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THE WIENER–HOPF EQUATION WITH PROBABILITY KERNEL OF OSCILLATING TYPE

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Abstract. We prove the existence of a solution to the inhomogeneous Wiener–Hopf equation whose kernel is a nonarithmetic probability distribution generating an oscillating random walk. Asymptotic properties of the solution are established depending on the properties of the inhomogeneous term of the equation.

Keywords: integral equation, inhomogeneous equation, Wiener-Hopf equation, asymptotic behavior, nonarithmetic distribution, oscillating type.

1. Introduction

Consider the inhomogeneous generalized Wiener–Hopf equation

$$z(x) = \int_{-\infty}^{x} z(x-y)\, F(dy) + f(x), \qquad x \geq 0, \tag{1}$$

where z is the function sought, F is a given probability distribution on $\mathbb{R}$ and f is a known function. We study equation (1) whose *kernel* F is a nonarithmetical probability distribution generating a random walk of *oscillating* type. Recall (see [1, § V.2, Definition 3]) that a probability distribution F on $\mathbb{R}$ is called *arithmetic* if it is concentrated on the set of points of the form $0, \pm\lambda, \pm 2\lambda, \ldots$ Let X_k, $k \geq 1$, be independent random variables with the same distribution F not concentrated at zero. These variables generate the random walk $S_0 = 0$, $S_n = X_1 + \ldots + X_n$, $n \geq 1$. By Theorem 1 in [1, § XII.2], there exist only two types of random walks: (i) the

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oscillating type (S_n oscillates with probability 1 between $-\infty$ and $+\infty$); (ii) the drifting type (S_n tends to $-\infty$ or $+\infty$ with probability 1). The random walk drifts to $-\infty$ (see [1, § XII.7, Theorem 2]) if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathsf{P}(S_n > 0) < \infty.$$

Thus, $\{S_n\}$ is an oscillating random walk if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathsf{P}(S_n > 0) = \sum_{n=1}^{\infty} \frac{1}{n} \mathsf{P}(S_n < 0) = \infty.$$

By this criterion, $\{S_n\}$ is an oscillating random walk if F is a distribution with symmetric density $k(x) = k(-x)$ since in this case $\mathsf{P}(S_n > 0) = \mathsf{P}(S_n < 0) = 1/2$. Moreover, $\{S_n\}$ is an oscillating random walk if $\mathsf{E}X_1 := \int_{\mathbb{R}} x\, F(dx) = 0$ (see [1, § XII.2, Theorem 2 and § XII.7, Theorem 3]).

Let ν and $\varkappa$ be finite measures on the σ-algebra $\mathscr{B}$ of Borel sets in $\mathbb{R}$. Their *convolution* is the measure

$$\nu * \varkappa(A) := \iint\limits_{\{x+y\in A\}} \nu(dx)\, \varkappa(dy) = \int_{\mathbb{R}} \nu(A-x)\, \varkappa(dx), \qquad A \in \mathscr{B};$$

here $A - x := \{y \in \mathbb{R} : x + y \in A\}$. Denote by F^{n*} the n-th convolution power of F:

$$F^{1*} := 1, \quad F^{(n+1)*} := F^{n*} * F, \quad n \ge 1,$$

and $F^{0*} := \delta_0$ (the atomic measure of unit mass concentrated at zero). Let U be the renewal measure generated by F: $U := \sum_{n=0}^{\infty} F^{n*}$. Denote by $\widehat{\nu}(s)$ the Laplace trnsform of an arbitrary complex-valued measure ν: $\widehat{\nu}(s) := \int_{\mathbb{R}} e^{sx}\, \nu(dx)$. Let ν be a measure defined on $\mathscr{B}$, and $a(x)$, $x \in \mathbb{R}$, a function. Define the convolution $\nu * a(x)$ as the function $\int_{\mathbb{R}} a(x-y)\, \nu(dy)$, $x \in \mathbb{R}$. For $c \in \mathbb{C}$, we assume that c/∞ is equal to zero. The relation $a(x) \sim cb(x)$ as $x \to \infty$ means that $a(x)/b(x) \to c$ as $x \to \infty$; if $c = 0$, then $a(x) = o(b(x))$. Let $\mathbb{R}_+$ be the set of all nonnegative numbers and $\mathbb{R}_- := \mathbb{R} \setminus \mathbb{R}_+$ be the set of all negative numbers.

2. Existence of a Solution and Its Explicit Form

Put $\overline{\mathscr{T}}_+ := \min\{n \ge 1 : S_n \ge 0\}$. The random variable $\overline{\mathscr{H}}_+ := S_{\overline{\mathscr{T}}_+}$ is called the *first weak ascending ladder height*. Similarly, $\overline{\mathscr{T}}_- := \min\{n \ge 1 : S_n < 0\}$ and $\overline{\mathscr{H}}_- := S_{\overline{\mathscr{T}}_-}$ is the *first strong descending ladder height*. We have the factorization identity (E stands for "expectation")

$$(2) \quad 1 - \xi \mathsf{E}(e^{sX_1}) = \big[1 - \mathsf{E}\big(\xi^{\overline{\mathscr{T}}_-} e^{s\overline{\mathscr{H}}_-}\big)\big]\big[1 - \mathsf{E}\big(\xi^{\overline{\mathscr{T}}_+} e^{s\overline{\mathscr{H}}_+}\big)\big], \quad |\xi| \le 1, \quad \Re s = 0.$$

Note that (2) was deduced in [2, Section 2] from an analogous identity in [1, § XVIII.3] for another collection of ladder variables.

Consider equation (1) with $f \in L_1(\mathbb{R}_+)$ It suffices to prove the existence of a solution z to (1) for nonnegative functions f; in the case of real functions f, we must use the representation $f = f^+ - f^-$ ($f^+ := \max(f, 0)$, $f^- := -\min(f, 0)$), while $f = \Re f + i \Im f$ for complex f. Then, for real functions, we have $z = z_+ - z_-$, where $z_\pm$ are two solutions to (1) with $f^\pm$ instead of f. Denote by $F_\pm$ the distributions of the random variables $\overline{\mathscr{H}}_\pm$ respectively. It follows from the identity (2) that

$$(3) \qquad \delta_0 - F = (\delta_0 - F_-) * (\delta_0 - F_+).$$

Let $U_\pm := \sum_{k=0}^\infty F_\pm^{k*}$ be the renewal measures generated by the distributions $F_\pm$ respectively. Denote by $\mathbf{1}_{\mathbb{R}_-}$ the indicator of the subset $\mathbb{R}_-$ in $\mathbb{R}$: $\mathbf{1}_{\mathbb{R}_-}(x) = 1$ for $x \in \mathbb{R}_-$ and $\mathbf{1}_{\mathbb{R}_-}(x) = 0$ for $x \in \mathbb{R}_+$. A similar meaning has the notation $\mathbf{1}_{\mathbb{R}_+}$. Extend the function f onto the whole line: $f(x) := 0$, $x < 0$. This convention will be valid throughout.

Theorem 1. *Let F be a probability distribution and $f \in L_1(\mathbb{R}_+)$. Then the function*

$$z(x) = U_+ * \big[(U_- * f)\mathbf{1}_{\mathbb{R}_+}\big](x), \qquad x \in \mathbb{R}_+, \tag{4}$$

is the solution to (1) *coinciding with the solution obtained by successive approximations.*

Proof. Put $\xi \in (0,1)$ in (2). We get

$$1 - \xi\widehat{F}(s) = [1 - \widehat{F}_{\xi-}(s)][1 - \widehat{F}_{\xi+}(s)], \qquad \Re s = 0,$$

where $F_{\xi\pm}$ are positive measures concentrated on the sets $\mathbb{R}_\pm$ respectively; moreover, $F_{\xi\pm}(\mathbb{R}_\pm) < 1$. Solve the equation

$$z_\xi(x) = \xi \int_{-\infty}^x z_\xi(x-y)\, F(dy) + f(x), \qquad x \geq 0, \tag{5}$$

by successive approximations:

$$z_\xi^{(0)}(x) = f(x), \quad z_\xi^{(n)}(x) = \xi \int_{-\infty}^x z_\xi^{(n-1)}(x-y)\, F(dy) + f(x), \quad x \geq 0, \quad n \geq 1.$$

Obviously, $z_\xi^{(n)}(x) \uparrow$ as $n \uparrow \infty$ for all $x \in \mathbb{R}_+$. Hence $z_\xi(x) = \lim_{n\to\infty} z_\xi^{(n)}(x)$, $x \in \mathbb{R}_+$, is a solution to (5). Let us show that $z_\xi \in L_1(\mathbb{R}_+)$. Consider the renewal equation

$$\zeta_\xi(x) = \xi \int_{\mathbb{R}} \zeta_\xi(x-y)\, F(dy) + f(x), \qquad x \in \mathbb{R}. \tag{6}$$

Construct its solution by successive approximations:

$$\zeta_\xi^{(0)}(x) = f(x), \quad \zeta_\xi^{(n)}(x) = \xi \int_{\mathbb{R}} \zeta_\xi^{(n-1)}(x-y)\, F(dy) + f(x), \quad n = 1, 2, \dots$$

Since $\zeta_\xi^{(n)}(x) \uparrow$ as $n \uparrow$, we can pass to the limit under the integral sign. Thus, the limits $\zeta_\xi(x) := \lim_{n\to\infty} \zeta_\xi^{(n)}(x)$, $x \in \mathbb{R}$, exist; moreover, $z_\xi(x) \leq \zeta_\xi(x)$, $x \in \mathbb{R}_+$. The function $\zeta_\xi(x)$ is a solution to (6). Since $\zeta_\xi^{(n)}(x) = \sum_{k=0}^n \xi^k F^{k*} * f(x)$, the function $\zeta_\xi(x)$ is representable as $\zeta_\xi(x) = U_\xi * f(x)$, where $U_\xi = \sum_{k=0}^\infty \xi^k F^{k*}$ is the renewal measure generated by the improper distribution ξF. The measure U_ξ is finite since $\xi F(\mathbb{R}) = \xi < 1$. Hence $\zeta_\xi \in L_1(\mathbb{R})$. Thus, we have proved the existence of a solution $z_\xi \in L_1(\mathbb{R}_+)$ to (5).

Let us first find the explicit form of the solution z_ξ. The article [3] contains a brief exposition of the formal scheme for solving the classical Wiener–Hopf equation (see also [4, Chapter II, § 5, Subsection 5.4]). By analogy with the first step of this scheme, write down (5) on the whole line. Let

$$v_\xi(x) = \begin{cases} z_\xi(x) & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases} \quad n_\xi(x) = \begin{cases} 0 & \text{for } x \geq 0, \\ -\xi \int_{-\infty}^x z_\xi(x-y)\, F(dy) & \text{for } x < 0. \end{cases}$$

The function $n_\xi(x)$ makes sense. Indeed, $|n_\xi(x)| \le \xi F * \zeta_\xi(x) \in L_1(\mathbb{R})$. Equation (5) becomes equivalent to the following renewal equation:

$$v_\xi(x) = \xi \int_{-\infty}^{\infty} v_\xi(x-y)\, F(dy) + f(x) + n_\xi(x), \qquad x \in \mathbb{R},$$

or, briefly,

$$(\delta_0 - \xi F) * v_\xi = f + n_\xi. \tag{7}$$

Put $U_{\xi\pm} := \sum_{n=0}^{\infty} F_{\xi\pm}^{n*}$. The measures $U_{\xi\pm}$ are finite. Form the convolutions of $U_{\xi-}$ with both sides of (7). The equality $\delta_0 - \xi F = (\delta_0 - F_{\xi-}) * (\delta_0 - F_{\xi+})$ implies that

$$U_{\xi-} * (\delta_0 - \xi F) = \delta_0 - F_{\xi+}.$$

Therefore,

$$(\delta_0 - F_{\xi+}) * v_\xi(x) = U_{\xi-} * (f + n_\xi)(x), \qquad x \in \mathbb{R}. \tag{8}$$

The finite measure $U_{\xi-}$ is nonnegative and concentrated on $\mathbb{R}_- \cup \{0\}$. The left-hand side of (8) is identically zero on $\mathbb{R}_-$. Consequently,

$$[U_{\xi-} * (f + n_\xi)]\mathbf{1}_{\mathbb{R}_-} = 0,$$

which implies that $U_{\xi-} * n_\xi = -(U_{\xi-} * f)\mathbf{1}_{\mathbb{R}_-}$. Form the convolutions of both sides of this equality with the measure $\delta_0 - F_{\xi-}$. We obtain

$$n_\xi = -(\delta_0 - F_{\xi-}) * \big[(U_{\xi-} * f)\mathbf{1}_{\mathbb{R}_-}\big].$$

Thus,

$$\begin{aligned} v_\xi = U_\xi * (f + n_\xi) &= U_\xi * \big\{f - (\delta_0 - F_{\xi-}) * \big[(U_{\xi-} * f)\mathbf{1}_{\mathbb{R}_-}\big]\big\} \\ &= U_\xi * \big\{f - (\delta_0 - F_{\xi-})(U_{\xi-} * f) + (\delta_0 - F_{\xi-}) * \big[(U_{\xi-} * f)\mathbf{1}_{\mathbb{R}_+}\big]\big\} \\ &= U_\xi * \big\{(\delta_0 - F_{\xi-}) * \big[(U_{\xi-} * f)\mathbf{1}_{\mathbb{R}_+}\big]\big\} = U_{\xi+} * \big[(U_{\xi-} * f)\mathbf{1}_{\mathbb{R}_+}\big]. \end{aligned}$$

Letting $\xi \uparrow 1$, we get (4). Let us prove the last assertion of the theorem. Form the successive approximations $z^{(n)}$ for the initial equation (1):

$$z^{(0)}(x) = f(x), \quad z^{(n)}(x) = \int_{-\infty}^{x} z^{(n-1)}(x-y)\, F(dy) + f(x), \qquad x \ge 0.$$

We have $z^{(n)}(x) \uparrow$ as $n \uparrow$. Consequently, there exist limits $\lim_{n\to\infty} z^{(n)}(x)$, $x \ge 0$. Show that they coincide with (4). Using monotonicity and induction on n, we have

$$z^{(n)}(x) = \lim_{\xi\uparrow 1} z_\xi^{(n)}(x) \le \lim_{\xi\uparrow 1} z_\xi(x) = z(x), \qquad x \ge 0.$$

It follows that

$$z_\xi(x) := \lim_{n\to\infty} z_\xi^{(n)}(x) \le \lim_{n\to\infty} z^{(n)}(x) \le \lim_{\xi\uparrow 1} z_\xi(x) = z(x).$$

Passing to the limit as $\xi \uparrow 1$, we see that $\lim_{n\to\infty} z^{(n)}(x) = z(x)$, i.e., the solution z defined by (4) coincides with the solution $\lim_{n\to\infty} z^{(n)}(x)$, obtained by successive approximations. □

Remark 1. Theorem 1 holds for all types of nonarithmetic probability distributions: both for distributions of drifting type and for distributions of oscillating type.

3. Absolutely continuous components

We shall need the following conditions on F and $F_\pm$.

Condition $(\mathfrak{S})$. For some $n = n(F) \ge 1$ the distribution F^{n*} has a nonzero absolutely continuous component.

Condition $(\mathfrak{S}_\pm)$. For some $n = n(F_\pm) \ge 1$ the distributions $F_\pm^{n*}$ have a nonzero absolutely continuous components.

Theorem 2. *Let F be a probability distribution such that $\mathsf{E}X_1 = 0$ and $\mathsf{E}X_1^2 = \sigma^2 < \infty$. Then $(\mathfrak{S}) \Longrightarrow (\mathfrak{S}_\pm)$.*

Proof. Let G be a distribution with nonzero mean. Then the distribution G^{n*} for some $n = n(G) \ge 1$ has a nonzero absolutely continuous component if and only if the function

$$\frac{s}{[1-\widehat{G}(s)](s-1)}, \quad \Re s = 0,$$

is the Laplace transform $\widehat{V}_G(s)$ of a finite measure V_G [5]. By Theorem 1 in [1, § XVIII.5] applied to the random walk $\{-S_n\}$, there exist finite expectations $\mu_\pm := \mathsf{E}\overline{\mathscr{H}}_\pm$; moreover, $\mu_+|\mu_-| = \sigma^2/2$. Show that condition $(\mathfrak{S})$ implies the following assertion: the function

$$\widehat{V}(s) = \frac{s^2}{[1-\widehat{F}(s)](s-1)^2}, \qquad \Re s = 0,$$

is the Laplace transform $\widehat{V}_G(s)$ of a finite measure V. Let ν be a finite complex measure. Consider the measure $T\nu$ with density

$$v(x,\nu) := \begin{cases} -\nu((-\infty, x]), & x \le 0, \\ \nu((x,\infty)), & x > 0. \end{cases}$$

Then $T\nu$ is a locally finite measure, i.e., the values $T\nu(A)$ are finite on bounded sets $A \in \mathscr{B}$. Denote by $|\nu|$ the total variatiion of ν. If $\int_{\mathbb{R}} |x|\,|\nu|(dx) < \infty$, then $T\nu$ is a locally finite measure and its Laplace transform is of the form $\widehat{T\nu}(s) = [\widehat{\nu}(s) - \widehat{\nu}(0)]/s$, $\Re s = 0$, where $\widehat{T\nu}(0)$ is defined by continuity as $\widehat{T\nu}(0) = \int_{\mathbb{R}} x\,\nu(dx)$. If $\int_{\mathbb{R}} |x|^k\,|\nu|(dx) < \infty$, where $k \ge 1$ is an integer, then $\int_{\mathbb{R}} |x|^{k-1}\,|T\nu|(dx) < \infty$ by Theorem 3 in [6], where we put $\varphi(x) = (1+|x|)^{k-1}$. We have

$$\begin{aligned}\frac{1}{\widehat{V}(s)} &= \frac{[1-\widehat{F}(s)](s^2-2s+1)}{s^2} \\ &= 1-\widehat{F}(s) + \frac{2[\widehat{F}(s)-1]}{s} + \frac{1-\widehat{F}(s)}{s^2} = 1-\widehat{F}(s) + 2\widehat{TF}(s) - \widehat{T^2F}(s),\end{aligned}$$

since $\widehat{TF}(0) = 0$. Consequently, the function $1/\widehat{V}(s)$ is the Laplace transform $\widehat{W}(s)$ of the finite measure $W = \delta_0 - F + 2TF - T^2F$ and $\widehat{W}(0) = -\sigma^2/2 \ne 0$. Show that the measure W is invertible in the Banach algebra $\mathfrak{B}$ of finite measures ν on $\mathscr{B}$, where the multiplication is the convolution of measures, the norm is $|\nu|(\mathbb{R})$, the unity of $\mathfrak{B}$ is the measure δ_0, the addition and multiplication of measures by numbers are defined in the usual way. Let $\mathscr{M}$ be the space of maximal ideals of $\mathfrak{B}$. The following facts are well known [7]. Each maximal ideal $M \in \mathscr{M}$ generates a homoorphism $h : \mathfrak{B} \to \mathbb{C}$ and M is the kernel of this homomorphism. Denote by $\nu(M)$ the value of h at $\nu \in \mathfrak{B}$. An element $\nu \in \mathfrak{B}$ is invertible if and only if ν does not belong to every maximal ideal $M \in \mathscr{M}$. In other words, ν is invertible if and only if $\nu(M) \neq 0$ for all $M \in \mathscr{M}$. The space $\mathscr{M}$ is split into two sets: $\mathscr{M}_1$ is the set of the maximal ideals which do not contain the collection $L(\mathfrak{B})$ of all absolutely continuous measures in $\mathfrak{B}$ and $\mathscr{M}_2 = \mathscr{M} \setminus \mathscr{M}_1$. If $M \in \mathscr{M}_1$, then the corresponding homomorphism is of the form $h(\nu) = \widehat{\nu}(s_0)$, where $\Re s_0 = 0$. In this case, $M = \{\nu \in \mathfrak{B} : \widehat{\nu}(s_0) = 0\}$ [8, Chapter IV, Section 4]. If $M \in \mathscr{M}_2$, then $\nu(M) = 0$ for all $\nu \in L(\mathfrak{B})$. Show that $W(M) \neq 0$ for all $M \in \mathscr{M}$, thus establishing the existence of an inverse element $W^{-1} \in \mathfrak{B}$. If $M \in \mathscr{M}_1$, then for some s_0, $\Re s_0 = 0$, we have $W(M) = \widehat{W}(s_0) \neq 0$, since $\widehat{F}(s) \neq 1$ on $i\mathbb{R} \setminus \{0\}$ for the nonarithmetic distribution F. Let $M \in \mathscr{M}_2$. Obviously, $W(M) = 1 - F(M)$ since TF and T^2F are absolutely continuous measures. Let $F^{n*} = F_c^{n*} + F_s^{n*}$ be the decomposition of the distribution F^{n*} into the absolutely continous and singular components with respect to Lebesgue measure (F_s^{n*} is the sum of the discrete and singular components in the usual sense). We have $F^n(M) = F^{n*}(M) = F_s^{n*}(M)$,

$$|F_s^{n*}(M)| \leq \|F_s^{n*}\| = F_s^{n*}(\mathbb{R}) < F_c^{n*}(\mathbb{R}) + F_s^{n*}(\mathbb{R}) = F^{n*}(\mathbb{R}) = 1.$$

Therefore, $|F(M)| < 1$ and $W(M) \neq 0$, hence there exists an inverse element $W^{-1} \in \mathfrak{B}$. Obviously, $W^{-1} = V$ since the Laplace transform of the measure W^{-1} is equal to $1/\widehat{W}(s) = \widehat{V}(s)$. Use the factorization (3):

$$\widehat{V}(s) = \frac{s}{[1 - \widehat{F}_-(s)](s-1)} \cdot \frac{s}{[1 - \widehat{F}_+(s)](s-1)} =: \widehat{V}_-(s) \cdot \widehat{V}_+(s). \tag{9}$$

Show that $\widehat{V}_+(s)$ is the Laplace transform of some finite measure V_+. By the aforementioned criterion from [5], it will follow that the distribution F_+ satisfies condition $(\mathfrak{S}_+)$. The function

$$\widehat{W}_-(s) = \frac{[1 - \widehat{F}_-(s)](s-1)}{s}, \qquad \Re s = 0,$$

is the Laplace transform of the finite measure $W_- = \delta_0 - F_- + TF_-$. Multiply both sides of equality (9) by $\widehat{W}_-(s)$. We obtain $\widehat{V}_+(s) = \widehat{V}(s)\widehat{W}_-(s)$, i.e., $\widehat{V}_+(s)$, $\Re s = 0$, is the Laplace transform of the finite measure $V_+ = V * W_-$. Thus, $(\mathfrak{S}) \Longrightarrow (\mathfrak{S}_+)$. Similarly, $(\mathfrak{S}) \Longrightarrow (\mathfrak{S}_-)$. □

4. Asymptotic properties of the solution

We shall need the following fact which is a consequence of [9, Theorem 2.6.4 (a)].

Theorem 3. *Let F be a probability distribution with positive mean μ. Let $a(x)$, $x \in \mathbb{R}$, be a bounded summable function such that $a(x) \to 0$ as $x \to \infty$. Suppose that condition $(\mathfrak{S})$ is satisfied. Then*

$$U * a(x) \to \frac{1}{\mu} \int_{\mathbb{R}} a(y)\, dy \qquad \text{as } x \to \infty\ .$$

Consider two cases depending on the properties of the inhomogeneous term f.

Theorem 4. *Let F be a probability distribution such that $\mathsf{E}X_1 = 0$, $\mathsf{E}X_1^2 < \infty$ and condition $(\mathfrak{S})$ holds. Suppose that the function $\int_x^\infty |f(y)|\, dy$, $x \in \mathbb{R}_+$, is summable and $f(x)$, $x \in \mathbb{R}$, is a bounded function such that $f(x) \to 0$ as $x \to \infty$. Then the solution z to (1) given by (4) satisfies the relation*

$$z(x) \to \frac{1}{\mu_+} \int_{-\infty}^{0} \int_{-y}^{\infty} f(u)\, du\, U_-(dy) \qquad \text{as } x \to \infty, \tag{10}$$

where $\mu_+ := \mathsf{E}\overline{\mathscr{H}}_+$.

Proof. By Theorem 2, the distributions $F_\pm$ satisfiy condition $(\mathfrak{S}_\pm)$. Show that $(U_- * f)\mathbf{1}_{\mathbb{R}_+} \in L_1(\mathbb{R}_+)$. As established in [10], the renewal measure U_- admits the decomposition $U_- = U_1 + U_2$, where the measure U_1 is absolutely continuous with a bounded continuous density $u_1(x)$ such that $u_1(x) \to 1/|\mu_-|$ as $x \to -\infty$, and the measure U_2 is finite. Choose a constant $C < \infty$ such that $|f(x)| \le C$, $x \in \mathbb{R}$, and $|u_1(x)| \le C$ for all $x \in \mathbb{R}_-$. We have

$$\begin{aligned}|U_- * f(x)| &\le \int_{-\infty}^0 |f(x-y)|\, U_-(dy) \\ &\le \int_{-\infty}^0 |f(x-y)| u_1(y)\, dy + \int_{-\infty}^0 |f(x-y)|\, |U_2|(dy) =: I_1(x) + I_2(x),\end{aligned}$$

$$I_1(x) \le C \int_{-\infty}^0 |f(x-y)|\, dy = C \int_x^\infty |f(y)|\, dy \in L_1(\mathbb{R}_+).$$

It is easily seen that $I_2(x) \in L_1(\mathbb{R}_+)$:

$$\begin{aligned}\int_0^\infty I_2(x)\, dx &= \int_0^\infty \int_{-\infty}^0 |f(x-y)|\, |U_2|(dy)\, dx \\ &= \int_{-\infty}^0 \int_0^\infty |f(x-y)|\, dx\, |U_2|(dy) = \int_{-\infty}^0 \int_{-y}^\infty |f(x)|\, dx\, |U_2|(dy) \\ &\le \int_{-\infty}^0 \int_0^\infty |f(x)|\, dx\, |U_2|(dy) = \|f\|_1 |U_2|(\mathbb{R}_- \cup \{0\}) < \infty.\end{aligned}$$

Moreover, the functions $I_k(x)$, $k = 1, 2$, are bounded:

$$I_1(x) \le C\|f\|_1, \qquad I_2(x) \le C|U_2|(\mathbb{R}_- \cup \{0\}).$$

Obviously, they tend to zero as $x \to \infty$. In order to establish relation (10), it remains to apply Theorem 3 to the function $U_- * f(x)$ and the distribution F_+ with mean μ_+: as $x \to \infty$

$$\begin{aligned}z(x) &= \int_0^x U_- * f(x-y)\, U_+(dy) \to \frac{1}{\mu_+} \int_0^\infty U_- * f(x)\, dx \\ &= \frac{1}{\mu_+} \int_0^\infty \int_{-\infty}^0 f(x-y)\, U_-(dy)\, dx = \frac{1}{\mu_+} \int_{-\infty}^0 \int_{-y}^\infty f(u)\, du\, U_-(dy). \quad \square\end{aligned}$$

Theorem 5. *Let F be a probability distribution such that $\mathsf{E}X_1 = 0$, $\mathsf{E}X_1^2 = \sigma^2 < \infty$ and condition $(\mathfrak{S})$ holds. Suppose that the function $f(x)$, $x \in \mathbb{R}$, is a bounded nonnegative summable function such that $f(x) \to 0$ as $x \to \infty$ and $\int_0^\infty \int_x^\infty f(y)\, dy\, dx = \infty$. Then the solution z to* (1) *given by* (4) *satisfies the relation*

$$z(x) \sim \frac{2}{\sigma^2} \int_0^x \int_y^\infty f(u)\, du\, dy \qquad \text{as } x \to \infty. \tag{11}$$

Proof. Denote by $\mathfrak{L}_-$ the restriction of Lebesgue measure to $\mathbb{R}_-$. We use the notation from the proof of Theorem 4. Represent the renewal measure U_- in the form

$$U_- = \frac{1}{|\mu_-|}\mathfrak{L}_- + U_2 + \Big(U_1 - \frac{1}{|\mu_-|}\mathfrak{L}_-\Big).$$

The measure $U_3 := U_1 - \mathfrak{L}_-/|\mu_-|$ is absolutely continuous with density

$$u_3(x) := u(x) - \frac{1}{|\mu_-|} \to 0 \qquad \text{as } x \to -\infty.$$

We have

$$U_- * f(x) = \frac{1}{|\mu_-|}\int_x^\infty f(y)\,dy + \int_{-\infty}^0 f(x-y)\,\big[U_2(dy) + u_3(y)\,dy\big] =: \sum_{k=1}^3 J_k(s).$$

By Theorem 4 in [11], where we put $K(x) = \int_x^\infty f(y)\,dy$,

$$U_+ * J_1(x) \sim \frac{1}{\mu_+|\mu_-|}\int_0^x \int_y^\infty f(u)\,du\,dy \to \infty \qquad \text{as } x \to \infty. \tag{12}$$

Since $\mu_+|\mu_-| = \sigma^2/2$, the right-hand side of (12) coincides with the right-hand side of (11). Show that $J_2(x) \in L_1(\mathbb{R}_+)$. We have

$$\begin{aligned}\int_0^\infty |J_2(x)|\,dx &\le \int_0^\infty \int_{-\infty}^0 f(x-y)\,|U_2|(dy)\,dx \\ &= \int_{-\infty}^0 \int_{-y}^\infty f(x)\,dx\,|U_2|(dy) \le \|f\|_1 |U_2|((-\infty,0]) < \infty.\end{aligned}$$

The function $J_2(x)$ is bounded and tends to zero as $x \to \infty$. By Theorems 2 and 3, it follows that

$$U_+ * J_2(x) \to \frac{1}{\mu_+}\int_0^\infty J_2(y)\,dy \qquad \text{as } x \to \infty. \tag{13}$$

Given an arbitrary $\varepsilon > 0$, choose $A = A(\varepsilon) < 0$ such that $|u_3(x)| \le \varepsilon$ for $x \le A$. Split $J_3(x)$ into the sum of two integrals:

$$J_3(x) = \left(\int_{-\infty}^A + \int_A^0\right) f(x-y)u_3(y)\,dy =: J_4(x) + J_5(x).$$

We have

$$\begin{aligned}|J_4(x)| &\le \int_{-\infty}^A f(x-u)|u_3(u)|\,du \\ &\le \varepsilon \int_{-\infty}^A f(x-u)\,du = \varepsilon \int_{x-A}^\infty f(u)\,du \le \varepsilon|\mu_-|J_1(x).\end{aligned}$$

Consequently,

$$\begin{aligned}|U_+ * J_4(x)| &\le U_+ * |J_4|(x) \le \varepsilon|\mu_-|U_+ * J_1(x) \\ &\sim \frac{\varepsilon|\mu_-|}{\mu_+}\int_0^x \int_y^\infty f(u)\,du\,dy \qquad \text{as } x \to \infty.\end{aligned} \tag{14}$$

Show that $J_5(x) \in L_1(\mathbb{R}_+)$. Let $|u_3(x)| \le C < \infty$. Then

$$\begin{aligned}\int_0^\infty |J_5(x)|\,dx &\le C\int_0^\infty \int_A^0 f(x-y)\,dy\,dx \\ &= C\int_0^\infty \int_x^{x-A} f(v)\,dv\,dx = C\int_0^\infty f(v)\int_{(v+A)^+}^v dx\,dv \le CA\|f\|_1.\end{aligned}$$

Obviously, $|J_5(x)| \leq C^2|A| < \infty$ and $J_5(x) \to 0$ as $x \to \infty$ by the Lebesgue Dominated Convergence Theorem. Apply Theorems 2 and 3 to the distribution F_+ and to the function $J_5(x)$:

$$U_+ * J_5(x) \to \frac{1}{\mu_+} \int_0^\infty J_5(y)\, dy \qquad \text{as } x \to \infty. \tag{15}$$

The relations (12)–(15) imply that, for $k = 2, \dots, 5$,

$$U_+ * J_k(x) = o[U_+ * J_1(x)] \qquad \text{as } x \to \infty,$$

which completes the proof of the theorem. □

Theorem 5 implies the following result without the nonnegativity of f.

Corollary 1. *Let F be a probability distribution such that $\mathsf{E}X_1 = 0$, $\mathsf{E}X_1^2 = \sigma^2 < \infty$ and condition $(\mathfrak{S})$ holds. Suppose that $f(x) = f_1(x) - f_2(x)$, $x \in \mathbb{R}_+$, where $f_1(x)$ and $f_2(x)$ are bounded nonnegative summable functions such that $f_i(x) \to 0$ as $x \to \infty$, $i = 1, 2$, and*

$$\int_0^x \int_y^\infty f_i(u)\, du\, dy \sim c_i d(x) \qquad \text{as } x \to \infty, \qquad i = 1, 2,$$

where c_1, $c_2 \geq 0$ are constants. Then the solution z to (1) given by (4) satisfies the relation

$$z(x) \sim \frac{2(c_1 - c_2)}{\sigma^2} d(x) \qquad \text{as } x \to \infty.$$

If $c_1 \neq c_2$, then relation (11) holds.

Proof. Assume for simplicity that c_1, $c_2 > 0$. Denote by z_1, z_2 the solutions to (1) corresponding to f_1, f_2 respectively. Then $z = z_1 - z_2$ is a solution to (1), corresponding to f. By Theorem 5,

$$\frac{z(x)}{d(x)} = \frac{z_1(x)}{\displaystyle\int_0^x \int_y^\infty f_1(u)\, du\, dy} \cdot \frac{1}{d(x)} \int_0^x \int_y^\infty f_1(u)\, du\, dy$$

$$- \frac{z_2(x)}{\displaystyle\int_0^x \int_y^\infty f_2(u)\, du\, dy} \cdot \frac{1}{d(x)} \int_0^x \int_y^\infty f_2(u)\, du\, dy \to \frac{2(c_1 - c_2)}{\sigma^2} \qquad \text{as } x \to \infty. \quad \square$$

5. Influence of the solution to the homogeneous equation

The solution z to (1) obtained in Theorem 1 is not the only solution to this equation. Consider the homogeneous equation

$$Z(x) = \int_{-\infty}^x Z(x - y)\, F(dy), \qquad x \in \mathbb{R}_+, \tag{16}$$

where Z is the function sought and F is a probability distribution that generates an oscillating random walk $\{S_n\}$. Put $\mathscr{T}_+ := \min\{n \geq 1 : S_n > 0\}$. Let U_{G+} be the renewal measure generated by the distribution G_+ of the *first strong ascending ladder height* $\mathscr{H}_+ := S_{\mathscr{T}_+}$. The renewal function $Z(x) := U_{G+}((-\infty, x])$, $x \in \mathbb{R}_+$, is a solution to (16) with normalization $Z(0) = 1$ (see [12, Theorem 1]) and asymptotics $Z(x) \sim x/\mathsf{E}\mathscr{H}_+$ as $x \to \infty$. Let $c \in \mathbb{C}$ be arbitrary. Obviously, the function $Z_c := z + cZ$ is a solution to the inhomogeneous equation (1). In what follows, we assume that $\mathsf{E}X_1 = 0$ and $\sigma^2 = \mathsf{E}X_1^2 < \infty$. Let us investigate the asymptotics of the solution Z_c under various assumptions on the kernel F and on the inhomogeneous term f. The following facts are known:

1) if $\mathsf{E}(X_1^+)^3 < \infty$, then $\mathsf{E}\mathscr{H}_+^2 < \infty$ (see [13, Theorem (i)] with $\phi(x) = x^2$);

2) if $\mathsf{E}\mathscr{H}_+^2 < \infty$, then (see [1, § XI.4])

$$Z(x) - \frac{x}{\mathsf{E}\mathscr{H}_+} \to \frac{\mathsf{E}\mathscr{H}_+^2}{2(\mathsf{E}\mathscr{H}_+)^2} \qquad \text{as } x \to \infty;$$

3) if $\mathsf{E}(X_1^+)^3 = \infty$, then [14, Theorem 1]

$$Z(x) - \frac{x}{\mathsf{E}\mathscr{H}_+} \sim \frac{2}{\sigma^2 \mathsf{E}\mathscr{H}_+} \int_0^x \int_y^\infty \int_v^\infty [1 - F(u)]\, du\, dv\, dy \qquad \text{as } x \to \infty.$$

The asymptotics of the difference $Z_c(x) - cx/\mathsf{E}\mathscr{H}_+$ as $x \to \infty$ is established by comparing the asymptotic behavior of the functions

$$A_1(x) := \int_0^x \int_y^\infty f(u)\, du\, dy, \qquad A_2(x) := \int_0^x \int_y^\infty \int_v^\infty [1 - F(u)]\, du\, dv\, dy.$$

In order to assess the rate of growth of $A_2(x)$, let us consider the case when the tail $1 - F(x) =: R(x)$ of F regularly varies at infinity with index $\alpha \in (-3, -2)$, i.e., when $R(tx)/R(x) \to t^\alpha$ as $x \to \infty$, $t > 0$ being fixed. Then

$$Z(x) - \frac{x}{\mathsf{E}\mathscr{H}_+} \sim \frac{2A_2(x)}{\sigma^2 \mathsf{E}\mathscr{H}_+} \sim \frac{2x^3[1 - F(x)]}{\sigma^2 \mathsf{E}\mathscr{H}_+(\alpha + 1)(\alpha + 2)(\alpha + 3)} \qquad \text{as } x \to \infty.$$

The last equivalence follows from the properties of regularly varying functions (see [1, § VIII.9, Theorem 1]). Consider the following possibilities

$$(17) \quad A_1(x) = o(A_2(x)), \quad A_2(x) = o(A_1(x)), \quad A_1(x) \sim K A_2(x) \qquad \text{as } x \to \infty,$$

where $K > 0$. The next results about the asymptotic behavior of the solution Z_c to (1) follow from the above facts and the already proved theorems.

Theorem 6. *Let the hypotheses of Theorem* 4 *be satisfied and let* $\mathsf{E}(X_1^+)^3 < \infty$. *Then*

$$Z_c(x) - \frac{cx}{\mathsf{E}\mathscr{H}_+} \to \frac{1}{\mu_+} \int_{-\infty}^0 \int_{-y}^\infty f(u)\, du\, U_-(dy) + \frac{c\,\mathsf{E}\mathscr{H}_+^2}{2(\mathsf{E}\mathscr{H}_+)^2} \qquad \text{as } x \to \infty.$$

Theorem 7. *Let F be a probability distribution such that condition* ($\mathfrak{S}$) *holds true and let f satisfy the hypotheses of either Theorem* 4 *or Theorem* 5. *Suppose that at least one of the functions $A_1(x)$ and $A_2(x)$ tends to infinity as $x \to \infty$. Then in all three cases of* (17) *the following relation holds*:

$$Z_c(x) - \frac{cx}{\mathsf{E}\mathscr{H}_+} \sim \frac{2}{\sigma^2} A_1(x) + \frac{2c}{\sigma^2 \mathsf{E}\mathscr{H}_+} A_2(x) \qquad \text{as } x \to \infty.$$

Remark 2. A probability distribution F is called *symmetric* if its distribution function $F(x) := F((-\infty, x])$ is equal to $1 - F(-x - 0)$ [15, § 3.1]. In other words, the distribution F of the random variable X_1 is symmetric if it coincides with the distribution of $-X_1$. A distribution F is called *continuous* if its distribution function is continuous. For symmetric distributions, the formulas in Theorems 4–7 become simplified since $\mathsf{E}\mathscr{H}_+ = \sigma/\sqrt{2\gamma_0}$ and $\mu_+ = \sigma\sqrt{\gamma_0/2}$, where γ_0 is equal to $\exp[-\sum_{n=0}^\infty \mathsf{P}(S_n = 0)/n]$ (see, for example, [12, Lemma 2]). For continuous distributions F the quantity γ_0 is equal to one.

6. Conclusion

The present paper concludes our investigations of the inhomogeneous Wiener-Hopf equation (1) started in papers [16] and [17]. Here we have considered the case when the distribution F generates a random walk of oscillating type whereas the papers [16, 17] deal with cases when the corresponding random walk drifts to $+\infty$ and to $-\infty$, respectively.

If the distribution F is absolutely continuous, i.e., $F(dx) = k(x)\,dx$, then equation (1) is equivalent to the classical Wiener-Hopf equation

$$z(x) = \int_0^\infty k(x-y)z(y)\,dy + f(x), \qquad x \geq 0.$$

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