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STONE LATTICES OF MULTIPLY Ω-CANONICAL FITTING CLASSES

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Abstract. Let L be a lattice with 0 and 1. A distributive lattice L with pseudocomplements, each element of which satisfies an identity $a^\circ \vee (a^\circ)^\circ = 1$, where a° is a pseudocomplement of an element a, is called a Stone lattice. The article describes multiply Ω-canonical Fitting classes with a Stone lattice of multiply Ω-canonical Fitting subclasses. It is shown that such Fitting classes are subclasses of the class $\mathfrak{D}_\Omega = \times_{A\in\Omega}\mathfrak{G}_A = (B_1 \times B_2 \times ... \times B_n : B_i \in \mathfrak{G}_{A_i}$ for some $A_i \in \Omega$, $i \in \{1, 2, \dots, n\}$, $n \in \mathbb{N})$.

Keywords: finite group, Fitting class, Ω-canonical Fitting class, lattice of Fitting classes, Stone lattice.

1. Introduction

Methods of the general lattice theory, described by G. Birkhoff [1], are widely used in research of various group classes. Specific techniques for studying lattices of τ-closed n-multiply local formations were developed by A. N. Skiba [2]. For lattices of subgroup-closed Fitting classes such techniques were created by S. Reifferscheid [3, 4]. The versatility of those methods allowed researchers to use ideas, expressed in the papers mentioned above, for studying lattices of various formations and Fitting classes. For instance, A. N. Skiba and N. N. Vorob'ev considered multiply local Fitting classes with a Stone lattice of multiply local Fitting subclasses [5]. Similar results were obtained by N. N. Vorob'ev and A. I. Titova for multiply ω-local Fitting classes [6] as well as by N. N. Vorob'ev and A.P. Mekhovich for multiply saturated formations [7].

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In this paper, we obtain necessary and sufficient conditions for n-multiply (totally) Ω-canonical Fitting classes, for the lattice of all of their n-multiply (totally) Ω-canonical Fitting subclasses to be a Stone lattice.

Throughout our paper we consider only finite groups. A class of groups $\mathfrak{F}$ is called a Fitting class if it is closed under taking normal subgroups and products of normal $\mathfrak{F}$-subgroups. We introduce the following notation: Let $\mathfrak{G}$ be the class of all finite groups, Ω a nonempty subclass of a class of all finite simple groups $\mathfrak{I}$ ($\Omega' = \mathfrak{I}\backslash\Omega$), $\mathcal{K}(G)$ a class of all simple groups, isomorphic to the composition factors of a group G and $\mathcal{K}(\mathfrak{X})$ the union of classes $\mathcal{K}(G)$ of all $G \in \mathfrak{X}$, where $\mathfrak{X}$ is a nonempty class of groups. Also let (G) be a class of all groups, isomorphic to a group G, and by $\mathfrak{G}_\Omega$ denote a class of all finite Ω-groups, i.e. groups G, such that $\mathcal{K}(G) \subseteq \Omega$ and $1 \in \mathfrak{G}_\Omega$. For $A \in \mathfrak{I}$ assume $\mathfrak{G}_A = \mathfrak{G}_{(A)}$. Furthermore, let $A' = \mathfrak{I} \setminus (A)$, $\mathfrak{D}_\Omega = \times_{A\in\Omega}\mathfrak{G}_A = (B_1 \times B_2 \times ... \times B_n : B_i \in \mathfrak{G}_{A_i}$ for some $A_i \in \Omega$, $i \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N})$.

A class of groups $\mathfrak{F}$ is called a Fitting formation if $\mathfrak{F}$ is both a formation and a Fitting class. All functions under consideration take the same values for isomorphic groups from their domain. A function $f : \Omega \cup \{\Omega'\} \to \{\text{groups' Fitting classes}\}$ is called a ΩR-function while a function $\varphi : \mathfrak{I} \to \{\text{nonempty Fitting formations}\}$ is said to be a FR-function. Let $O^\Omega(G)$ be a $\mathfrak{G}_\Omega$-coradical of a group G, and let $G^{\varphi(A)}$ be its $\varphi(A)$-coradical. A Fitting class $\mathfrak{F} = \Omega R(f, \varphi) = (G : O^\Omega(G) \in f(\Omega')$ and $G^{\varphi(A)} \in f(A)$ for all $A \in \Omega \cap \mathcal{K}(G))$ is said to be Ω-foliated with an Ω-satellite f and a direction φ. A Fitting class $\mathfrak{F} = \Omega R(f, \varphi)$ is said to be Ω-canonical or an ΩK-Fitting class ($\mathfrak{F} = \Omega KR(f, \varphi)$) if $\varphi(A) = \mathfrak{G}_A\mathfrak{G}_{A'}$ for all $A \in \mathfrak{I}$ [8, 9].

Let $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. Every Fitting class is considered 0-multiply Ω-canonical. Given $n \geqslant 1$, a Fitting class $\mathfrak{F}$ is said to be n-multiply Ω-canonical if $\mathfrak{F}$ has at least one Ω satellite, all nonempty values of which are $(n-1)$-multiply Ω-canonical Fitting classes. Similarly, $\mathfrak{F}$ is called totally Ω-canonical if it is n-multiply Ω-canonical for all $n \in \mathbb{N}$ [2, 10].

The terminology not defined here can be found in [2, 11].

2. Preliminaries

Lemma 2.1. [8, Theorem 6] *Let f be an inner Ω-satellite of an ΩK-Fitting class $\mathfrak{F}$. Then an ΩK-Fitting class $\mathfrak{F}$ has a unique maximal inner Ω-satellite h. Moreover, $h(\Omega') = \mathfrak{F}$ and $h(A) = h(A)\mathfrak{G}_A = f(A)\mathfrak{G}_A$ for every $A \in \Omega$.*

The intersection of all n-multiply Ω-canonical Fitting classes, which contain a non-empy class of groups $\mathfrak{X}$ is said to be an n-multiply Ω-canonical Fitting class *generated by* $\mathfrak{X}$ and is denoted by $\Omega KR^n(\mathfrak{X})$.

Lemma 2.2. [10, Corollary 2] *Let $\mathfrak{X}$ be a nonempty class of groups. Then an n-multiply Ω-canonical Fitting class $\mathfrak{F} = \Omega KR^n(\mathfrak{X})$ has a unique minimal n-multiply Ω-canonical satellite f, such that $f(\Omega') = \Omega KR^{n-1}(O^\Omega(G) : G \in \mathfrak{X})$, $f(A) = \Omega KR^{n-1}(O^{A,A'}(G) : G \in \mathfrak{X})$ for all $A \in \Omega \cap \mathcal{K}(\mathfrak{X})$ and $f(A) = \varnothing$, if $A \in \Omega \setminus \mathcal{K}(\mathfrak{X})$.*

A collection $\{\mathfrak{F}_i \mid i \in I\}$ of nonempty classes of groups $\mathfrak{F}_i$, such that $\mathfrak{F}_i \cap \mathfrak{F}_j = (1)$ for all $i \neq j$, $i, j \in I$, is called *an orthogonal system of classes*. Denote by $\otimes_{i\in I}\mathfrak{F}_i$ a collection of groups of the form $A_1 \times ... \times A_t$, where $A_1 \in \mathfrak{F}_{i_1}$, ..., $A_t \in \mathfrak{F}_{i_t}$, $i_1, \ldots$, $i_t \in I$.

Lemma 2.3. [12, Lemma 4] *Let $\mathfrak{F}_1$, $\mathfrak{F}_2$, ..., $\mathfrak{F}_t$ be an orthogonal system of Fitting classes. Then $\mathfrak{F}_1 \otimes \mathfrak{F}_2 \otimes ... \otimes \mathfrak{F}_t$ is a Fitting class.*

Lemma 2.4. [13, Corollary 3] *Let* $\mathfrak{F} = \otimes_{i\in I}\mathfrak{F}_i$ *for some system of nonempty Fitting classes* $\{\mathfrak{F}_i \mid i \in I\}$. *If every* $\mathfrak{F}_i$ *is an* n*-multiply* Ω*-canonical Fitting class then* $\mathfrak{F}$ *is an* n*-multiply* Ω*-canonical Fitting class as well.*

Lemma 2.5. [12, Lemma 1] *Suppose that* $\mathfrak{F} = \otimes_{i\in I}\mathfrak{F}_i$ *and* $\mathfrak{M}$ *a nonempty Fitting subclass of* $\mathfrak{F}$. *Then* $\mathfrak{M} = \otimes_{i\in I}(\mathfrak{M} \cap \mathfrak{F}_i)$.

Let $\mathfrak{F}$ be an n-multiply Ω-canonical Fitting class. Denote by $\Omega K^n(\mathfrak{F})$ a lattice of all of its n-multiply Ω-canonical Fitting subclasses.

We now provide essential notions of the general lattice theory for the lattice $\Omega K^n(\mathfrak{F})$.

An n-multiply Ω-canonical Fitting subclass $\mathfrak{M}$ of a class $\mathfrak{F}$ is said to be *complementable* in the lattice $\Omega K^n(\mathfrak{F})$ if there exists an n-multiply Ω-canonical Fitting subclass $\mathfrak{H}$ of $\mathfrak{F}$, such that $\mathfrak{M} \cap \mathfrak{H} = (1)$ and $\mathfrak{F} = \Omega KR^n(\mathfrak{M} \cup \mathfrak{H})$. The lattice $\Omega K^n(\mathfrak{F})$ is said to be *distributive* if for all classes $\mathfrak{M}$, $\mathfrak{H}$, $\mathfrak{K} \in \Omega K^n(\mathfrak{F})$ it is true that

$$\mathfrak{M} \cap \Omega KR^n(\mathfrak{H} \cup \mathfrak{K}) = \Omega KR^n((\mathfrak{M} \cap \mathfrak{H}) \cup (\mathfrak{M} \cap \mathfrak{K})).$$

A distributive lattice with complements is called *a Boolean lattice*.

Lemma 2.6. [14, Corollary 3] *Given a non-trivial* n*-multiply* Ω*-canonical Fitting class* $\mathfrak{F}$, *the following statements are equivalent:*

1) $\Omega K^n(\mathfrak{F})$ *is a Boolean lattice.*

2) $\mathfrak{F} = \otimes_{i\in I}\mathfrak{F}_i$, *where* $\{\mathfrak{F}_i \mid i \in I\}$ *is a set of all atoms of* $\Omega K^n(\mathfrak{F})$.

3) All Fitting subclasses of $\mathfrak{F}$, *which are the elements of* $\Omega K^n(\mathfrak{F})$, *are complementable.*

In a lattice $\Omega K^n(\mathfrak{F})$ a class $\mathfrak{M}^\circ$ is called a *pseudocomplement* of a class $\mathfrak{M}$, if $\mathfrak{M}^\circ$ is the greatest element of $\Omega K^n(\mathfrak{F})$, such that $\mathfrak{M} \cap \mathfrak{M}^\circ = (1)$. A lattice $\Omega K^n(\mathfrak{F})$ is said to be a *lattice with pseudocomplements*, if all of its elements have a pseudocomplement. A distributive lattice with pseudocomplements is called a *Stone lattice*, if for all of its elements it is true that

$$\Omega KR^n(\mathfrak{M}^\circ \cup (\mathfrak{M}^\circ)^\circ) = \mathfrak{F}.$$

Let $\mathfrak{F}$ be a totally Ω-canonical Fitting class. Denote by $\Omega K^\infty(\mathfrak{F})$ a lattice of its totally Ω-canonical Fitting subclasses. The above-mentioned notions of the general lattice theory are similar for $\Omega K^\infty(\mathfrak{F})$.

Lemma 2.7. [15, Corollary 2.1] *Let* $\mathfrak{X}$ *be a nonempty class of groups. Then a totally* Ω*-canonical Fitting class* $\mathfrak{F} = \Omega KR^\infty(\mathfrak{X})$ *has a unique minimal totally* Ω*-canonical satellite* f *such that* $f(\Omega') = \Omega KR^\infty(O^\Omega(G) : G \in \mathfrak{X})$, $f(A) = \Omega KR^\infty(O^{A,A'}(G) : G \in \mathfrak{X})$ *for all* $A \in \Omega \cap \mathcal{K}(\mathfrak{X})$ *and* $f(A) = \varnothing$ *if* $A \in \Omega \setminus \mathcal{K}(\mathfrak{X})$.

Lemma 2.8. [9, Lemma 12] *Let* φ *be an* FR*-function,* $\mathfrak{F} = \cap_{i\in I}\mathfrak{F}_i$, *where* $\mathfrak{F}_i = \Omega R(f_i, \varphi)$ $(\mathfrak{F}_i = R(f_i, \varphi))$, $i \in I$. *Then* $\mathfrak{F} = \Omega R(f, \varphi)$ $(\mathfrak{F} = R(f, \varphi))$, *where* $f = \cap_{i\in I} f_i$.

3. Main Results

Lemma 3.1. *Let* $\mathfrak{F}$ *be an* n*-multiply (totally)* Ω*-canonical Fitting class. If* $A \in \Omega \cap \mathcal{K}(\mathfrak{F})$, *then* $\mathfrak{G}_A \subseteq \mathfrak{F}$.

Proof. Assume f is an inner Ω-satellite of a Fitting class $\mathfrak{F}$. Then by Lemma 2.1, it follows that $\mathfrak{F}$ has an inner Ω-satellite h, such that $h(\Omega') = \mathfrak{F}$ and $h(A) = h(A)\mathfrak{G}_A = f(A)\mathfrak{G}_A$ for all $A \in \Omega$.

If $A \in \Omega \cap \mathcal{K}(\mathfrak{F})$, then $\mathfrak{G}_A \subseteq h(A)\mathfrak{G}_A = f(A)\mathfrak{G}_A = h(A) \subseteq \mathfrak{F}$. □

Lemma 3.2. *Let $\mathfrak{F} = \Omega KR^n(G)$ be a singly generated n-multiply Ω-canonical Fitting class. Then the lattice $\Omega K^n(\mathfrak{F})$ contains only a finite number of atoms.*

Proof. Assume $\mathfrak{M}$ is an atom in $\Omega K^n(\mathfrak{F})$. Then $\mathfrak{M} = \Omega KR^n(A)$, where A is a simple group from $\mathfrak{M}$.

We will show that $A \in \mathcal{K}(G)$. The proof will be by induction on multiplicity of n.

1) If $n = 0$, then $\Omega KR^n(G) = fit(G)$. Since $\mathcal{K}(fit(G)) = \mathcal{K}(G)$ and $A \in \mathfrak{M} \subseteq \mathfrak{F} = fit(G)$, then $A \in \mathcal{K}(G)$.

2) Let $n > 0$ and assume that the statement is true for all singly generated $(n-1)$-multiply Ω-canonical Fitting classes.

3) Assume f is a minimal $(n-1)$-multiply Ω-canonical satellite of $\mathfrak{F} = \Omega KR^n(G)$.

If A is an Ω'-group, then, taking into account the structure of f, and by Lemma 2.2, we obtain that

$$A = O^{\Omega}(A) \in f(\Omega') = \Omega KR^{n-1}(O^{\Omega}(G)).$$

Then by induction $A \in \mathcal{K}(O^{\Omega}(G))$. Since $\mathcal{K}(O^{\Omega}(G) \subseteq \mathcal{K}(fit(G)) = \mathcal{K}(G)$, we derive that $A \in \mathcal{K}(G)$.

Assume that A is an Ω-group and $A \notin \mathcal{K}(G)$, i.e. $A \in \Omega \setminus \mathcal{K}(G)$. Taking into account the structure of f and by Lemma 2.2, we obtain that

$$1 = O^{A,A'}(A) \in f(A) = \varnothing,$$

which is a contradiction, and thus $A \in \mathcal{K}(G)$. Since G is a finite group, it only has a finite number of composition factors, therefore a lattice $\Omega K^n(\mathfrak{F})$ contains only a finite number of atoms. □

Lemma 3.3. *Let $\mathfrak{F}$ be an n-multiply Ω-canonical Fitting class. If for every $A \in \Omega \cap \mathcal{K}(\mathfrak{F})$ the class $\mathfrak{G}_A$ is complementable in the lattice $\Omega K^n(\mathfrak{F})$, then $\mathfrak{F} \subseteq \mathfrak{D}_{\Omega}$.*

Proof. Consider a n-multiply Ω-canonical Fitting class. It can be considered as a lattice union of all of its singly generated n-multiply Ω-canonical Fitting subclasses, i.e.

$$\mathfrak{F} = \Omega KR^n(\cup_{G \in \mathfrak{F}} \Omega KR^n(G)).$$

Hence, it suffices to prove the lemma for all singly generated n-multiply Ω-canonical Fitting subclasses $\mathfrak{M} = \Omega KR^n(G)$ of $\mathfrak{F}$.

We will first show that for every $A \in \Omega \cap \mathcal{K}(\mathfrak{M})$ the class $\mathfrak{G}_A$ is complementable in the lattice $\Omega K^n(\mathfrak{M})$. By Lemma 3.1, we obtain that $\mathfrak{G}_A \subseteq \mathfrak{M}$.

Since $A \in \Omega \cap \mathcal{K}(\mathfrak{M}) \subseteq \Omega \cap \mathcal{K}(\mathfrak{F})$ and $\mathfrak{G}_A$ is complementable in the lattice $\Omega K^n(\mathfrak{F})$, there exists an n-multiply Ω-canonical Fitting subclass $\mathfrak{H}$ of $\mathfrak{F}$, such that $\mathfrak{G}_A \cap \mathfrak{H} = (1)$, $\Omega KR^n(\mathfrak{G}_A \cup \mathfrak{H}) = \mathfrak{F}$. Therefore, by Lemmas 2.3 and 2.4, we have $\mathfrak{F} = \mathfrak{G}_A \otimes \mathfrak{H}$.

Moreover, it follows from Lemma 2.5 that

$$\mathfrak{M} = \mathfrak{M} \cap \mathfrak{F} = \mathfrak{M} \cap (\mathfrak{G}_A \otimes \mathfrak{H}) = (\mathfrak{M} \cap \mathfrak{G}_A) \otimes (\mathfrak{M} \cap \mathfrak{H}) =$$

$$= \mathfrak{G}_A \otimes (\mathfrak{M} \cap \mathfrak{H}) = \Omega KR^n(\mathfrak{G}_A \cup (\mathfrak{M} \cap \mathfrak{H})).$$

Since $\mathfrak{G}_A \cap (\mathfrak{M} \cap \mathfrak{H}) \subseteq \mathfrak{G}_A \cap \mathfrak{H} = (1)$, by definition it follows that $\mathfrak{M} \cap \mathfrak{H}$ is a complement of the class $\mathfrak{G}_A$ in $\Omega K^n(\mathfrak{M})$.

We will finish our proof using induction on the number of atoms of the lattice $\Omega K^n(\mathfrak{M})$. By the previous lemma, $\Omega K^n(\mathfrak{M})$ only contains a finite number m of atoms.

We will prove that $\mathfrak{M} \subseteq \mathfrak{D}_\Omega$ (1) by induction on m.

1) First, we verify that (1) is true for $m = 1$. In this case, $\Omega K^n(\mathfrak{M})$ contains a single atom $\mathfrak{G}_A$, where $A \in \Omega \cap \mathcal{K}(\mathfrak{M})$. Earlier, it was shown that a class $\mathfrak{G}_A$ is complementable in $\Omega K^n(\mathfrak{M})$, hence there exists an n-multiply Ω-canonical Fitting subclass $\mathfrak{H}$ of $\mathfrak{M}$, such that $\mathfrak{G}_A \cap \mathfrak{H} = (1)$ and $\Omega KR^n(\mathfrak{G}_A \cup \mathfrak{H}) = \mathfrak{M}$. Since $\Omega K^n(\mathfrak{M})$ only contains one atom, we derive that $\mathfrak{H} = (1)$. Then $\mathfrak{M} = \Omega KR^n(\mathfrak{G}_A \cup \mathfrak{H}) = \Omega KR^n(\mathfrak{G}_A) = \mathfrak{G}_A \subseteq \mathfrak{D}_\Omega$. Therefore, (1) is true for $m = 1$.

2) Let $m > 1$. Assume (1) is true for all singly generated n-multiply Ω-canonical Fitting subclasses with lattices containing less than m atoms.

3) We will prove that (1) is true for m. Let $\mathfrak{G}_A \subseteq \mathfrak{M}$ and $\mathfrak{H}$ is a complement to $\mathfrak{G}_A$ in the lattice $\Omega K^n(\mathfrak{M})$. Then $\mathfrak{G}_A \nsubseteq \mathfrak{H}$ and the lattice $\Omega K^n(\mathfrak{H})$ contains less atoms than m; therefore $\mathfrak{H}$ is a singly generated n-multiply Ω-canonical Fitting class. Then, by induction hypothesis, (1) is true for $\mathfrak{H}$, i.e. $\mathfrak{H} \subseteq \mathfrak{D}_\Omega$. Since $\mathfrak{G}_A \subseteq \mathfrak{D}_\Omega$, we can see that $\mathfrak{M} = \Omega KR^n(\mathfrak{G}_A \cup \mathfrak{H}) = \mathfrak{G}_A \otimes \mathfrak{H} \subseteq \mathfrak{D}_\Omega$.

Therefore the lemma is true for every singly generated n-multiply Ω-canonical Fitting subclass $\mathfrak{M} = \Omega KR^n(G)$ of $\mathfrak{F}$. Hence $\mathfrak{F} = \Omega KR^n(\cup_{G \in \mathfrak{F}} \Omega KR^n(G)) \subseteq \mathfrak{D}_\Omega$. □

Theorem 3.1. *Let $\mathfrak{F}$ be an n-multiply Ω-canonical Fitting class. The lattice $\Omega K^n(\mathfrak{F})$ is a Stone lattice iff $\mathfrak{F} \subseteq \mathfrak{D}_\Omega$.*

Proof of necessity. Suppose that $\mathfrak{F}$ is an n-multiply Ω-canonical Fitting class, $\Omega K^n(\mathfrak{F})$ a Stone lattice.

Then there exists a pseudocomplement for every element $\mathfrak{M}$ of the lattice $\Omega K^n(\mathfrak{F})$. We will show that $\mathfrak{F} \cap \mathfrak{G}_{\Omega_1'}$, where $\Omega_1 = \Omega \cap \mathcal{K}(\mathfrak{M})$, is that pseudocomplement. Since $\Omega_1 \cap \Omega_1' = \varnothing$, it follows that $\mathfrak{M} \cap (\mathfrak{F} \cap \mathfrak{G}_{\Omega_1'}) = (1)$. Let $\mathfrak{H}$ be an n-multiply Ω-canonical Fitting subclass of $\mathfrak{F}$ and $\mathfrak{H} \cap \mathfrak{M} = (1)$. We can see that $\Omega_1 \cap \mathcal{K}(\mathfrak{H}) = \Omega \cap \mathcal{K}(\mathfrak{M}) \cap \mathcal{K}(\mathfrak{H}) = \varnothing$. Thus $\mathfrak{H} \subseteq \mathfrak{F} \cap \mathfrak{G}_{\Omega_1'}$ and, by definition, $\mathfrak{F} \cap \mathfrak{G}_{\Omega_1'}$ is a pseudocomplement of $\mathfrak{M}$ in the lattice $\Omega K^n(\mathfrak{F})$.

Assume $\mathfrak{M} = \mathfrak{G}_A \subseteq \mathfrak{F}$, $A \in \Omega \cap \mathcal{K}(\mathfrak{F})$. Then, by the previous statement, $\mathfrak{F} \cap \mathfrak{G}_{A'}$ is a pseudocomplement of $\mathfrak{G}_A$ in the lattice $\Omega K^n(\mathfrak{F})$.

We will now show that $\mathfrak{G}_A$ is a pseudocomplement of $\mathfrak{F} \cap \mathfrak{G}_{A'}$ in the lattice $\Omega K^n(\mathfrak{F})$. Since $(A) \cap (A)' = \varnothing$, it follows that $\mathfrak{G}_A \cap (\mathfrak{F} \cap \mathfrak{G}_{A'}) = (1)$. Let $\mathfrak{H}_1$ be an n-multiply Ω-canonical Fitting subclass of $\mathfrak{F}$ such that $\mathfrak{H}_1 \cap (\mathfrak{F} \cap \mathfrak{G}_{A'}) = (1)$. Therefore $\mathcal{K}(\mathfrak{H}_1) \cap (A)' = \varnothing$, thus $\mathfrak{H}_1 \subseteq \mathfrak{G}_A$.

Then, by the definition of a Stone lattice, $\mathfrak{F} = \Omega KR^n(\mathfrak{G}_A \cup (\mathfrak{F} \cap \mathfrak{G}_{A'}))$. Moreover, $\mathfrak{G}_A \cap (\mathfrak{F} \cap \mathfrak{G}_{A'}) = (1)$. Hence for all $A \in \Omega \cap \mathcal{K}(\mathfrak{F})$ the class $\mathfrak{G}_A$ is complementable in the lattice $\Omega K^n(\mathfrak{F})$. From Lemma 3.3, it follows that $\mathfrak{F} \subseteq \mathfrak{D}_\Omega$.

Sufficiency. Let $\mathfrak{F} \subseteq \mathfrak{D}_\Omega$ and $\mathfrak{M} \in \Omega K^n(\mathfrak{F})$. Set $\Omega_1 = \Omega \cap \mathcal{K}(\mathfrak{M})$ and $\Omega_2 = (\Omega \cap \mathcal{K}(\mathfrak{F})) \setminus \Omega_1$. If $\mathfrak{M} = \mathfrak{F}$, then (1) is a complement of $\mathfrak{M}$ in the lattice $\Omega K^n(\mathfrak{F})$. If $\mathfrak{M} \neq \mathfrak{F}$, then $\mathfrak{D}_{\Omega_2}$ is a complement of $\mathfrak{M}$ in $\Omega K^n(\mathfrak{F})$. Indeed, since $\Omega_1 \cap \Omega_2 = \varnothing$ we can see that $\mathfrak{M} \cap \mathfrak{D}_{\Omega_2} = (1)$. Assume that $\mathfrak{K} = \Omega KR^n(\mathfrak{M} \cup \mathfrak{D}_{\Omega_2}) \neq \mathfrak{F}$. Because $\Omega_2 \subseteq \Omega \cap \mathcal{K}(\mathfrak{F})$, it follows from Lemma 3.1 that $\mathfrak{G}_A \subseteq \mathfrak{F}$ for every $A \in \Omega_2$. Then $\mathfrak{D}_{\Omega_2} = \times_{A \in \Omega_2} \mathfrak{G}_A \subseteq \mathfrak{F}$. Moreover, $\mathfrak{M} \subseteq \mathfrak{F}$. Hence $\mathfrak{K} = \Omega KR^n(\mathfrak{M} \cup \mathfrak{D}_{\Omega_2}) \subseteq \mathfrak{F}$. Let $\mathfrak{F} \nsubseteq \mathfrak{K}$ and let G be a group of the minimal order from $\mathfrak{F} \setminus \mathfrak{K}$. Then G is a comonolithic group. Since $G \in \mathfrak{F} \subseteq \mathfrak{D}_\Omega = \times_{A \in \Omega} \mathfrak{G}_A$, we obtain that $G \in \mathfrak{G}_A$ for some $A \in \Omega \cap \mathcal{K}(\mathfrak{F})$. Hence A is either in $\Omega_1 = \Omega \cap \mathcal{K}(\mathfrak{M})$, or $A \in \Omega_2$. In the first case, by Lemma 3.1, we obtain that $G \in \mathfrak{G}_A \subseteq \mathfrak{M} \subseteq \mathfrak{K}$, which is a contradiction. In the second case, $G \in \mathfrak{G}_A \subseteq \mathfrak{D}_{\Omega_2} = \times_{A \in \Omega_2} \mathfrak{G}_A \subseteq \mathfrak{K}$, which is also a contradiction. Therefore $\mathfrak{K} = \mathfrak{F}$. Then, by Lemma 2.6, $\Omega K^n(\mathfrak{F})$ is a Boolean lattice and is, therefore, distributive.

We will show that $\mathfrak{D}_{\Omega_2}$, a complement of $\mathfrak{M}$, is also its pseudocomplement in the lattice $\Omega K^n(\mathfrak{F})$. Let $\mathfrak{H}$ be an n-multiply Ω-canonical Fitting subclass of $\mathfrak{F}$ such that $\mathfrak{H} \cap \mathfrak{M} = (1)$. Then $\mathcal{K}(\mathfrak{H}) \cap \Omega_1 = \varnothing$. Thus $\Omega \cap \mathcal{K}(\mathfrak{H}) \subseteq \Omega_2$. Since $\mathfrak{H} \subseteq \mathfrak{F} \subseteq \mathfrak{D}_\Omega$, it can be seen that $\mathfrak{H} \subseteq \mathfrak{D}_{\Omega_2}$.

Now we will show that $\mathfrak{M}$ is a pseudocomplement of $\mathfrak{D}_{\Omega_2}$ in the lattice $\Omega K^n(\mathfrak{F})$. We will demonstrate that from Lemma 3.1 it follows that $\mathfrak{D}_{\Omega_1} \subseteq \mathfrak{M}$. Indeed, let $A \in \Omega_1 = \Omega \cap \mathcal{K}(\mathfrak{M})$. Then $\mathfrak{G}_A \subseteq \mathfrak{M}$ for every $A \in \Omega_1$. Hence $\mathfrak{D}_{\Omega_1} = \times_{A\in\Omega_1}\mathfrak{G}_A \subseteq \mathfrak{M}$. Let $\mathfrak{H}_1$ be an n-multiply Ω-canonical Fitting subclass of $\mathfrak{F}$, such that $\mathfrak{H}_1 \cap \mathfrak{D}_{\Omega_2} = (1)$. Then $\mathcal{K}(\mathfrak{H}_1) \cap \Omega_2 = \varnothing$, therefore $\Omega \cap \mathcal{K}(\mathfrak{H}_1) \subseteq \Omega_1$. Since $\mathfrak{H}_1 \subseteq \mathfrak{F} \subseteq \mathfrak{D}_\Omega$ it can be seen that $\mathfrak{H}_1 \subseteq \mathfrak{D}_{\Omega_1} \subseteq \mathfrak{M}$.

Then, by definition, we obtain that $\Omega K^n(\mathfrak{F})$ is a Stone lattice.

□

For $\Omega = \mathfrak{J}$, Theorem 3.1 yields the following statement.

Corollary 3.1. *Let $\mathfrak{F}$ be an n-multiply Ω-canonical Fitting class. Then $K^n(\mathfrak{F})$ is a Stone lattice iff $\mathfrak{F} \subseteq \mathfrak{D}_\Omega$.*

From Lemma 2.4, we immediately obtain the following result.

Lemma 3.4. *Suppose that $\mathfrak{F} = \otimes_{i\in I}\mathfrak{F}_i$ for a system of nonempty Fitting classes $\{\mathfrak{F}_i \mid i \in I\}$. If all $\mathfrak{F}_i$ are totally ΩK-Fitting classes, then $\mathfrak{F}$ is also a totally ΩK-Fitting class.*

Lemma 3.5. *Let $\mathfrak{F} = \Omega K R^\infty(G)$ be a singly generated totally Ω-canonical Fitting class. Then the lattice $\Omega K^\infty(\mathfrak{F})$ contains only a finite number of atoms.*

Proof. Let $\mathfrak{M}$ be an atom of $\Omega K^\infty(\mathfrak{F})$. Then $\mathfrak{M} = \Omega K R^\infty(A)$, where A is a simple group, which lies in $\mathfrak{M}$. Since $A \in \mathfrak{M} \subseteq \mathfrak{F}$ and $\mathfrak{F} = \Omega K R(f)$, where f is a totally Ω-canonical satellite of $\mathfrak{F}$, then $O^\Omega(A) \in f(\Omega')$ and $O^{B,B'}(A) \in f(B)$ for all $B \in \Omega \cap \mathcal{K}(A)$. Then $B \cong A$ и $1 = O^{A,A'}(A) \in f(A)$. Moreover, by definition, we obtain that A is an Ω-group.

We will show that $A \in \mathcal{K}(G)$. Let f be a minimal totally Ω-canonical satellite of $\mathfrak{F}$.

Assume that $A \notin \mathcal{K}(G)$, i.e. $A \in \Omega \setminus \mathcal{K}(G)$. Considering the structure of the satellite from Lemma 2.7, we obtain that

$$1 = O^{A,A'}(A) \in f(A) = \varnothing,$$

which is a contradiction. Therefore $A \in \mathcal{K}(G)$.

Since G is a finite group, it only has a finite number of composition factors, therefore the lattice $\Omega K^\infty(\mathfrak{F})$ contains a finite number of atoms. □

Lemma 3.6. *Let $\mathfrak{F}$ be a totally Ω-canonical Fitting class. If for every $A \in \Omega \cap \mathcal{K}(\mathfrak{F})$ the class $\mathfrak{G}_A$ is complementable in the lattice $\Omega K^\infty(\mathfrak{F})$, then $\mathfrak{F} \subseteq \mathfrak{D}_\Omega$.*

The proof of this statement is similar to that of Lemma 3.3, but based on Lemmas 3.4 and 3.5 instead of Lemmas 2.4 and 3.2.

Theorem 3.2. *Let $\mathfrak{F}$ be a totally Ω-canonical Fitting class. Then $\Omega K^\infty(\mathfrak{F})$ is a Stone lattice iff $\mathfrak{F} \subseteq \mathfrak{D}_\Omega$.*

Proof. We will show that if $\mathfrak{F} \subseteq \mathfrak{D}_\Omega$, then the lattice $\Omega K^\infty(\mathfrak{F})$ is distributive.

Let $\mathfrak{M}$, $\mathfrak{H}$ and $\mathfrak{K}$ be totally Ω-canonical Fitting subclasses of $\mathfrak{F}$. By Lemma 2.8, using induction over the multiplicity, we get that that the intersection of totally Ω-canonical Fitting classes is also a totally Ω-canonical Fitting class. Hence it is trivial that

$$\Omega KR^\infty((\mathfrak{M} \cap \mathfrak{H}) \cup (\mathfrak{M} \cap \mathfrak{K})) \subseteq \mathfrak{M} \cap \Omega KR^\infty(\mathfrak{H} \cup \mathfrak{K}).$$

Assume that the converse inclusion is not true and

$$G \in \mathfrak{M} \cap \Omega KR^\infty(\mathfrak{H} \cup \mathfrak{K}) \setminus \Omega KR^\infty((\mathfrak{M} \cap \mathfrak{H}) \cup (\mathfrak{M} \cap \mathfrak{K}))$$

is a group of minimal order, which satisfies that property. Then G is a comonolithic group. Since $G \in \mathfrak{M} \subseteq \mathfrak{F} \subseteq \mathfrak{D}_\Omega = \times_{A \in \Omega} \mathfrak{G}_A$, we obtain that $G \in \mathfrak{G}_A$ for some $A \in \Omega \cap \mathcal{K}(\mathfrak{F})$. Moreover, $G \in \Omega KR^\infty(\mathfrak{H} \cup \mathfrak{K})$, hence $A \in \Omega \cap \mathcal{K}(\Omega KR^\infty(\mathfrak{H} \cup \mathfrak{K}))$. If $A \notin \Omega \cap \mathcal{K}(\mathfrak{H} \cup \mathfrak{K})$ then it follows from Lemma 2.7 that $f(A) = \varnothing$, where f is a minimal totally Ω-canonical satellite of a Fitting class $\Omega KR^\infty(\mathfrak{H} \cup \mathfrak{K})$. On the other hand, since $G \in \Omega KR^\infty(\mathfrak{H} \cup \mathfrak{K}) = \Omega KR^\infty(f)$, it can be shown that $O^{A,A'}(G) \in f(A)$. But $G \in \mathfrak{G}_A$, and therefore $O^{A,A'}(G) = 1$. Thus $1 \in \varnothing$, which is impossible. Hence we can see that $A \in \Omega \cap \mathcal{K}(\mathfrak{H} \cup \mathfrak{K})$, and by Lemma 3.1 $\mathfrak{G}_A \subseteq \mathfrak{H}$ or $\mathfrak{G}_A \subseteq \mathfrak{K}$. Therefore

$$G \in (\mathfrak{M} \cap \mathfrak{H}) \cup (\mathfrak{M} \cap \mathfrak{K}) \subseteq \Omega KR^\infty((\mathfrak{M} \cap \mathfrak{H}) \cup (\mathfrak{M} \cap \mathfrak{K})).$$

The contradiction proves that $\Omega K^\infty(\mathfrak{F})$ is a distributive lattice.

Further proof is similar to the proof of Theorem 3.1, but based on Lemma 3.6 instead of Lemma 3.3. □

For $\Omega = \mathfrak{I}$, Theorem 3.2 yields the following statement.

Corollary 3.2. *Let $\mathfrak{F}$ be a totally canonical Fitting class. Then $\Omega K^\infty(\mathfrak{F})$ is a Stone lattice iff $\mathfrak{F} \subseteq \mathfrak{D}_\Omega$.*

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