathbf{z}.\end{aligned}$$

Hence $\overrightarrow{\mathbf{C}_1\mathbf{C}_2}$ and $\overrightarrow{\mathbf{C}_1\mathbf{C}_3}$ are parallel, and so $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_3$ are collinear. □

As another consequence of the formula in the proof, we may deduce

Theorem 21 (Apollonian center quadrances)**.** *The quadrances between the points* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *is*

$$Q\left(\mathbf{C}_1, \mathbf{C}_2\right) = \frac{Q_1Q_2Q_3}{\left(Q_1-Q_3\right)^2\left(Q_2-Q_3\right)^2}T$$

where

$$T \equiv Q_1^2 + Q_2^2 + Q_3^2 - Q_1Q_2 - Q_1Q_3 - Q_2Q_3.$$

Proof. Clearly

$$Q\left(\mathbf{C}_1, \mathbf{C}_2\right) = Q\left(\overrightarrow{\mathbf{C}_1\mathbf{C}_2}\right) = \frac{1}{\left(Q_1-Q_3\right)^2\left(Q_2-Q_3\right)^2}Q\left(\mathbf{z}\right).$$

But from the Triangle vector quadrance corollary,

$$\begin{aligned}Q\left(\mathbf{z}\right) &= -Q_2\left(Q_3-Q_1\right)Q_3\left(Q_1-Q_2\right)Q_1 - Q_1\left(Q_2-Q_3\right)Q_3\left(Q_1-Q_2\right)Q_2 \\ &\quad -Q_1\left(Q_2-Q_3\right)Q_2\left(Q_3-Q_1\right)Q_3 \\ &= Q_1Q_2Q_3\left(Q_1^2+Q_2^2+Q_3^2-Q_1Q_2-Q_1Q_3-Q_2Q_3\right) \\ &= Q_1Q_2Q_3T\end{aligned}$$

so the result follows. □

The common line $\mathbf{C}_1\mathbf{C}_2 = \mathbf{C}_2\mathbf{C}_3$ is also known as the **Lemoine axis** ℓ_L of the triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$.

3.4. Brocard axis.

There is another important and famous line related to the Apollonian circles of a triangle, called the Brocard axis, but our derivation of it must depart from the usual description, which does not work generally.

Theorem 22 (Brocard axis)**.** *The three radical axes of the three Apollonian circles of a triangle coincide, and this common axis is perpendicular to the Lemoine axis. The radical base of any two of the Apollonian circles is the Schoute center*

$$\mathbf{S} \equiv \frac{Q_1\left(2Q_1-Q_2-Q_3\right)}{2T}\mathbf{A}_1 + \frac{Q_2\left(2Q_2-Q_1-Q_3\right)}{2T}\mathbf{A}_2 + \frac{Q_3\left(2Q_3-Q_1-Q_2\right)}{2T}\mathbf{A}_3$$

where

$$T \equiv Q_1^2 + Q_2^2 + Q_3^2 - Q_1Q_2 - Q_1Q_3 - Q_2Q_3.$$

Proof. From the previous theorem, and the fact that the centers $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ of the Apollonian circles are collinear, it follows that the three radical axes of the three circles must all be parallel, perpendicular to the Lemoine axis. If we apply the above equation to the case of $\mathcal{C}_1$ and $\mathcal{C}_2$ we obtain that the radical base of these two circles is the point

$$\mathbf{P} \equiv \left(\frac{1}{2} + \frac{R_2 - R_1}{2\left(C_2 - C_1\right)^2}\right)\mathbf{C}_1 + \left(\frac{1}{2} + \frac{R_1 - R_2}{2\left(C_2 - C_1\right)^2}\right)\mathbf{C}_2$$

From the Apollonian center quadrances theorem we have

$$\left(\mathbf{C}_2 - \mathbf{C}_1\right)^2 = Q\left(\mathbf{C}_1, \mathbf{C}_2\right) = \frac{Q_1 Q_2 Q_3}{\left(Q_1 - Q_3\right)^2 \left(Q_2 - Q_3\right)^2} T$$

and from the Triangle Apollonian circles theorem we know that

$$R_1 = \frac{Q_1 Q_2 Q_3}{\left(Q_2 - Q_3\right)^2} \qquad \text{and} \qquad R_2 = \frac{Q_1 Q_2 Q_3}{\left(Q_1 - Q_3\right)^2}.$$

Putting these together gives

$$\begin{aligned}
\mathbf{P} &= \frac{1}{2}\left(Q_3 - Q_2\right)\frac{Q_1 - 2Q_2 + Q_3}{T}\mathbf{C}_1 + \frac{1}{2}\left(Q_3 - Q_1\right)\frac{Q_2 - 2Q_1 + Q_3}{T}\mathbf{C}_2 \\
&= \frac{1}{2}\left(Q_3 - Q_2\right)\frac{Q_1 - 2Q_2 + Q_3}{T}\left(\frac{Q_2}{Q_2 - Q_3}\mathbf{A}_2 + \frac{Q_3}{Q_3 - Q_2}\mathbf{A}_3\right) \\
&\quad + \frac{1}{2}\left(Q_3 - Q_1\right)\frac{Q_2 - 2Q_1 + Q_3}{T}\left(\frac{Q_3}{Q_3 - Q_1}\mathbf{A}_3 + \frac{Q_1}{Q_1 - Q_3}\mathbf{A}_1\right) \\
&= \frac{Q_1\left(2Q_1 - Q_2 - Q_3\right)}{2T}\mathbf{A}_1 + \frac{Q_2\left(2Q_2 - Q_1 - Q_3\right)}{2T}\mathbf{A}_2 + \frac{Q_3\left(2Q_3 - Q_1 - Q_2\right)}{2T}\mathbf{A}_3 \\
&= \mathbf{S}.
\end{aligned}$$

Since this is now a symmetric expression in the indices, we may deduce that the radical bases of the other two pairs of circles are also $\mathbf{S}$. Since all three radical axes are parallel and pass through $\mathbf{S}$, they are identical. $\mathbf{S}$ is identified as X(187) in [5], [6]. ☐

This common radical axis of the Apollonian circles of a triangle is the **Brocard axis** ℓ_B of the triangle $\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$. The traditional description is the line joining the common intersections of the Apollonian circles ([2, p.218]), but such intersections may not exist, as we show in the Isodynamic points criterion theorem. The common radical axis is therefore a broader definition that establishes this line of a triangle over general fields and metrical geometries.

Corollary 4. *The Lemoine axis ℓ_L and the Brocard axis ℓ_B can now be respectively expressed in terms of a variable point $\mathbf{V}$ as*

$$\ell_L : \mathbf{z} \times (\mathbf{V} - \mathbf{S}) = 0 \qquad \textit{and} \qquad \ell_B : \mathbf{z} \cdot (\mathbf{V} - \mathbf{S}) = 0.$$

Proof. Since $\mathbf{S}$ is the common radical base of the Apollonian circles, ℓ_L and ℓ_B meet perpendicularly at $\mathbf{S}$. From the Lemoine axis theorem, $\mathbf{z}$ is the direction vector of ℓ_L, so the two axes can be expressed as above. ☐

3.5. **Intersection with Apollonian circles.** By the Dividing point theorem we know the Apollonian circle $\mathcal{C}_1$ intersects the side $\mathbf{A}_2\mathbf{A}_3$ precisely when Q_2Q_3 is a square. Similarly, $\mathcal{C}_2$ and $\mathcal{C}_3$ respectively intersect $\mathbf{A}_3\mathbf{A}_1$ and $\mathbf{A}_1\mathbf{A}_2$ precisely when Q_3Q_1 and Q_1Q_2 are squares. The points where $\mathcal{C}_1$ intersects $\mathbf{A}_2\mathbf{A}_3$ (when Q_2Q_3 is a square) also lie on the bilines of $\mathbf{A}_1$ which are classically known as the interior and exterior angle bisectors ([1, p.261]). This result is consistent with the conditions for bilines ([7, p.8]).

While a picture might suggest that circles do or do not meet, in fact this is a number theoretical condition, depending on the field as well as the respective equations.

Theorem 23 (Isodynamic points criterion)**.** *For a given geometry over a given field $\mathbb{F}$, the Apollonian circles of a triangle meet precisely when 3δ is a square in $\mathbb{F}$, where $\delta = \det B$ is the determinant of the quadratic form of the geometry.*

Proof. If two circles in a pencil meet in two points, then the line through those two points is the radical axis of the pencil. So finding the meets of two such circles can be replaced by the somewhat simpler problem of finding the meet of either circle with the common radical axis. In this case, we need to find the intersection between the Brocard axis and one of the Apollonian circles.

Let $\mathbf{w}$ be a vector perpendicular to $\mathbf{z}$, then a point on the Brocard axis has the form $\mathbf{V} = \mathbf{S} + \lambda\mathbf{w}$ and this lies on the Apollonian circle $\mathcal{C}_1$ with center $\mathbf{C}_1$ precisely when

$$(\mathbf{V} - \mathbf{C}_1)^2 = \left(\mathbf{S} + \lambda\mathbf{w} - \left(\frac{Q_2}{Q_2 - Q_3}\mathbf{A}_2 + \frac{Q_3}{Q_3 - Q_2}\mathbf{A}_3\right)\right)^2 = \frac{Q_1Q_2Q_3}{(Q_2 - Q_3)^2}.$$

By the triangle normal vector theorem, we have

$$\begin{aligned}\mathbf{w} \quad = \quad & \frac{1}{2}Q_1\left(Q_2^2 + Q_3^2 - Q_1Q_2 - Q_1Q_3\right)\mathbf{A}_1 + \frac{1}{2}Q_2\left(Q_1^2 + Q_3^2 - Q_2Q_1 - Q_2Q_3\right)\mathbf{A}_2 \\ & + \frac{1}{2}Q_3\left(Q_1^2 - Q_3Q_1 + Q_2^2 - Q_3Q_2\right)\mathbf{A}_3.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{V} - \mathbf{C}_1 \quad = \quad & \mathbf{S} + \lambda\mathbf{w} - \left(\frac{Q_2}{Q_2 - Q_3}\mathbf{A}_2 + \frac{Q_3}{Q_3 - Q_2}\mathbf{A}_3\right) \\ = \quad & \frac{Q_1(2Q_1 - Q_2 - Q_3)}{2T}\mathbf{A}_1 + \frac{Q_2(2Q_2 - Q_1 - Q_3)}{2T}\mathbf{A}_2 + \frac{Q_3(2Q_3 - Q_1 - Q_2)}{2T}\mathbf{A}_3 \\ & + \frac{\lambda}{2}Q_1\left(Q_2^2 + Q_3^2 - Q_1Q_2 - Q_1Q_3\right)\mathbf{A}_1 + \frac{\lambda}{2}Q_2\left(Q_1^2 + Q_3^2 - Q_2Q_1 - Q_2Q_3\right)\mathbf{A}_2 \\ & + \frac{\lambda}{2}Q_3\left(Q_1^2 - Q_3Q_1 + Q_2^2 - Q_3Q_2\right)\mathbf{A}_3 - \left(\frac{Q_2}{Q_2 - Q_3}\mathbf{A}_2 + \frac{Q_3}{Q_3 - Q_2}\mathbf{A}_3\right)\end{aligned}$$

This can be simplified to $\alpha_1\mathbf{A}_1+\alpha_2\mathbf{A}_2+\alpha_3\mathbf{A}_3$ where

$$\begin{aligned}
\alpha_1 &= -\frac{\left(Q_1^2Q_2^2-Q_1^2Q_3^2-Q_1Q_2^3+Q_1Q_2^2Q_3-Q_1Q_2Q_3^2+Q_1Q_3^3\right)T\lambda}{2T\left(Q_2-Q_3\right)}\\
&\quad -\frac{\left(2Q_1^2Q_3-2Q_1^2Q_2+Q_1Q_2^2-Q_1Q_3^2\right)}{2T\left(Q_2-Q_3\right)}\\
\alpha_2 &= -\frac{\left(Q_1^2Q_2Q_3-Q_1^2Q_2^2+Q_1Q_2^3-Q_1Q_2^2Q_3+Q_2^3Q_3-2Q_2^2Q_3^2+Q_2Q_3^3\right)T\lambda}{2T\left(Q_2-Q_3\right)}\\
&\quad -\frac{2Q_2T+\left(3Q_2^2Q_3-2Q_2^3+Q_1Q_2^2-Q_2Q_3^2-Q_1Q_2Q_3\right)}{2T\left(Q_2-Q_3\right)}\\
\alpha_3 &= \frac{\left(Q_1^2Q_2Q_3-Q_1^2Q_3^2-Q_1Q_2Q_3^2+Q_1Q_3^3+Q_2^3Q_3-2Q_2^2Q_3^2+Q_2Q_3^3\right)T\lambda}{2T\left(Q_2-Q_3\right)}\\
&\quad +\frac{2Q_3T+\left(3Q_2Q_3^2-Q_2^2Q_3-Q_1Q_2Q_3-2Q_3^3+Q_1Q_3^2\right)}{2T\left(Q_2-Q_3\right)}.
\end{aligned}$$

From the definition of T,

$$\alpha_1+\alpha_2+\alpha_3=-\frac{1}{T}\left(-Q_1^2+Q_1Q_2+Q_1Q_3-Q_2^2+Q_2Q_3-Q_3^2+T\right)=0$$

so by the Triangle vector quadrance corollary we may compute that

$$Q\left(\mathbf{V}-\mathbf{C}_1\right)=\left(\mathbf{V}-\mathbf{C}_1\right)^2=-\alpha_2\alpha_3Q_1-\alpha_1\alpha_3Q_2-\alpha_1\alpha_2Q_3$$

and

$$\begin{aligned}
&\left(\mathbf{V}-\mathbf{C}_1\right)^2-\frac{Q_1Q_2Q_3}{\left(Q_2-Q_3\right)^2}\\
&= -\alpha_2\alpha_3Q_1-\alpha_1\alpha_3Q_2-\alpha_1\alpha_2Q_3-\frac{Q_1Q_2Q_3}{\left(Q_2-Q_3\right)^2}\\
&= \frac{Q_1Q_2Q_3}{4T}\left(\mathcal{A}T^2\lambda^2-3\right)=0
\end{aligned}$$

where

$$\mathcal{A}=2Q_1Q_2+2Q_1Q_3+2Q_2Q_3-Q_1^2-Q_2^2-Q_3^2.$$

By the Quadrea theorem, we have

$$\begin{aligned}
\mathcal{A} &= 16\,\delta\,\mathrm{area}^2\left(\overline{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}\right)\\
\delta &= \det B.
\end{aligned}$$

Hence, λ has solution precisely when 3δ is a square in the field, and the intersections, when exist, are called the **Isodynamic Points**. We also note that

$$2T=\left(Q_1-Q_2\right)^2+\left(Q_2-Q_3\right)^2+\left(Q_3-Q_1\right)^2$$

so $T=0$ over the rational field precisely when $Q_1=Q_2=Q_3$, i.e. when the triangle is equilateral. However, this condition is not necessarily true in finite fields, and there are triangles whose Apollonian circles do not meet. □

Corollary 5. *The Brocard axis does not intersect the Apollonian circles over $\mathbb{Q}$. In fact, the Apollonian circles do not intersect in red or green geometry even over $\mathbb{R}$.*

Theorem 24 (Lemoine axis intersection)**.** *The Apollonian circles meet the Lemoine axis precisely when T is a square.*

Proof. A point on the Lemoine axis has the form $\mathbf{V} = \mathbf{S} + \lambda \mathbf{z}$ and this lies on the Apollonian circle $\mathcal{C}_1$ with center $\mathbf{C}_1$ precisely when

$$(\mathbf{V} - \mathbf{C}_1)^2 = \left(\mathbf{S} + \lambda \mathbf{z} - \left(\frac{Q_2}{Q_2 - Q_3}\mathbf{A}_2 + \frac{Q_3}{Q_3 - Q_2}\mathbf{A}_3\right)\right)^2 = \frac{Q_1 Q_2 Q_3}{(Q_2 - Q_3)^2}.$$

Now from the Brocard axis theorem and the definition of $\mathbf{z} = Q_1 (Q_2 - Q_3) \mathbf{A}_1 + Q_2 (Q_3 - Q_1) \mathbf{A}_2 + Q_3 (Q_1 - Q_2) \mathbf{A}_3$, we may compute

$$\begin{aligned}
& \mathbf{V} - \mathbf{C}_1 = \mathbf{S} + \lambda \mathbf{z} - \left(\frac{Q_2}{Q_2 - Q_3}\mathbf{A}_2 + \frac{Q_3}{Q_3 - Q_2}\mathbf{A}_3\right) \\
= \; & \frac{Q_1 (2Q_1 - Q_2 - Q_3)}{2T}\mathbf{A}_1 + \frac{Q_2 (2Q_2 - Q_1 - Q_3)}{2T}\mathbf{A}_2 + \frac{Q_3 (2Q_3 - Q_1 - Q_2)}{2T}\mathbf{A}_3 \\
& +\lambda (Q_1 (Q_2 - Q_3) \mathbf{A}_1 + Q_2 (Q_3 - Q_1) \mathbf{A}_2 + Q_3 (Q_1 - Q_2) \mathbf{A}_3) \\
& -\left(\frac{Q_2}{Q_2 - Q_3}\mathbf{A}_2 + \frac{Q_3}{Q_3 - Q_2}\mathbf{A}_3\right) \\
= \; & \left(\frac{1}{T}Q_1^2 - \frac{1}{2T}Q_1 Q_2 - \frac{1}{2T}Q_1 Q_3 + \lambda Q_1 Q_2 - \lambda Q_1 Q_3\right)\mathbf{A}_1 \\
& + \left(\frac{1}{T}Q_2^2 - \frac{Q_2}{Q_2 - Q_3} - \frac{1}{2T}Q_1 Q_2 - \frac{1}{2T}Q_2 Q_3 - \lambda Q_1 Q_2 + \lambda Q_2 Q_3\right)\mathbf{A}_2 \\
& + \left(\frac{1}{T}Q_3^2 + \frac{Q_3}{Q_2 - Q_3} - \frac{1}{2T}Q_1 Q_3 - \frac{1}{2T}Q_2 Q_3 + \lambda Q_1 Q_3 - \lambda Q_2 Q_3\right)\mathbf{A}_3 \\
= \; & \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2 + \alpha_3 \mathbf{A}_3.
\end{aligned}$$

Now from the definition of T,

$$\begin{aligned}
\alpha_1 + \alpha_2 + \alpha_3 &= \left(\frac{1}{T}Q_1^2 - \frac{1}{2T}Q_1 Q_2 - \frac{1}{2T}Q_1 Q_3 + \lambda Q_1 Q_2 - \lambda Q_1 Q_3\right) \\
&\quad + \left(\frac{1}{T}Q_2^2 - \frac{Q_2}{Q_2 - Q_3} - \frac{1}{2T}Q_1 Q_2 - \frac{1}{2T}Q_2 Q_3 - \lambda Q_1 Q_2 + \lambda Q_2 Q_3\right) \\
&\quad + \left(\frac{1}{T}Q_3^2 + \frac{Q_3}{Q_2 - Q_3} - \frac{1}{2T}Q_1 Q_3 - \frac{1}{2T}Q_2 Q_3 + \lambda Q_1 Q_3 - \lambda Q_2 Q_3\right) \\
&= -\frac{1}{T}\left(T - Q_1^2 - Q_2^2 - Q_3^2 + Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3\right) = 0
\end{aligned}$$

so by the Triangle vector quadrance corollary we may compute that

$$\begin{aligned}
Q(\mathbf{V} - \mathbf{C}_1) &= (\mathbf{V} - \mathbf{C}_1)^2 \\
&= -\alpha_2 \alpha_3 Q_1 - \alpha_1 \alpha_3 Q_2 - \alpha_1 \alpha_2 Q_3 \\
&= \frac{1}{4T}\frac{Q_1 Q_2 Q_3}{(Q_3 - Q_2)^2}\left(4T^2 (Q_2 - Q_3)^2 \lambda^2 + 4T (Q_2 - Q_3)(2Q_1 - Q_2 - Q_3)\lambda\right) \\
&\quad + \frac{1}{4T}\frac{Q_1 Q_2 Q_3}{(Q_3 - Q_2)^2}\left(4T - 3 (Q_2 - Q_3)^2\right) \\
&= \frac{1}{4T}\frac{Q_1 Q_2 Q_3}{(Q_3 - Q_2)^2}\left(2 (Q_3 - Q_2) T\lambda + (Q_2 - 2Q_1 + Q_3)\right)^2
\end{aligned}$$

So the Brocard axis and the Apollonian circle $\mathcal{C}_1$ will meet precisely when we can solve the quadratic equation in λ

$$\frac{1}{4T}\frac{Q_1Q_2Q_3}{(Q_3-Q_2)^2}\left(2\left(Q_3-Q_2\right)T\lambda+\left(Q_2-2Q_1+Q_3\right)\right)^2=\frac{Q_1Q_2Q_3}{(Q_2-Q_3)^2}$$

or more simply

$$\left(2\left(Q_3-Q_2\right)T\lambda+\left(Q_2-2Q_1+Q_3\right)\right)^2=4T.$$

This clearly has a solution precisely when T is a square. □

4. Triangles in Finite Fields

Since for the blue geometry $\delta=1$ while for the red and green geometries $\delta=-1$, the condition that a triangle have isodynamic points in all three colours is that -1 and 3 are both squares in the field. This does not happen over the rational numbers, nor over any real extension of the rational numbers. It is, however, possible with triangles over finite fields. A quick computer search shows both 3 and -1 are squares in the following prime fields (among the first 100 primes):

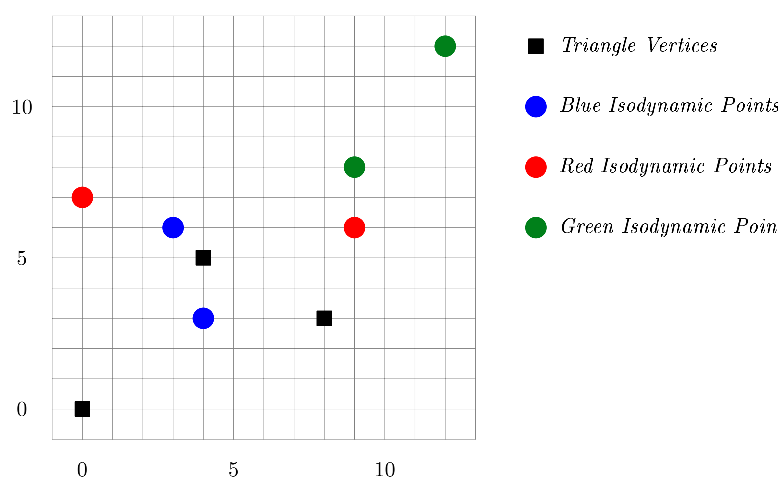
$$13,37,61,73,97,109,157,181,193,229,241,277,$$
$$313,337,349,373,397,409,421,433,457,541\ldots$$

Example 25. *Over $\mathbb{F}_{13}$ a triangle with vertices*

$$\mathbf{A}_1=[4,5], \qquad \mathbf{A}_2=[0,0], \qquad \mathbf{A}_3=[8,3]$$

has

	Blue	*Red*	*Green*
Lemoine axis	$6+11x+6y=0$	$11+12x+10y=0$	$5+12x+9y=0$
Brocard axis	$4+7x+11y=0$	$6+3x+y=0$	$3+12x+4y=0$
Isodynamic points	$[3,6],[4,3]$	$[0,7],[9,6]$	$[12,12],[9,8]$



Another aspect is to look for circum-radial triangles. While it is easy to do so in any color of geometry, finding a triangle that is circum-radial in all colors is, perhaps, a number theoretical challenge on its own. For example, we can choose three points on a circle of rational radius in blue geometry (say $x^2+y^2=25$), such as $[3,4]$ or $[0,5]$ and $[4,-3]$, but such triangle is not circum-radial in other colors.

We found better news in finite fields. A computer search found the following results of the number of circum-radial triangles over some small prime fields. These are unique circum-radial triangles in all three colors of geometry with non-null sides.

$\mathbb{F}_{11}$	15
$\mathbb{F}_{13}$	0
$\mathbb{F}_{17}$	256
$\mathbb{F}_{19}$	378
$\mathbb{F}_{23}$	968
$\mathbb{F}_{29}$	2,632
$\mathbb{F}_{31}$	4,530
$\mathbb{F}_{37}$	8,424
$\mathbb{F}_{41}$	15,040
$\mathbb{F}_{43}$	20,790
$\vdots$	
$\mathbb{F}_{127}$	2,273,922
$\vdots$	

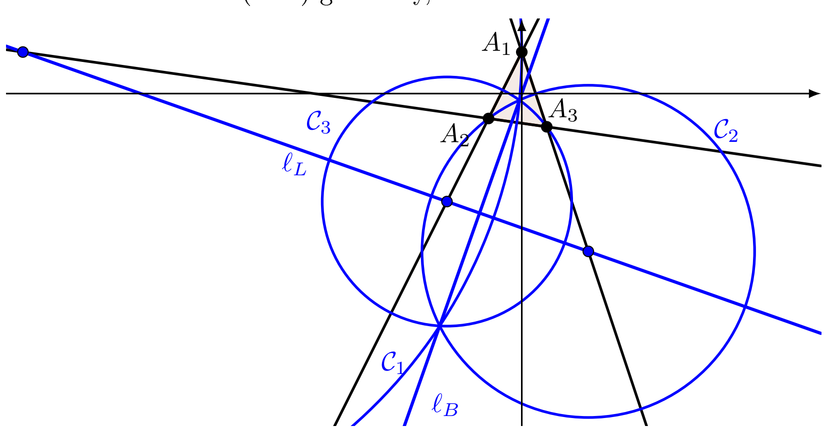
It is unclear why there is no triangle in $\mathbb{F}_{13}$ that is circum-radial in all colors, especially when the number in $\mathbb{F}_{11}$ is not trivial.

5. Explicit Examples

(1) In the triangle with vertices

$$\mathbf{A}_1=[0,5]\,,\qquad \mathbf{A}_2=[-4,-3]\,,\qquad \mathbf{A}_3=[3,-4]$$

then in the Euclidean (blue) geometry,



we have the following quadrances

$$Q_1=50\qquad Q_2=90\qquad Q_3=80.$$

The Apollonian circles, Lemoine and Brocard Axes have equations

$$\begin{aligned}
\mathcal{C}_1 &: (x+60)^2+(y-5)^2 = 3600 = (60)^2\\
\mathcal{C}_2 &: (x-8)^2+(y+19)^2 = 400 = (20)^2\\
\mathcal{C}_3 &: (x+9)^2+(y+13)^2 = 225 = (15)^2\\
\ell_L &: 6x+17y+275=0\\
\ell_B &: 17x-6y=0
\end{aligned}$$

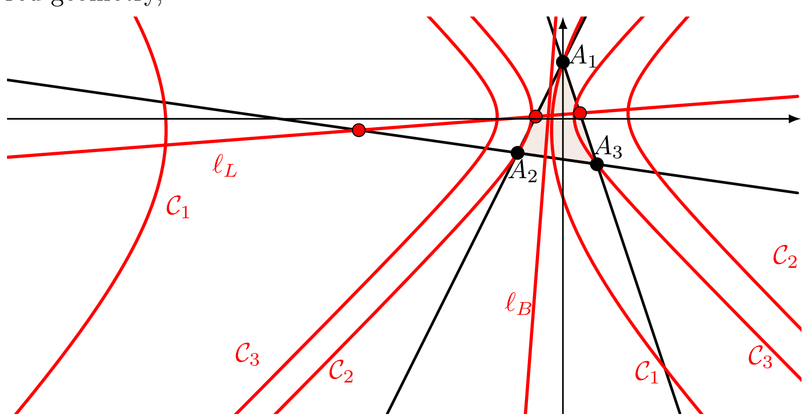
with quadratic curvatures and corresponding (positive) curvatures

$$\begin{aligned}
K_1 &= \frac{1}{3600} \quad \text{and} \quad k_1 = \frac{1}{60}\\
K_2 &= \frac{1}{400} \quad \text{and} \quad k_2 = \frac{1}{20}\\
K_3 &= \frac{1}{225} \quad \text{and} \quad k_3 = \frac{1}{15}.
\end{aligned}$$

We may check that indeed

$$k_1 + k_2 = k_3.$$

In red geometry,



we have the following quadrances

$$Q_1 = 48 \qquad Q_2 = -72 \qquad Q_3 = -48.$$

The Apollonian circles, Lemoine and Brocard Axes have equations

$$\begin{aligned}
\mathcal{C}_1 &: (x+18)^2-(y+1)^2 = 288 = (12r_2)^2\\
\mathcal{C}_2 &: \left(x-\frac{3}{2}\right)^2-\left(y-\frac{1}{2}\right)^2 = 18 = (3r_2)^2\\
\mathcal{C}_3 &: \left(x+\frac{12}{5}\right)^2-\left(y-\frac{1}{5}\right)^2 = \frac{288}{25} = \left(\frac{12r_2}{5}\right)^2\\
\ell_L &: x-13y+5=0\\
\ell_B &: 13x-y+17=0
\end{aligned}$$

where

$$r_2^2 = 2$$

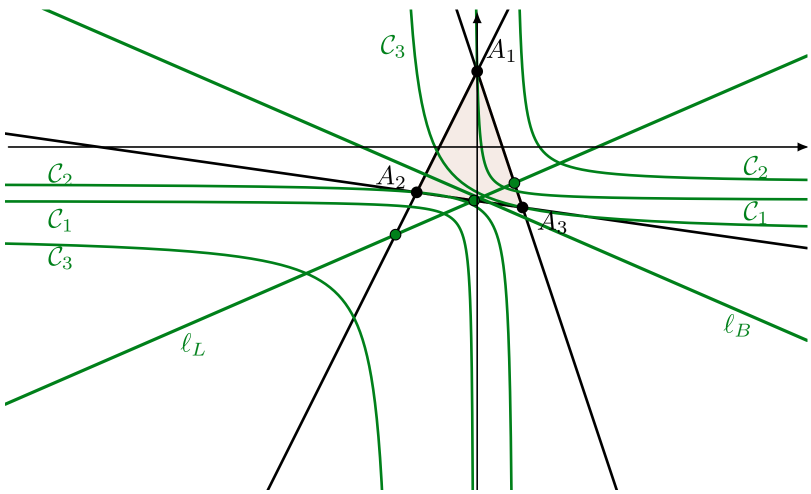
with quadratic curvatures and corresponding (positive) curvatures

$$K_1 = \frac{1}{288} \quad \text{and} \quad k_1 = \frac{1}{12r_2}$$
$$K_2 = \frac{1}{18} \quad \text{and} \quad k_2 = \frac{1}{3r_2}$$
$$K_3 = \frac{25}{288} \quad \text{and} \quad k_3 = \frac{5}{12r_2}.$$

and indeed

$$k_1 + k_2 = k_3.$$

In green geometry,



we have the following quadrances

$$Q_1 = -14 \qquad Q_2 = -54 \qquad Q_3 = 64.$$

The Apollonian circles, Lemoine and Brocard Axes have equations

$$\mathcal{C}_1 \ : \ 2\left(x+\frac{12}{59}\right)\left(y+\frac{209}{59}\right) = \frac{12096}{3481} = \left(\frac{24r_{21}}{59}\right)^2$$
$$\mathcal{C}_2 \ : \ 2\left(x-\frac{32}{13}\right)\left(y+\frac{31}{13}\right) = \frac{1344}{169} = \left(\frac{8r_{21}}{13}\right)^2$$
$$\mathcal{C}_3 \ : \ 2\left(x+\frac{27}{5}\right)\left(y+\frac{29}{5}\right) = \frac{756}{25} = \left(\frac{6r_{21}}{5}\right)^2$$
$$\ell_L \ : \ 222x - 511y - 1765 = 0$$
$$\ell_B \ : \ 222x + 511y + 1693 = 0$$

where

$$r_{21}^2 = 21$$

with quadratic curvatures and corresponding (positive) curvatures

$$K_1 = \frac{3481}{12096} \quad \text{and} \quad k_1 = \frac{59}{24r_{21}}$$
$$K_2 = \frac{169}{1344} \quad \text{and} \quad k_2 = \frac{13}{8r_{21}}$$
$$K_3 = \frac{25}{756} \quad \text{and} \quad k_3 = \frac{5}{6r_{21}}.$$

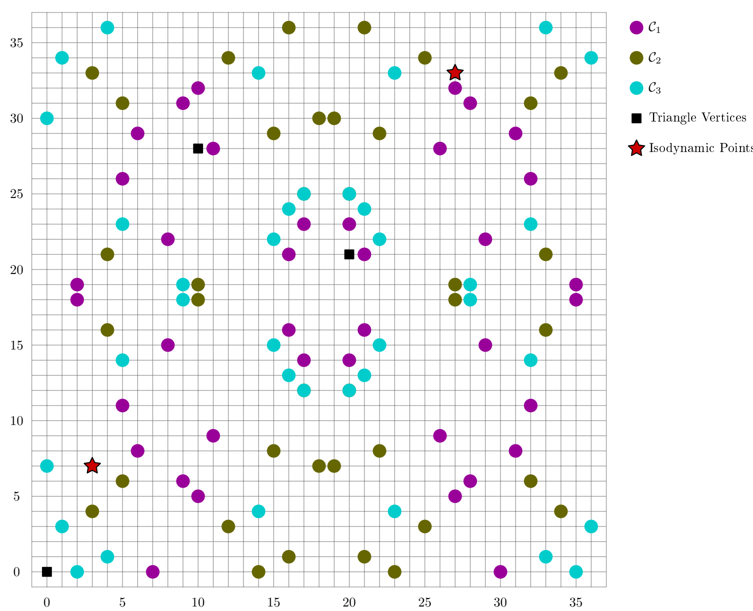
and indeed

$$k_2 + k_3 = k_1.$$

(2) Over $\mathbb{F}_{37}$ in the triangle with vertices

$$\mathbf{A}_1 = [0,0], \qquad \mathbf{A}_2 = [10,28], \qquad \mathbf{A}_3 = [20,21]$$

we have the three circles in the blue geometry



where I_1, I_2 are the Isodynamic Points.

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