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A REMARK ON NORMALIZATIONS IN A LOCAL LARGE DEVIATIONS PRINCIPLE FOR INHOMOGENEOUS BIRTH – AND – DEATH PROCESS

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Abstract. This work is a continuation of [13]. We consider a continuous-time birth – and – death process in which the transition rates are regularly varying function of the process position. We establish rough exponential asymptotic for the probability that a sample path of a normalized process lies in a neighborhood of a given nonnegative continuous function. We propose a variety of normalization schemes for which the large deviation functional preserves its natural integral form.

Keywords: birth – and – death process, normalization (scaling), large deviations principle, local large deviations principle, rate function.

1. Introduction

The study of birth – and – death processes provides an interesting topic, both theoretically and in a number of applications. As examples, the processes are popular modeling tools in evolution, population biology, genetics, and ecology, see, for example, the review [1], and [2]. Many important models in queuing theory, operations research, demography, economics and engineering can be represented by

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these processes, (see, [3], [4] and many others); we also would mention here models of competitive production and pricing, [6], [7]. For statistical inference in birth and death processes we recommend [5], where the authors also provided a good review about application of these processes.

We consider a continuous – time Markov process $\xi(t)$, $t \geq 0$, with state space $\mathbb{Z}^+ := \{0\} \cup \mathbb{N}$, and with $\xi(0) = 0$. The evolution of the process ξ is governed by the transition rates $\lambda(x) > 0$ for the jump $x \to x+1$, $x \in \mathbb{Z}^+$, and $\mu(x) > 0$ for the jump $x \to x-1$, $x \in \mathbb{N}$. For $x = 0$ we set $\mu(x) = 0$.

Further we need to consider functions $\lambda(x)$, $\mu(x)$ for all positive large enough values $x \in \mathbb{R}$ (not only integer). Thus, one can choose any reasonable extension for the functions (e.g. step – wise interpolation).

A key assumption is that the functions $\lambda(x)$, $\mu(x)$ are continuous and *regularly varying* at infinity:

$$\lambda(x) := y(x)x^l, \quad \mu(x) := z(x)x^m, \tag{1}$$

where $l, m \geq 0, l \neq m$ and hence $l \vee m > 0$ (here and below $l \vee m$ stands for the maximum of numbers l, m), $y(x)$, $z(x)$ are the *slowly varying* functions at infinity. Recall that function $a(x)$ is called slowly varying at infinity, if $\lim\limits_{x\to\infty} \frac{a(\beta x)}{a(x)} = 1$ for all $\beta > 0$ (see, e.g., [15] for more details).

When $l \geq 1$, the process ξ, generally speaking, can go to infinity ("explode") during a random time, finite with probability 1. There are two approaches to construct such processes: (i) one can stop the process at a random time point (the time of explosion) (see, e.g., [8, ch. 15, Section 4], [9, ch. 6]); (ii) one can extend the phase space $\mathbb{Z}^+$ by adding an absorbing state, denoted by ∞ (see, e.g., [10, ch. 4, Section 48]). We will work with events that exclude an explosion of the process in a given time – slot $0 \leq t \leq T$. Thus, for our results it makes no difference which approach is used.

We are interested in a *local large deviation principle* (LLDP) for the family of scaled processes

$$\xi_{\varphi,T}(t) := \frac{\xi(tT)}{\varphi(T)}, \ 0 \leq t \leq 1. \tag{2}$$

Here $T > 0$ is parameter and φ a positive function. The conditions upon φ is stated as follows:

$$\lim_{T\to\infty} \varphi(T) = \infty \ \text{ and } \ \lim_{T\to\infty} \frac{\varphi(T)\ln\big(\varphi(T)\big)}{\psi(T)} = 0. \tag{3}$$

where

$$\psi(T) := T\big(\lambda(\varphi(T)) \vee \mu(\varphi(T))\big). \tag{4}$$

Note that if $l \vee m > 1$ and $\lim\limits_{T\to\infty} \varphi(T) = \infty$ then obviously that the second equality in condition (3) holds.

Let $\mathbb{D}[0,1]$ denote the space of right – continuous functions with left – limit at each $t \in [0,1]$ (*càdlàg functions*). For any $f, g \in \mathbb{D}[0,1]$, set

$$\rho(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|.$$

Let us recall the definition of local large deviation principle.

Definition 1. *The family of random processes $\xi_{\varphi,T}$ satisfies the LLDP on the set $G \subseteq \mathbb{D}[0,1]$ with a rate functional $I = I(f) : \mathbb{D}[0,1] \to [0,\infty]$ and a normalising function $\psi(T)$ with $\lim\limits_{T\to\infty} \psi(T) = \infty$ if, for any function $f \in G$, the following equality holds true:*

$$\lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\xi_{\varphi,T} \in U_\varepsilon(f)) = \lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\xi_{\varphi,T} \in U_\varepsilon(f)) = -I(f),$$

where

$$U_\varepsilon(f) = \{g \in \mathbb{D}[0,1] : \ \rho(f,g) < \varepsilon\}.$$

Let $\mathbb{C}[0,1]$ denote the space of all continuous functions on $[0,1]$. We set

$$\mathbb{C}_+ := \{f \in \mathbb{C}[0,1] : \ f(0) = 0 \text{ and } f(t) > 0, \text{ when } 0 < t \leq 1\}.$$

Under conditions (1) and (3) we study the LLDP for the family of random processes $\xi_{\varphi,T}$ on the set $\mathbb{C}_+$. The point is that under the above formalism (1), (3) the rate functional $I(f)$ does not depend on the choice of φ and has a natural integral form:

$$I(f) = \int_0^1 f^{l\vee m}(t)dt. \tag{5}$$

In an earlier paper by the authors [13], a similar result was proved for constant functions $y(x)$, $z(x)$ and $\varphi(T) = T$. The present work is an attempt to answer the question to what extent the result of [13] can be generalized without changing the form of the functional $I(f)$. The second motivation comes from a comparison with the case of constant values $\lambda(x) \equiv \lambda$ and $\mu(x) \equiv \mu$ (the latter for $x \geq 1$). In our scheme, this happens when $l = m = 0$. Here, depending on the choice of the space−scaling factor $\varphi(T)$, one distinguishes between *moderate* (when $\varphi(T)/\sqrt{T} \to \infty$ and $\varphi(T)/T \to 0$), *large* (when $\varphi(T)/T \to C \in (0,\infty)$) and *super−large* (when $\varphi(T)/T \to \infty$) deviations, with different forms of $I(f)$ (see [11] for more details). It turns out that under the conditions introduced in the current paper, the large deviation functional preserves its form regardless of the choice of function φ.

The idea and the method of proof goes back to [6, 13, 14]; this provides certain limitations for the parameters of the scheme. We would like to note that the case $l = m$ is not covered by our condition (1) and hence is not considered in this paper, although it was included in [13] in a more specific situation. (In some sense, $l = m$ it is the most difficult case within the above formalism.)

The paper is organized as follows: in Section 2 we introduce our main result (Theorem 1) and key lemmas: Lemma 1 – 3. In Section 3 we prove Theorem 1 and the lemmas. In Section 4 we prove the auxiliary results.

2. Basic definitions and the main result

Theorem 1. *Under conditions* (1), (3) *the family of random processes $\xi_{\varphi,T}$ satisfies the* LLDP *on the set $\mathbb{C}_+$, with the normalized function $\psi(T)$ as in* (4) *and the rate function $I(f)$ as in* (5).

Remark 1. *For the Yule pure birth process ($l > 0$, $\mu(x) \equiv 0$; see for example* [12] *for the definition of the process) the rate function has the form*

$$I(f) = \int_0^1 f^l(t)dt, \quad f \in \mathbb{C}_M.$$

Here $\mathbb{C}_M$ is the set of continuous monotone increasing functions on $[0,1]$ starting from 0.

As in [6, 13], we consider an auxiliary Markov process $\zeta(t)$, $t \in [0,T]$, on $\mathbb{Z}$, homogeneous in time and space $\mathbb{Z}$, with rate 1 and equiprobable 1/2 jumps ± 1. Denote by $\mathbb{D}_{\pm 1}[0,T]$ the set of piecewise-constant càdlàg functions on the interval $[0,T]$ starting at zero with jumps ± 1.

For the function $u \in \mathbb{D}_{\pm 1}[0,T]$ define the number of jumps in the interval $[0,T]$ as $N_T(u)$ and the jump moments as $t_1, t_2, \dots, t_{N_T(u)}$ such that $0 = t_0 < t_1 < \dots < t_{N_T(u)} < T$. Further, let $\nu(u(t_{i-1}), u(t_i))$ is given by

$$\nu(u(t_{i-1}), u(t_i)) := \begin{cases} \lambda(u(t_{i-1})), & \text{if } u(t_i) - u(t_{i-1}) = 1, \\ \mu(u(t_{i-1})), & \text{if } u(t_i) - u(t_{i-1}) = -1. \end{cases} \tag{6}$$

Denote by $\tau_i = t_i - t_{i-1}$, $1 \le i \le N_T(u)$ time intervals between jumps of the function u.

The first auxiliary statement is Lemma 1 below; we give it without proof as it is straightforward, it follows from an independence of the waiting times between jumps of the processes ξ and ζ.

Lemma 1. (Cf. [6, 13].) *The distribution of the random process ξ on $\mathbb{D}_{\pm 1}[0,T]$ is absolutely continuous with respect to that of a process ζ. The corresponding density $\mathbf{p} = \mathbf{p}_T$ on $\mathbb{D}_{\pm 1}[0,T]$ (the Radon-Nikodym derivative) has the form:*

$$\mathbf{p}(u) = \begin{cases} 2^{N_T(u)} \left(\prod_{i=1}^{N_T(u)} e^{-(h(u(t_{i-1}))-1)\tau_i} \nu(u(t_{i-1}), u(t_i)) \right) & \\ \qquad \times e^{-(h(u(t_{N_T(u)}))-1))(T - t_{N_T(u)})}, & \text{if } N_T(u) \ge 1, \\ e^{-(h(0)-1)T}, & \text{if } N_T(u) = 0, \end{cases} \tag{7}$$

where $h := \lambda + \mu$.

Let $N_T(\zeta)$ be the number of jumps of $\zeta(t)$ on the interval $[0,T]$. The claim of Lemma 1 is equivalent to the fact that for any measurable set $G \subseteq \mathbb{D}_{\pm 1}[0,T]$

$$\mathbf{P}(\xi \in G) = e^T \mathbf{E}\big(e^{-A_T(\zeta)} e^{B_T(\zeta) + N_T(\zeta)\ln 2}; \zeta \in G\big), \tag{8}$$

where

$$\begin{aligned} A_T(\zeta) &:= \int_0^T h(\zeta(t))dt \\ &= \begin{cases} \sum_{i=1}^{N_T(\zeta)} h(\zeta(t_{i-1}))\tau_i + h(\zeta(t_{N_T(\zeta)}))(T - t_{N_T(\zeta)}), & \text{if } N_T(\zeta) \ge 1, \\ h(0)T, & \text{if } N_T(\zeta) = 0, \end{cases} \end{aligned} \tag{9}$$

and

$$B_T(\zeta) := \begin{cases} \sum_{i=1}^{N_T(\zeta)} \ln(\nu(\zeta(t_{i-1}), \zeta(t_i))), & \text{if } N_T(\zeta) \ge 1, \\ 0, & \text{if } N_T(\zeta) = 0. \end{cases} \tag{10}$$

Below we use (8) in the study of asymptotic behavior of $\ln \mathbf{P}(\xi_{\varphi,T} \in U_\varepsilon(f))$.

The proof of Theorem 1 shows that in the case $l \ne m$ the main contribution to this asymptotic comes from $A_T(\zeta)$.

Consider the sequence of scaled processes

$$\zeta_{\varphi,T}(t) := \frac{\zeta(tT)}{\varphi(T)}, \quad t \in [0,1]. \tag{11}$$

Further on, we write, for brevity, N_T, A_T, B_T instead of $N_T(\zeta), A_T(\zeta), B_T(\zeta)$.

Lemma 2. *Let the conditions of Theorem* 1 *be fulfilled. Then*

$$\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{E}\big(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f)\big)\le 0,$$

where $f\in\mathbb{C}_+$.

Lemma 3. *Let the conditions of Theorem* 1 *be fulfilled. Then*

$$\lim_{\varepsilon\to 0}\liminf_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{E}\big(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f)\big)\ge 0,$$

where $f\in\mathbb{C}_+$.

Thus, in the lemmas above we can write an equality instead of inequality.

3. Proof of Theorem 1 and Lemmas 2, 3

Proof of Theorem 1. First, let us estimate the term A_T

$$A_T=\int_0^T h(\zeta(t))dt=T\int_0^1 h(\varphi(T)\zeta_{\varphi,T}(s))ds.$$

We consider a set of trajectories ζ where $\zeta_{\varphi,T}\in U_\varepsilon(f)$.

For fixed ε let $\delta:=\delta(\varepsilon)=\max\limits_{0\le t\le 1}\{t: f(t)\le 2\varepsilon\}$. We note that $\lim\limits_{\varepsilon\to 0}\delta=0$ for all functions from the set $\mathbb{C}_+$ and on the event $\{\omega:\zeta_{\varphi,T}\in U_\varepsilon(f)\}$ we have

$$\min_{s\in[\delta,1]}\zeta_{\varphi,T}(s)\ge\varepsilon,\quad \max_{s\in[0,\delta]}\zeta_{\varphi,T}(s)\le 3\varepsilon.$$

By (1) for any $\gamma_0>0$, $s\in[\delta,1]$ and sufficiently large $T>0$

$$1-\gamma_0\le\frac{h(\varphi(T)\zeta_{\varphi,T}(s))}{V(\varphi(T))(\zeta_{\varphi,T}(s))^{l\vee m}}\le 1+\gamma_0, \tag{12}$$

where $V(x):=\lambda(x)\vee\mu(x)$.

By (12) for all sufficiently large T

$$\begin{aligned} T\int_\delta^1(1-\gamma_0)V(\varphi(T))(f(s)-\varepsilon)^{l\vee m}ds\le{}& A_T\\ \le T\int_0^\delta h(\varphi(T)\zeta_{\varphi,T}(s))ds&+T\int_\delta^1(1+\gamma_0)V(\varphi(T))(f(s)+\varepsilon)^{l\vee m}ds.\end{aligned}\tag{13}$$

From Lemma 5 it follows that

$$\limsup_{T\to\infty}\sup_{s\in[0,\delta]}\frac{h(\varphi(T)\zeta_{\varphi,T}(s))}{h(3\varepsilon\varphi(T))}\le\limsup_{T\to\infty}\sup_{b\in[0,1]}\frac{h(3\varepsilon\varphi(T)b)}{h(3\varepsilon\varphi(T))}\le 1.$$

Thus, using (13) we get that for all sufficiently large T

$$\begin{aligned}\psi(T)\int_\delta^1(1-\gamma_0)(f(s)-\varepsilon)^{l\vee m}ds\ \le\ & A_T\\ \le T\delta(1+\gamma_0)V(3\varepsilon\varphi(T))&+\psi(T)\int_\delta^1(1+\gamma_0)(f(s)+\varepsilon)^{l\vee m}ds.\end{aligned}\tag{14}$$

Using (8) and the inequalities (14), we shift to logarithms obtaining that

$$
\begin{aligned}
&-\int_\delta^1 (1-\gamma_0)(f(s)-\varepsilon)^{l\vee m}ds + \limsup_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{E}\big(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f)\big)\\
&\quad\ge \limsup_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{P}(\xi_{\varphi,T}\in U_\varepsilon(f)) \ge \liminf_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{P}(\xi_{\varphi,T}\in U_\varepsilon(f))\\
&\quad\ge -\int_\delta^1 (1+\gamma_0)(f(s)+\varepsilon)^{l\vee m}ds - \limsup_{T\to\infty}\delta(1+\gamma_0)\frac{V(3\varepsilon\varphi(T))}{V(\varphi(T))}\\
&\qquad + \liminf_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{E}\big(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f)\big).
\end{aligned}
\tag{15}
$$

Since (15) is fulfilled for any $\gamma_0>0$, letting $\varepsilon\to 0$, $\gamma_0\to 0$ we receive

$$
\begin{aligned}
&-\int_0^1 f^{l\vee m}(s)ds + \lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{E}\big(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f)\big)\\
&\quad\ge \lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{P}(\xi_{\varphi,T}(\cdot)\in U_\varepsilon(f))\\
&\quad\ge \lim_{\varepsilon\to 0}\liminf_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{P}(\xi_{\varphi,T}(\cdot)\in U_\varepsilon(f))\\
&\quad\ge -\int_0^1 f^{l\vee m}(s)ds + \lim_{\varepsilon\to 0}\liminf_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{E}\big(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f)\big).
\end{aligned}
\tag{16}
$$

Applying Lemmas 2 and Lemma 3 to inequalities (16) finishes the proof of the theorem. □

Proof of Lemma 2. In this lemma the goal is to establish the claimed upper bound for the expected value $\mathbf{E}\big(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f)\big)$. Obviously,

$$
\begin{aligned}
&\mathbf{E}\big(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f)\big) := E_1+E_2,\quad\text{with}\\
&E_1 := \mathbf{E}\big(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f); N_T\le\theta(T)\big),\\
&E_2 := \mathbf{E}\big(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f); N_T>\theta(T)\big),
\end{aligned}
\tag{17}
$$

where

$$
\theta(T) := \sqrt{\frac{\psi(T)\varphi(T)}{\ln(\varphi(T))}}.
$$

Let us first find an upper bound for E_1. Denote

$$
M := \max_{t\in[0,1]} f(t)\vee 1.
$$

If $\zeta_{\varphi,T}\in U_\varepsilon(f)$ and $N_T\le\theta(T)$ then for any $\gamma_1>0$ and for all sufficiently large T

$$
\begin{aligned}
B_T &= \sum_{i=1}^{N_T}\ln\big(\nu(\zeta(t_{i-1}),\zeta(t_i))\big) \le \sum_{i=1}^{N_T}\big(\ln(1\vee\lambda(\zeta(t_{i-1}))+\ln(1\vee\mu(\zeta(t_{i-1}))\big)\\
&\le \theta(T)\Big(\ln\big(\lambda(\varphi(T))(M+\varepsilon)^l(1+\gamma_1)\big)+\ln\big(\mu(\varphi(T))(M+\varepsilon)^m(1+\gamma_1)\big)\Big).
\end{aligned}
$$

Denote $M_1 := (M+\varepsilon)^{l+m}(1+\gamma_1)^2$. As

$$
\lambda(\varphi(T))\vee\mu(\varphi(T)) \le y(\varphi(T))\vee z(\varphi(T))\varphi^{l\vee m}(T)
$$

and for sufficiently large T the inequality $y(\varphi(T)) \vee z(\varphi(T)) \le \varphi^{l\vee m}(T)$ holds, we obtain the inequalities

$$
\begin{aligned}
E_1 &\le \exp\Big\{\theta(T)\ln\big(M_1\lambda(\varphi(T))\mu(\varphi(T))\big)\Big\}2^{\theta(T)}\\
&\le \exp\Big\{\theta(T)\ln\big(2M_1\lambda(\varphi(T))\mu(\varphi(T))\big)\Big\}\\
&\le \exp\Big\{\theta(T)\ln\big(2M_1\big(\lambda(\varphi(T))\vee\mu(\varphi(T))\big)^2\big)\Big\}\\
&\le \exp\Big\{\theta(T)\ln\big(2M_1\varphi^{4(l\vee m)}(T)\big)\Big\}.
\end{aligned} \tag{18}
$$

Now we find an upper bound for E_2. Denote by k_+ and k_- the number of positive and negative jumps of the process $\zeta_{\varphi,T}$ and let $L = k_+ - k_-$. For $\zeta_{\varphi,T} \in U_\varepsilon(f)$ the following inequality holds

$$
f(1) - \varepsilon \le \zeta_{\varphi,T}(1) \le f(1) + \varepsilon. \tag{19}
$$

Since the jumps of the process $\zeta_{\varphi,T}(\cdot)$ are $\pm 1/\varphi(T)$, by inequality (19) we have

$$
(f(1)-\varepsilon)\varphi(T) \le L \le (f(1)+\varepsilon)\varphi(T), \tag{20}
$$

and

$$
k_+ + k_- = N_T, \quad k_+ = \frac{N_T + L}{2}, \quad k_- = \frac{N_T - L}{2}. \tag{21}
$$

As $\zeta_{\varphi,T} \in U_\varepsilon(f)$, we obtain from (21) that for any $\gamma_1 > 0$ and for T large enough,

$$
\begin{aligned}
B_T = \sum_{i=1}^{N_T} \ln\big(\nu(\zeta(t_{i-1}),\zeta(t_i))\big) &\le \frac{N_T + L}{2}\ln\big(\lambda(\varphi(T))(M+\varepsilon)^l(1+\gamma_1)\big)\\
&\quad + \frac{N_T - L}{2}\ln\big(\mu(\varphi(T))(M+\varepsilon)^m(1+\gamma_1)\big).
\end{aligned} \tag{22}
$$

Since $N_T > \theta(T)$, we get, by using (20) and the condition (3), that

$$
\lim_{T\to\infty} \frac{N_T}{L} = \infty. \tag{23}
$$

Thus, by (22) and (23), for any $\gamma_1 > 0$ and all sufficiently T we obtain

$$
\begin{aligned}
B_T &\le \frac{N_T}{2}\ln\big(M_1\lambda(\varphi(T))\mu(\varphi(T))\big) + \frac{L}{2}\ln\left(\frac{\lambda(\varphi(T))}{\mu(\varphi(T))}(M+\varepsilon)^{l-m}\right)\\
&\le \frac{N_T}{2}(1+\gamma_1)\ln\big(M_1\lambda(\varphi(T))\mu(\varphi(T))\big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E_2 \quad &\le \quad \mathbf{E}\Big(e^{B_T + N_T\ln 2}; N_T \ge \theta(T)+1\Big)\\
&\le \quad \mathbf{E}\exp\left\{\frac{N_T}{2}(1+\gamma_1)\ln\big(4M_1\lambda(\varphi(T))\mu(\varphi(T))\big)\right\}.
\end{aligned} \tag{24}
$$

Since N_T has the Poisson distribution with parameter T, then for any $r \in \mathbb{R}$

$$
\mathbf{E}e^{rN_T} = e^{T(e^r - 1)} \le e^{Te^r}.
$$

Therefore, from (24) it follows that

$$
E_2 \le \exp\Big\{M_2 T\big(\lambda(\varphi(T))\mu(\varphi(T))\big)^{(1+\gamma_1)/2}\Big\}, \tag{25}
$$

where $M_2 := (4M_1)^{(1+\gamma_1)/2}$.

Now let us choose $\gamma_1 < \dfrac{|l-m|}{l+m} \neq 0$. Using inequalities (18), (25), condition (3) and an obvious inequality $\ln(E_1+E_2) \le \ln(2(E_1 \vee E_1))$, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{\psi(T)}\ln \mathbf{E}\bigl(e^{B_T(\zeta)+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f)\bigr)\\
&\le \lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{\Bigl[\theta(T)\ln\bigl(2M_1\varphi^{4(l\vee m)}(T)\bigr)\Bigr]\vee\Bigl[M_2T\bigl(\lambda(\varphi(T))\mu(\varphi(T))\bigr)^{(1+\gamma_1)/2}\Bigr]}{\psi(T)}\\
&\le \lim_{\varepsilon\to 0}\limsup_{T\to\infty}\left(\frac{\sqrt{\varphi(T)}\ln\bigl(2M_1\varphi^{4(l\vee m)}(T)\bigr)}{\sqrt{\psi(T)\ln(\varphi(T))}}\vee\frac{M_2\bigl(y(\varphi(T))z(\varphi(T))\varphi^{l+m}(T)\bigr)^{(1+\gamma_1)/2}}{\lambda(\varphi(T))\vee\mu(\varphi(T))}\right)\\
&= 0.
\end{aligned}$$

□

Proof of Lemma 3. The aim is to introduce the lower−bound for the term $\mathbf{E}\bigl(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f)\bigr)$. Set $M_3 := \inf_{x\in\mathbb{Z}^+}\lambda(x)\wedge\inf_{x\in\mathbb{N}}\mu(x)$, where $v\wedge w$ is a minimum of numbers v, w. By global assumptions we have $M_3>0$.

Observe that always $B_T \ge N_T\ln M_3$, thus for any constant $C>0$

$$\begin{aligned}
\mathbf{E}\bigl(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f)\bigr) &\ge \mathbf{E}\bigl(e^{N_T\ln M_3};\zeta_{\varphi,T}\in U_\varepsilon(f);N_T\le C\varphi(T)\bigr)\\
&\ge e^{C\varphi(T)(0\wedge\ln M_3)}\mathbf{P}\bigl(\zeta_{\varphi,T}\in U_\varepsilon(f);N_T\le C\varphi(T)\bigr).
\end{aligned}\tag{26}$$

From (26) it follows that

$$\begin{aligned}
\liminf_{T\to\infty}\frac{1}{\psi(T)}&\ln \mathbf{E}\bigl(e^{B_T+N_T\ln 2};\zeta_{\varphi,T}\in U_\varepsilon(f);N_T\le C\varphi(T)\bigr)\\
&\ge \liminf_{T\to\infty}\frac{1}{\psi(T)}\ln\mathbf{P}\bigl(\zeta_{\varphi,T}\in U_\varepsilon(f);N_T\le C\varphi(T)\bigr).
\end{aligned}$$

By Lemma 4 from the appendix, we can choose the constant $C>0$ such that

$$\liminf_{T\to\infty}\frac{1}{\psi(T)}\ln\mathbf{P}\bigl(\zeta_{\varphi,T}\in U_\varepsilon(f);N_T\le C\varphi(T)\bigr)=0,$$

This completes the proof of Lemma 3. □

4. Appendix

Lemma 4. *Let the condition* (3) *be fulfilled. Then for any function* $f\in\mathbb{C}_+$ *and any* $\varepsilon>0$ *there exists a constant* C $(C=C(\varepsilon))$ *such that*

$$\liminf_{T\to\infty}\frac{1}{\psi(T)}\ln\mathbf{P}\bigl(\zeta_{\varphi,T}\in U_\varepsilon(f);N_T\le C\varphi(T)\bigr)=0.$$

Proof. The process $\zeta(t)$ can be represented as

$$\zeta(t)=\zeta^{(1)}(t)-\zeta^{(2)}(t),$$

where $\zeta^{(1)}(t)$ and $\zeta^{(2)}(t)$ are independent Poisson processes with rate $1/2$.

Since f is continuous on $[0,1]$ there exists a continuous function of finite variation g, defined on $[0,1]$, such that $\rho(f,g)<\varepsilon/2$, $g(0)=0$. Moreover, there exist continuous monotone non-decreasing functions g_+ and g_- such that

$$g(t)=g_+(t)-g_-(t),\quad g_+(0)=g_-(0)=0.$$

Define in analogy to (11),

$$\zeta^{(1)}_{\varphi,T}(t) = \frac{\zeta^{(1)}(tT)}{\varphi(T)}, \quad \zeta^{(2)}_{\varphi,T}(t) = \frac{\zeta^{(2)}(tT)}{\varphi(T)}.$$

Furthermore, let $N^{(r)}_T$ stands for the number of jumps of $\zeta^{(r)}$ on $[0,T]$, $r = 1,2$. Finally, denote

$$C_1 = g_+(1), \quad C_2 = g_-(1), \quad C = C_1 + C_2.$$

Because of independence of processes $\zeta^{(1)}$ and $\zeta^{(2)}$ we can write

$$\begin{aligned}\mathbf{P}\big(\zeta_{\varphi,T} \in U_\varepsilon(f); N_T \le C\varphi(T)\big) \ge & \mathbf{P}\big(\zeta^{(1)}_{\varphi,T} \in U_{\varepsilon/4}(g_+); N^{(1)}_T \le C_1\varphi(T)\big) \\ & \times \mathbf{P}\big(\zeta^{(2)}_{\varphi,T} \in U_{\varepsilon/4}(g_-); N^{(2)}_T \le C_2\varphi(T)\big) \\ =: & P_1 P_2.\end{aligned} \tag{27}$$

To derive the lower−bound for the probability P_1, consider a partition of the unit interval by points $0 = t_0 < t_1 < \cdots < t_K = 1$ such that

$$\max_{1\le i\le K}(g_+(t_i) - g_+(t_{i-1})) < \frac{\varepsilon}{8}.$$

Since $\zeta^{(1)}$ is a process with independent increments, we get that for a sufficiently large T

$$\begin{aligned}P_1 &\ge \prod_{i=1}^K \mathbf{P}\left(\zeta^{(1)}(Tt_i) - \zeta^{(1)}(Tt_{i-1}) = \lfloor (g_+(t_i) - g_+(t_{i-1}))\varphi(T)\rfloor\right) \\ &= \prod_{i=1}^K \frac{e^{-T(t_i-t_{i-1})/2}(T(t_i-t_{i-1})/2)^{\lfloor (g_+(t_i)-g_+(t_{i-1}))\varphi(T)\rfloor}}{\lfloor (g_+(t_i) - g_+(t_{i-1}))\varphi(T)\rfloor!} \\ &\ge \prod_{i=1}^K \exp\left\{-\frac{T(t_i-t_{i-1})}{2} - (g_+(t_i) - g_+(t_{i-1}))\varphi(T)\ln\big((g_+(t_i) - g_+(t_{i-1}))\varphi(T)\big)\right\} \\ &\ge \prod_{i=1}^K \exp\left\{-\frac{T(t_i-t_{i-1})}{2} - (g_+(t_i) - g_+(t_{i-1}))\varphi(T)\ln\big(g_+(1)\varphi(T)\big)\right\} \\ &\ge \exp\left\{-T - g_+(1)\varphi(T)\ln(g_+(1)\varphi(T))\right\},\end{aligned}$$

where $\lfloor b\rfloor$ is the integer part of the number b.

In the same way we obtain a lower bound for P_2:

$$P_2 \ge \exp\big\{-T - g_-(1)\varphi(T)\ln\big(g_-(1)\varphi(T)\big)\big\}.$$

Then from (3) it follows that

$$\begin{aligned}\liminf_{T\to\infty} \ln \mathbf{P}\big(\zeta_{\varphi,T} \in U_\varepsilon(f); N_T \le C\varphi(T)\big) &\ge \liminf_{T\to\infty} \ln(\mathbf{P}_1\mathbf{P}_2) \\ \ge \liminf_{T\to\infty} \frac{-2T - (g_-(1) + g_+(1))\varphi(T)\ln\big((g_-(1) + g_+(1))\varphi(T)\big)}{\psi(T)} &= 0.\end{aligned}$$

This completes the proof of Lemma 4. □

Lemma 5. *Let the non-negative function $h(x), x \in [0,\infty)$ satisfies the following conditions:*

1) $\sup_{x\in[0,a]} h(x) < \infty$ *for any* $a > 0$*;*
2) $h(x) = L(x)x^p, p > 0$*, where $L(x)$ is the slowly varying at infinity function.*

Then, for any $c > 0$

$$\limsup_{x\to\infty} \sup_{b\in[0,c]} \frac{h(bx)}{h(x)} \le c^p. \tag{28}$$

Proof. Let us, first, note that according the uniform convergence theorem for regularly varying function, [15, Theorem 1.1], it follows that for any $d \in (0, c]$

$$\limsup_{x\to\infty} \sup_{b\in[d,c]} \frac{h(bx)}{h(x)} = \limsup_{x\to\infty} \sup_{b\in[d,c]} \frac{(bx)^p L(bx)}{x^p L(x)} \le c^p \limsup_{x\to\infty} \sup_{b\in[d,c]} \frac{L(bx)}{L(x)} = c^p. \tag{29}$$

We prove (28) by contradiction. Suppose the inequality (28) does not hold, then there exist two sequences b_k, $b_k \in [0, c]$ and x_k, $\lim\limits_{k\to\infty} x_k = \infty$ such that

$$\lim_{k\to\infty} \frac{h(b_k x_k)}{h(x_k)} > c^p \text{ (maybe the limit is infinity).} \tag{30}$$

Since $b_k \in [0, c]$ we suppose that the sequence b_k has a limit (if not we choose a subsequence). Consider different cases.

- If $\lim\limits_{k\to\infty} b_k = d > 0$, then due to (29) the inequality (30) does not true. Thus, $\lim\limits_{k\to\infty} b_k = 0$.
- Let $\lim\limits_{k\to\infty} b_k = 0$. Consider the sequence $b_k x_k$. Suppose there exists finite limit

  $$\limsup_{k\to\infty} b_k x_k = a < \infty.$$

  Then, due to the condition 1), the left hand side of (30) is equal to 0; thus, the inequality (30) does not hold.

Hereby, if (30) holds, then $\lim\limits_{k\to\infty} b_k = 0$ and $\limsup\limits_{k\to\infty} b_k x_k = \infty$ (we could set $\lim\limits_{k\to\infty} b_k x_k = \infty$, indeed, if $\limsup\limits_{k\to\infty} b_k x_k = \infty$, then we can find such subsequence).

Thus, if (30) holds, then there exist sequences b_k and x_k such that

$$\lim_{k\to\infty} b_k = 0, \quad \lim_{k\to\infty} b_k x_k = \infty. \tag{31}$$

Now, according the representation theorem for slowly varying functions, [15, Theorem 1.2], it follows that there exists a constant $B > 0$ such that for all $x \ge B$

$$L(x) = \exp\left\{ u(x) + \int_B^x \frac{v(t)}{t} dt \right\}, \tag{32}$$

where $u(x)$ is a bounded measurable function on $x \ge B$, such that $\lim\limits_{x\to\infty} u(x) = m$, $|m| < \infty$, and $v(x)$ is continuous function on $x \ge B$ such that $\lim\limits_{x\to\infty} v(x) = 0$.

Using (31), (32), we obtain

$$\lim_{k\to\infty}\frac{h(b_kx_k)}{h(x_k)}=\lim_{k\to\infty}\frac{(b_k)^pL(b_kx_k)}{L(x_k)}\le\lim_{k\to\infty}(b_k)^p\exp\left\{\int\limits_{b_kx_k}^{x_k}\frac{|v(t)|}{t}dt\right\}$$

$$\le\lim_{k\to\infty}(b_k)^p\exp\left\{\int\limits_{b_kx_k}^{x_k}\frac{p/2}{t}dt\right\}\le\lim_{k\to\infty}(b_k)^p\exp\left\{\frac{p}{2}\big(\ln(x_k)-\ln(b_kx_k)\big)\right\}$$

$$\le\lim_{k\to\infty}(b_k)^p\exp\left\{\ln\left(\frac{1}{b_k^{p/2}}\right)\right\}=0\le c^p.$$

It contradicts supposition (30). □

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