THE WIENER–HOPF EQUATION WITH PROBABILITY KERNEL OF OSCILLATING TYPE

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Abstract. We prove the existence of a solution to the inhomogeneous Wiener–Hopf equation whose kernel is a nonarithmetic probability distribution generating an oscillating random walk. Asymptotic properties of the solution are established depending on the properties of the inhomogeneous term of the equation.

Keywords: integral equation, inhomogeneous equation, Wiener-Hopf equation, asymptotic behavior, nonarithmetic distribution, oscillating type.

1. Introduction

Consider the inhomogeneous generalized Wiener–Hopf equation

\[ z(x) = \int_{-\infty}^{x} z(x-y) F(dy) + f(x), \quad x \geq 0, \]

where \( z \) is the function sought, \( F \) is a given probability distribution on \( \mathbb{R} \) and \( f \) is a known function. We study equation (1) whose kernel \( F \) is a nonarithmetic probability distribution generating a random walk of oscillating type. Recall (see [1, § V.2, Definition 3]) that a probability distribution \( F \) on \( \mathbb{R} \) is called arithmetic if it is concentrated on the set of points of the form \( 0, \pm \lambda, \pm 2\lambda, \ldots \). Let \( X_k, k \geq 1, \) be independent random variables with the same distribution \( F \) not concentrated at zero. These variables generate the random walk \( S_0 = 0, S_n = X_1 + \ldots + X_n, n \geq 1. \) By Theorem 1 in [1, § XII.2], there exist only two types of random walks: (i) the

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oscillating type ($S_n$ oscillates with probability 1 between $-\infty$ and $+\infty$); (ii) the drifting type ($S_n$ tends to $-\infty$ or $+\infty$ with probability 1). The random walk drifts to $-\infty$ (see [1, § XII.7, Theorem 2]) if and only if
\[ \sum_{n=1}^{\infty} \frac{1}{n} P(S_n > 0) < \infty. \]
Thus, $\{S_n\}$ is an oscillating random walk if and only if
\[ \sum_{n=1}^{\infty} \frac{1}{n} P(S_n > 0) = \sum_{n=1}^{\infty} \frac{1}{n} P(S_n < 0) = \infty. \]
By this criterion, $\{S_n\}$ is an oscillating random walk if $F$ is a distribution with symmetric density $k(x) = k(-x)$ since in this case $P(S_n > 0) = P(S_n < 0) = 1/2$.
Moreover, $\{S_n\}$ is an oscillating random walk if $EX_1 := \int_{\mathbb{R}} x F(dx) = 0$ (see [1, § XII.2, Theorem 2 and § XII.7, Theorem 3]).

Let $\nu$ and $\varepsilon$ be finite measures on the $\sigma$-algebra $\mathcal{B}$ of Borel sets in $\mathbb{R}$. Their convolution is the measure
\[ \nu \ast \varepsilon(A) := \int_{\{x+y\in A\}} \nu(dx) \varepsilon(dy) = \int_{\mathbb{R}} \nu(A-x) \varepsilon(dx), \quad A \in \mathcal{B}; \]
here $A - x := \{y \in \mathbb{R} : x + y \in A\}$. Denote by $F^n*$ the $n$-th convolution power of $F$:
\[ F^{1*} := 1, \quad F^{(n+1)*} := F^n* F, \quad n \geq 1, \]
and $F^n0* := \delta_0$ (the atomic measure of unit mass concentrated at zero). Let $U$ be the renewal measure generated by $F$: $U := \sum_{n=0}^{\infty} F^n*$. Denote by $\hat{\nu}(s)$ the Laplace transform of an arbitrary complex-valued measure $\nu$: $\hat{\nu}(s) := \int_{\mathbb{R}} e^{sx} \nu(dx)$. Let $\nu$ be a measure defined on $\mathcal{B}$, and $a(x), x \in \mathbb{R}$, a function. Define the convolution $\nu \ast a(x)$ as the function $\int_{\mathbb{R}} a(x-y) \nu(dy), x \in \mathbb{R}$. For $c \in \mathbb{C}$, we assume that $c/\infty$ is equal to zero. The relation $a(x) \sim cb(x)$ as $x \to \infty$ means that $a(x)/b(x) \to c$ as $x \to \infty$; if $c = 0$, then $a(x) = o(b(x))$.

Let $\mathbb{R}_+$ be the set of all nonnegative numbers and $\mathbb{R}_- := \mathbb{R} \setminus \mathbb{R}_+$ be the set of all negative numbers.

2. Existence of a Solution and Its Explicit Form

Put $\mathcal{F}_+ := \min\{n \geq 1 : S_n \geq 0\}$. The random variable $\mathcal{F}_+ := S_{\mathcal{F}_+}$ is called the first weak ascending ladder height. Similarly, $\mathcal{F}_- := \min\{n \geq 1 : S_n < 0\}$ and $\mathcal{F}_- := S_{\mathcal{F}_-}$ is the first strong descending ladder height. We have the factorization identity ($E$ stands for “expectation”)
\[ 1 - \xi E(e^{\xi X_1}) = [1 - E(\xi^{2\xi}) - \xi E(e^{\xi X_1})][1 - E(\xi^2 + e^{\xi X_1 + \xi^2})], \quad |\xi| \leq 1, \quad \Re \xi = 0. \]
Note that (2) was deduced in [2, Section 2] from an analogous identity in [1, § XVIII.3] for another collection of ladder variables.

Consider equation (1) with $f \in L_1(\mathbb{R}_+)$ It suffices to prove the existence of a solution $z$ to (1) for nonnegative functions $f$; in the case of real functions $f$, we must use the representation $f = f^+ - f^-$ ($f^+ := \max(f, 0)$, $f^- := -\min(f, 0)$), while $f = \Re f + i \Im f$ for complex $f$. Then, for real functions, we have $z = z_+ - z_-$, where $z_\pm$ are two solutions to (1) with $f^\pm$ instead of $f$. Denote by $F_n^\pm$ the distributions of the random variables $\mathcal{F}_n^\pm$ respectively. It follows from the identity (2) that
\[ \delta_0 = F = (\delta_0 - F_-) \ast (\delta_0 - F_+). \]
Let $U_\pm := \sum_{k=0}^\infty F_\pm^k$ be the renewal measures generated by the distributions $F_\pm$ respectively. Denote by $\mathbf{1}_{\mathbb{R}_-}$ the indicator of the subset $\mathbb{R}_-$ in $\mathbb{R}$: $\mathbf{1}_{\mathbb{R}_-}(x) = 1$ for $x \in \mathbb{R}_-$ and $\mathbf{1}_{\mathbb{R}_-}(x) = 0$ for $x \in \mathbb{R}_+$. A similar meaning has the notation $\mathbf{1}_{\mathbb{R}_+}$.

Extend the function $f$ onto the whole line: $f(x) := 0$, $x < 0$. This convention will be valid throughout.

**Theorem 1.** Let $F$ be a probability distribution and $f \in L_1(\mathbb{R}_+)$. Then the function

$$z(x) = U_+ \ast [(U_- \ast f)\mathbf{1}_{\mathbb{R}_+}](x), \quad x \in \mathbb{R}_+,$$

is the solution to (1) coinciding with the solution obtained by successive approximations.

**Proof.** Put $\xi \in (0, 1)$ in (2). We get

$$1 - \xi \hat{F}(s) = [1 - \hat{F}_{\xi-}(s)][1 - \hat{F}_{\xi+}(s)], \quad \Re s = 0,$$

where $F_{\xi \pm}$ are positive measures concentrated on the sets $\mathbb{R}_{\xi \pm}$ respectively; moreover, $F_{\xi \pm}(\mathbb{R}_{\xi \pm}) < 1$. Solve the equation

$$z_\xi(x) = \xi \int_{-\infty}^x z_\xi(x - y) F(dy) + f(x), \quad x \geq 0,$$

by successive approximations:

$$z_\xi^{(0)}(x) = f(x), \quad z_\xi^{(n)}(x) = \xi \int_{-\infty}^x z_\xi^{(n-1)}(x - y) F(dy) + f(x), \quad x \geq 0, \quad n \geq 1.$$

Obviously, $z_\xi^{(n)}(x) \uparrow$ as $n \uparrow \infty$ for all $x \in \mathbb{R}_+$. Hence $z_\xi(x) = \lim_{n \to \infty} z_\xi^{(n)}(x)$, $x \in \mathbb{R}_+$, is a solution to (5). Let us show that $z_\xi \in L_1(\mathbb{R}_+)$. Consider the renewal equation

$$\zeta_\xi(x) = \xi \int_{\mathbb{R}} \zeta_\xi(x - y) F(dy) + f(x), \quad x \in \mathbb{R}.$$

Construct its solution by successive approximations:

$$\zeta_\xi^{(0)}(x) = f(x), \quad \zeta_\xi^{(n)}(x) = \xi \int_{\mathbb{R}} \zeta_\xi^{(n-1)}(x - y) F(dy) + f(x), \quad n = 1, 2, \ldots$$

Since $\zeta_\xi^{(n)}(x) \uparrow$ as $n \uparrow$, we can pass to the limit under the integral sign. Thus, the limits $\zeta_\xi(x) := \lim_{n \to \infty} \zeta_\xi^{(n)}(x)$, $x \in \mathbb{R}$, exist; moreover, $z_\xi(x) \leq \zeta_\xi(x)$, $x \in \mathbb{R}_+$. The function $\zeta_\xi(x)$ is a solution to (6). Since $\zeta_\xi^{(n)}(x) = \sum_{k=0}^n \xi^k F_\xi^k \ast f(x)$, the function $\zeta_\xi(x)$ is representable as $\zeta_\xi(x) = U_\xi \ast f(x)$, where $U_\xi = \sum_{k=0}^{\infty} \xi^k F_\xi^k$ is the renewal measure generated by the improper distribution $\xi F$. The measure $U_\xi$ is finite since $\xi F(\mathbb{R}) = \xi < 1$. Hence $\zeta_\xi \in L_1(\mathbb{R})$. Thus, we have proved the existence of a solution $z_\xi \in L_1(\mathbb{R}_+)$ to (5).

Let us first find the explicit form of the solution $z_\xi$. The article [3] contains a brief exposition of the formal scheme for solving the classical Wiener–Hopf equation (see also [4, Chapter II, § 5, Subsection 5.4]). By analogy with the first step of this scheme, write down (5) on the whole line. Let

$$v_\xi(x) = \begin{cases} z_\xi(x) & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases} \quad n_\xi(x) = \begin{cases} 0 & \text{for } x \geq 0, \\ -\xi \int_{-\infty}^x z_\xi(x - y) F(dy) & \text{for } x < 0. \end{cases}$$
The function \( n_\xi(x) \) makes sense. Indeed, \( |n_\xi(x)| \leq \xi F \ast \xi \in L_1(\mathbb{R}) \). Equation (5) becomes equivalent to the following renewal equation:

\[
v_\xi(x) = \xi \int_{-\infty}^{\infty} v_\xi(x-y) F(dy) + f(x) + n_\xi(x), \quad x \in \mathbb{R},
\]

or, briefly,

\[
(\xi F + n_\xi) \ast v_\xi = f + n_\xi. \tag{7}
\]

Put \( U_\xi := \sum_{n=0}^{\infty} F_\xi^n \). The measures \( U_\xi \) are finite. Form the convolutions of \( U_\xi \) with both sides of (7). The equality \( \delta_0 - \xi F = (\delta_0 - F_\xi-) \ast (\delta_0 - F_\xi+) \) implies that

\[
U_\xi - (\delta_0 - \xi F) = \delta_0 - F_\xi_+.
\]

Therefore,

\[
(\delta_0 - F_\xi+) \ast v_\xi(x) = U_\xi - (f + n_\xi)(x), \quad x \in \mathbb{R}. \tag{8}
\]

The finite measure \( U_\xi \) is nonnegative and concentrated on \( \mathbb{R} \). The left-hand side of (8) is identically zero on \( \mathbb{R} \). Consequently,

\[
[U_\xi - (f + n_\xi)]1_{\mathbb{R}_-} = 0,
\]

which implies that \( U_\xi - n_\xi = -(U_\xi - f)1_{\mathbb{R}_-} \). Form the convolutions of both sides of this equality with the measure \( \delta_0 - F_\xi_- \). We obtain

\[
n_\xi = -(\delta_0 - F_\xi-) \ast [(U_\xi - f)1_{\mathbb{R}_-}].
\]

Thus,

\[
v_\xi = U_\xi \ast (f + n_\xi) = U_\xi \ast \left\{ f - (\delta_0 - F_\xi-) \ast [(U_\xi - f)1_{\mathbb{R}_-}] \right\}
\]

\[
= U_\xi \ast \left\{ f - (\delta_0 - F_\xi_-)(U_\xi - f) + (\delta_0 - F_\xi_-) \ast [(U_\xi - f)1_{\mathbb{R}_-}] \right\}
\]

\[
= U_\xi \ast \left\{ (\delta_0 - F_\xi_-) \ast [(U_\xi - f)1_{\mathbb{R}_-}] \right\} = U_\xi \ast [(U_\xi - f)1_{\mathbb{R}_+}].
\]

Letting \( \xi \uparrow 1 \), we get (4). Let us prove the last assertion of the theorem. Form the successive approximations \( z^{(n)} \) for the initial equation (1):

\[
z^{(0)}(x) = f(x), \quad z^{(n)}(x) = \int_{-\infty}^{x} z^{(n-1)}(x-y) F(dy) + f(x), \quad x \geq 0.
\]

We have \( z^{(n)}(x) \uparrow \) as \( n \uparrow \). Consequently, there exist limits \( \lim_{n \to \infty} z^{(n)}(x), \quad x \geq 0 \). Show that they coincide with (4). Using monotonicity and induction on \( n \), we have

\[
z^{(n)}(x) = \lim_{\xi \uparrow 1} z^{(n)}(x) \leq \lim_{\xi \uparrow 1} z_\xi(x) = z(x), \quad x \geq 0.
\]

It follows that

\[
z_\xi(x) := \lim_{n \to \infty} z^{(n)}_\xi \leq \lim_{n \to \infty} z^{(n)}(x) \leq \lim_{\xi \uparrow 1} z_\xi(x) = z(x).
\]

Passing to the limit as \( \xi \uparrow 1 \), we see that \( \lim_{n \to \infty} z^{(n)}(x) = z(x) \), i.e., the solution \( z \) defined by (4) coincides with the solution \( \lim_{n \to \infty} z^{(n)}(x) \), obtained by successive approximations. \( \square \)

**Remark 1.** Theorem 1 holds for all types of nonarithmetic probability distributions: both for distributions of drifting type and for distributions of oscillating type.
3. Absolutely continuous components

We shall need the following conditions on $F$ and $F_{\pm}$.

**Condition** $(\mathcal{S})$. For some $n = n(F) \geq 1$ the distribution $F^{n*}$ has a nonzero absolutely continuous component.

**Condition** $(\mathcal{S}_{\pm})$. For some $n = n(F_{\pm}) \geq 1$ the distributions $F_{\pm}^{n*}$ have a nonzero absolutely continuous component.

**Theorem 2.** Let $F$ be a probability distribution such that $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$. Then $(\mathcal{S}) \implies (\mathcal{S}_{\pm})$.

**Proof.** Let $G$ be a distribution with nonzero mean. Then the distribution $G^{n*}$ for some $n = n(G) \geq 1$ has a nonzero absolutely continuous component if and only if the function

$$s \frac{[1 - G(s)](s - 1)}{s^2}, \quad \Re s = 0,$$

is the Laplace transform $\hat{V}_G(s)$ of a finite measure $V_G$ [5]. By Theorem 1 in [1, § XVIII.5] applied to the random walk $\{-S_n\}$, there exist finite expectations $\mu_\pm := \mathbb{E}F_{\pm}$; moreover, $\mu_+ + \mu_- = \sigma^2/2$. Show that condition $(\mathcal{S})$ implies the following assertion: the function

$$\hat{V}(s) = \frac{s^2}{[1 - F(s)](s - 1)^2}, \quad \Re s = 0,$$

is the Laplace transform $\hat{V}_G(s)$ of a finite measure $V$. Let $\nu$ be a finite complex measure. Consider the measure $Tv$ with density

$$v(x, \nu) := \begin{cases} -\nu((\infty, x]), & x \leq 0, \\ \nu((x, \infty)), & x > 0. \end{cases}$$

Then $Tv$ is a locally finite measure, i.e., the values $Tv(A)$ are finite on bounded sets $A \in \mathcal{B}$. Denote by $|\nu|$ the total variation of $\nu$. If $\int_{\mathbb{R}} |x| |\nu|(dx) < \infty$, then $Tv$ is a locally finite measure and its Laplace transform is of the form $\hat{T}\nu(s) = \hat{\nu}(s) / s$, $\Re s = 0$, where $\hat{T}\nu(0)$ is defined by continuity as $\hat{T}\nu(0) = \int_{\mathbb{R}} x \nu(dx)$. If $\int_{\mathbb{R}} |x|^k |\nu|(dx) < \infty$, where $k \geq 1$ is an integer, then $\int_{\mathbb{R}} |x|^{k-1} |Tv|(dx) < \infty$ by Theorem 3 in [6], where we put $\varphi(x) = (1 + |x|)^{k-1}$. We have

$$\frac{1}{\hat{V}(s)} = \frac{[1 - \hat{F}(s)](s^2 - 2s + 1)}{s^2} = 1 - \hat{F}(s) + \frac{2[\hat{F}(s) - 1]}{s^2} + \frac{1 - \hat{F}(s)}{s^2} = 1 - \hat{F}(s) + 2\hat{TF}(s) - \hat{T^2F}(s),$$

since $\hat{T}\hat{F}(0) = 0$. Consequently, the function $1/\hat{V}(s)$ is the Laplace transform $\hat{W}(s)$ of the finite measure $W = \delta_0 - F + 2TF - T^2F$ and $\hat{W}(0) = -\sigma^2/2 \neq 0$. Show that the measure $W$ is invertible in the Banach algebra $\mathcal{B}$ of finite measures $\nu$ on $\mathcal{B}$, where the multiplication is the convolution of measures, the norm is $|\nu|([\mathbb{R}])$, the unity of $\mathcal{B}$ is the measure $\delta_0$, the addition and multiplication of measures by numbers are defined in the usual way. Let $\mathcal{M}$ be the space of maximal ideals of $\mathcal{B}$. The following facts are well known [7]. Each maximal ideal $M \in \mathcal{M}$ generates a homomorphism $h : \mathcal{B} \to \mathcal{C}$ and $M$ is the kernel of this homomorphism. Denote by $\nu(M)$ the value of $h$ at $\nu \in \mathcal{B}$. An element $\nu \in \mathcal{B}$ is invertible if and only if $\nu$ does not belong to every maximal ideal $M \in \mathcal{M}$. In other words, $\nu$ is invertible
if and only if \( \nu(M) \neq 0 \) for all \( M \in \mathcal{M} \). The space \( \mathcal{M} \) is split into two sets: \( \mathcal{M}_1 \) is the set of the maximal ideals which do not contain the collection \( L(\mathbb{B}) \) of all absolutely continuous measures in \( \mathcal{B} \) and \( \mathcal{M}_2 = \mathcal{M} \setminus \mathcal{M}_1 \). If \( M \in \mathcal{M}_1 \), then the corresponding homomorphism is of the form \( h(\nu) = \tilde{\nu}(s_0) \), where \( \Re s_0 = 0 \).

In this case, \( M = \{ \nu \in \mathcal{B} : \tilde{\nu}(s_0) = 0 \} \) [8, Chapter IV, Section 4]. If \( M \in \mathcal{M}_2 \), then \( \nu(M) = 0 \) for all \( \nu \in L(\mathbb{B}) \). Show that \( W(M) \neq 0 \) for all \( M \in \mathcal{M} \), thus establishing the existence of an inverse element \( W^{-1} \in \mathcal{B} \). If \( M \in \mathcal{M}_1 \), then for some \( s_0 \), \( \Re s_0 = 0 \), we have \( W(M) = \tilde{W}(s_0) \neq 0 \), since \( \tilde{F}(s) \neq 1 \) on \( i\mathbb{R} \setminus \{0\} \) for the nonarithmetic distribution \( F \). Let \( M \in \mathcal{M}_2 \). Obviously, \( W(M) = 1 - F(M) \) since \( TF \) and \( T^2F \) are absolutely continuous measures. Let \( F^\ast = F_c^\ast + F_s^\ast \) be the decomposition of the distribution \( F^\ast \) into the absolutely continuous and singular components with respect to Lebesgue measure (\( F_s^\ast \) is the sum of the discrete and singular components in the usual sense). We have \( F^\ast(M) = F_c^\ast(M) + F_s^\ast(M) \),

\[
|F_s^\ast(M)| \leq \|F_s^\ast\| = F_c^\ast(\mathbb{R}) + F_s^\ast(\mathbb{R}) = F^\ast(\mathbb{R}) = 1.
\]

Therefore, \( F(M) < 1 \) and \( W(M) \neq 0 \), hence there exists an inverse element \( W^{-1} \in \mathcal{B} \). Obviously, \( W^{-1} = V \) since the Laplace transform of the measure \( W^{-1} \) is equal to \( 1/\tilde{W}(s) = \tilde{V}(s) \). Use the factorization (3):

\[
\tilde{V}(s) = \frac{s}{[1 - \tilde{F}_-(s)](s - 1)}, \quad \frac{s}{[1 - \tilde{F}_+(s)](s - 1)} =: \tilde{V}_-(s) \cdot \tilde{V}_+(s).
\]

Show that \( \tilde{V}_+(s) \) is the Laplace transform of some finite measure \( V_+ \). By the aforementioned criterion from [5], it will follow that the distribution \( F_+ \) satisfies condition \((\mathcal{S}_+)\). The function

\[
\tilde{W}_-(s) = \frac{[1 - \tilde{F}_-(s)](s - 1)}{s}, \quad \Re s = 0,
\]

is the Laplace transform of the finite measure \( W_- = \delta_0 - F_- + TF_- \). Multiply both sides of equality (9) by \( \tilde{W}_-(s) \). We obtain \( \tilde{V}_+(s) = \tilde{V}(s)\tilde{W}_-(s) \), i.e., \( \tilde{V}_+(s), \Re s = 0 \), is the Laplace transform of the finite measure \( V_+ = V \ast W_- \). Thus, \((\mathcal{S}) \iff (\mathcal{S}_+)\). Similarly, \((\mathcal{S}) \iff (\mathcal{S}_-). \]

4. Asymptotic properties of the solution

We shall need the following fact which is a consequence of [9, Theorem 2.6.4 (a)].

**Theorem 3.** Let \( F \) be a probability distribution with positive mean \( \mu \). Let \( a(x), x \in \mathbb{R}, \) be a bounded summable function such that \( a(x) \to 0 \) as \( x \to \infty \). Suppose that condition \((\mathcal{S})\) is satisfied. Then

\[
U \ast a(x) \to \frac{1}{\mu} \int_{\mathbb{R}} a(y) \, dy \quad \text{as } x \to \infty.
\]

Consider two cases depending on the properties of the inhomogeneous term \( f \).

**Theorem 4.** Let \( F \) be a probability distribution such that \( EX_1 = 0, \quad EX_1^2 < \infty \) and condition \((\mathcal{S})\) holds. Suppose that the function \( \int_{-\infty}^{\infty} |f(y)| \, dy, x \in \mathbb{R}_+, \) is summable and \( f(x), x \in \mathbb{R}, \) is a bounded function such that \( f(x) \to 0 \) as \( x \to \infty \). Then the solution \( z \) to (1) given by (4) satisfies the relation

\[
z(x) \to \frac{1}{\mu_+} \int_{-\infty}^{\infty} \int_{-\infty}^{y} f(u) \, du \, U_-(dy) \quad \text{as } x \to \infty,
\]

where \( \mu_+ := \mathbb{E}F_+ \).
Proof. By Theorem 2, the distributions $F_0$ satisfy condition $(\mathcal{S}_+).$ Show that $(U_- * f)1_{\mathbb{R}_+} \in L_1(\mathbb{R}_+).$ As established in [10], the renewal measure $U_-$ admits the decomposition $U_- = U_1 + U_2,$ where the measure $U_1$ is absolutely continuous with a bounded continuous density $u_1(x)$ such that $u_1(x) \to 1/|\mu_-|$ as $x \to -\infty,$ and the measure $U_2$ is finite. Choose a constant $C < \infty$ such that $|f(x)| \leq C,$ $x \in \mathbb{R},$ and $|u_1(x)| \leq C$ for all $x \in \mathbb{R}.$ We have

$$|U_- * f(x)| \leq \int_{-\infty}^{0} |f(x-y)| U_-(dy) \leq \int_{-\infty}^{0} |f(x-y)| u_1(y) dy + \int_{-\infty}^{0} |f(x-y)| |U_2|(dy) =: I_1(x) + I_2(x),$$

$$I_1(x) \leq C \int_{-\infty}^{0} |f(x-y)| dy = C \int_{-\infty}^{0} |f(y)| dy \in L_1(\mathbb{R}_+).$$

It is easily seen that $I_2(x) \in L_1(\mathbb{R}_+):$

$$\int_{0}^{\infty} I_2(x) dx = \int_{0}^{\infty} \int_{-\infty}^{0} |f(x-y)| |U_2|(dy) dx \leq \int_{-\infty}^{0} \int_{-\infty}^{\infty} |f(x-y)| dx |U_2|(dy) \leq \int_{-\infty}^{\infty} \int_{0}^{\infty} |f(x)| dx |U_2|(dy) = \|f\|_1 |U_2|(\mathbb{R}_- \cup \{0\}) < \infty.$$

Moreover, the functions $I_k(x), k = 1, 2,$ are bounded:

$$I_1(x) \leq C\|f\|_1, \quad I_2(x) \leq C|U_2|(\mathbb{R}_- \cup \{0\}).$$

Obviously, they tend to zero as $x \to \infty.$ In order to establish relation (10), it remains to apply Theorem 3 to the function $U_- * f(x)$ and the distribution $F_0$ with mean $\mu_0:$ as $x \to \infty$

$$z(x) = \int_{0}^{x} U_- * f(x-y) U_+(dy) \to \frac{1}{\mu_+} \int_{0}^{\infty} U_- * f(x) dx = \frac{1}{\mu_+} \int_{-\infty}^{0} f(x-y) U_-(dy) dx = \frac{1}{\mu_+} \int_{-\infty}^{0} f(u) du U_-(dy).$$

Theorem 5. Let $F$ be a probability distribution such that $EX_1 = 0,$ $EX_1^2 = \sigma^2 < \infty$ and condition $(\mathcal{S})$ holds. Suppose that the function $f(x), x \in \mathbb{R},$ is a bounded nonnegative summable function such that $f(x) \to 0$ as $x \to \infty$ and $\int_{0}^{\infty} f(y) dy = \infty.$ Then the solution $z$ to (1) given by (4) satisfies the relation

$$z(x) \sim \frac{2}{\sigma^2} \int_{y}^{\infty} f(u) du dy \quad \text{as} \quad x \to \infty. \quad (11)$$

Proof. Denote by $\mathcal{L}_-$ the restriction of Lebesgue measure to $\mathbb{R}_-.$ We use the notation from the proof of Theorem 4. Represent the renewal measure $U_-$ in the form

$$U_- = \frac{1}{|\mu_-|} \mathcal{L}_- + U_2 + \left( U_1 - \frac{1}{|\mu_-|} \mathcal{L}_- \right).$$
The measure \( U_3 := U_1 - \mathcal{L}_-/|\mu_-| \) is absolutely continuous with density
\[
    u_3(x) := u(x) - \frac{1}{|\mu_-|} \xrightarrow{\text{as } x \to -\infty} 0.
\]
We have
\[
    U_- * f(x) = \frac{1}{|\mu_-|} \int_x^\infty f(y) \, dy + \int_{-\infty}^0 f(x-y) [U_2(dy) + u_3(y) \, dy] =: \sum_{k=1}^3 J_k(x).
\]
By Theorem 4 in [11], where we put \( K(x) = \int_x^\infty f(y) \, dy, \)
\[
    U_+ * J_1(x) = \frac{1}{\mu_+|\mu_-|} \int_y^x \int_y^\infty f(u) \, du \, dy \xrightarrow{\text{as } x \to \infty} \infty.
\]
Since \( \mu_-|\mu_-| = \sigma^2/2, \) the right-hand side of (12) coincides with the right-hand side of (11). Show that \( J_2(x) \in L_1(\mathbb{R}_+). \) We have
\[
    \int_0^\infty |J_2(x)| \, dx \leq \int_0^\infty \int_0^x f(x-y) |U_2|(dy) \, dx
    = \int_0^\infty \int_{-y}^\infty f(x) \, dx \, |U_2|(dy) \leq \|f\|_1 |U_2|((-\infty,0]) < \infty.
\]
The function \( J_2(x) \) is bounded and tends to zero as \( x \to \infty. \) By Theorems 2 and 3, it follows that
\[
    U_+ * J_2(x) \to \frac{1}{\mu_+} \int_0^\infty J_2(y) \, dy \xrightarrow{\text{as } x \to \infty} \infty.
\]
Given an arbitrary \( \varepsilon > 0, \) choose \( A = A(\varepsilon) < 0 \) such that \( |u_3(x)| \leq \varepsilon \) for \( x \leq A. \)
Split \( J_3(x) \) into the sum of two integrals:
\[
    J_3(x) = \left( \int_{-\infty}^A + \int_0^0 \right) f(x-y) u_3(y) \, dy =: J_4(x) + J_5(x).
\]
We have
\[
    |J_4(x)| \leq \int_{-\infty}^A f(x-u) |u_3(u)| \, du
    \leq \varepsilon \int_{-\infty}^A f(x-u) \, du = \varepsilon \int_{x-A}^\infty f(u) \, du \leq \varepsilon |\mu_-| J_1(x).
\]
Consequently,
\[
    |U_+ * J_4(x)| \leq U_+ * |J_4|(x) \leq \varepsilon |\mu_-| U_+ * J_1(x)
    \sim \varepsilon |\mu_-| \int_0^\infty f(u) \, du \, dy \xrightarrow{\text{as } x \to \infty} \infty.
\]
Show that \( J_5(x) \in L_1(\mathbb{R}_+). \) Let \( |u_3(x)| \leq C < \infty. \) Then
\[
    \int_0^\infty |J_5(x)| \, dx \leq C \int_0^\infty \int_A^0 f(x-y) \, dy \, dx
    = C \int_0^\infty \int_x^{x-A} f(v) \, dv \, dx = C \int_0^\infty f(v) \int_{(v+A)^+}^0 \, dx \, dv \leq CA\|f\|_1.
Obviously, $|J_5(x)| \leq C^2 |A| < \infty$ and $J_5(x) \to 0$ as $x \to \infty$ by the Lebesgue Dominated Convergence Theorem. Apply Theorems 2 and 3 to the distribution $F_+$ and to the function $J_5(x)$:

$$U_+ * J_5(x) \to \frac{1}{\mu_+} \int_0^\infty J_5(y) dy \quad \text{as } x \to \infty.$$  

The relations (12)–(15) imply that, for $k = 2, \ldots, 5$,

$$U_+ * J_k(x) = o(U_+ * J_1(x)) \quad \text{as } x \to \infty,$$

which completes the proof of the theorem. \hfill $\Box$

Theorem 5 implies the following result without the nonnegativity of $f$.

**Corollary 1.** Let $F$ be a probability distribution such that $EX_1 = 0$, $EX_1^2 = \sigma^2 < \infty$ and condition (S) holds. Suppose that $f(x) = f_1(x) - f_2(x)$, $x \in \mathbb{R}_+$, where $f_1(x)$ and $f_2(x)$ are bounded nonnegative summable functions such that $f_1(x) \to 0$ as $x \to \infty$, $i = 1, 2$, and

$$\int_0^x \int_y^\infty f_1(u) du dy \sim c_i d(x) \quad \text{as } x \to \infty, \quad i = 1, 2,$$

where $c_1, c_2 \geq 0$ are constants. Then the solution $z$ to (1) given by (4) satisfies the relation

$$z(x) \sim \frac{2(c_1 - c_2)}{\sigma^2} d(x) \quad \text{as } x \to \infty.$$

If $c_1 \neq c_2$, then relation (11) holds.

**Proof.** Assume for simplicity that $c_1$, $c_2 > 0$. Denote by $z_1$, $z_2$ the solutions to (1) corresponding to $f_1$, $f_2$ respectively. Then $z = z_1 - z_2$ is a solution to (1), corresponding to $f$. By Theorem 5,

$$\frac{z(x)}{d(x)} = \frac{z_1(x)}{d(x)} \frac{1}{\int_0^x \int_y^\infty f_1(u) du dy} - \frac{z_2(x)}{d(x)} \frac{1}{\int_0^x \int_y^\infty f_2(u) du dy} \to \frac{2(c_1 - c_2)}{\sigma^2} \quad \text{as } x \to \infty. \quad \square$$

### 5. Influence of the Solution to the Homogeneous Equation

The solution $z$ to (1) obtained in Theorem 1 is not the only solution to this equation. Consider the homogeneous equation

$$Z(x) = \int_{-\infty}^x Z(x - y) F(dy), \quad x \in \mathbb{R}_+, \tag{16}$$

where $Z$ is the function sought and $F$ is a probability distribution that generates an oscillating random walk $\{S_n\}$. Put $\tau := \min\{n \geq 1 : S_n > 0\}$. Let $U_{G+}$ be the renewal measure generated by the distribution $G_+$ of the first strong ascending ladder height $\mathcal{H}_+ := S_{\mathcal{H}_1}$. The renewal function $Z(x) := U_{G+}((-\infty, x])$, $x \in \mathbb{R}_+$, is a solution to (16) with normalization $Z(0) = 1$ (see [12, Theorem 1]) and asymptotics $Z(x) \sim x/E\mathcal{H}_+$ as $x \to \infty$. Let $c \in \mathbb{C}$ be arbitrary. Obviously, the function $Z_c := z + cZ$ is a solution to the inhomogeneous equation (1). In what follows, we assume that $EX_1 = 0$ and $\sigma^2 = EX_1^2 < \infty$. Let us investigate the
asymptotics of the solution $Z_e$ under various assumptions on the kernel $F$ and on the inhomogeneous term $f$. The following facts are known:
1) if $E(X_1^+)^3 < \infty$, then $E \mathcal{K}^2 < \infty$ (see [13, Theorem (i)] with $\phi(x) = x^2$); 
2) if $E \mathcal{K}^2 < \infty$, then (see [1, § XI.4])

$$Z(x) = \frac{x}{E \mathcal{K}^2} \rightarrow \frac{E \mathcal{K}^2}{2(E \mathcal{K}^2)^2} \quad \text{as } x \rightarrow \infty;$$

3) if $E(X_1^+)^3 = \infty$, then [14, Theorem 1]

$$Z(x) = \frac{x}{E \mathcal{K}_+} \sim \frac{2}{\sigma^2 E \mathcal{K}_+} \int_0^x \int_y^\infty \int_v^\infty [1 - F(u)] du dv dy \quad \text{as } x \rightarrow \infty.$$ 

The asymptotics of the difference $Z_e(x) - cx/E \mathcal{K}_+$ as $x \rightarrow \infty$ is established by comparing the asymptotic behavior of the functions

$$A_1(x) := \int_0^x \int_y^\infty f(u) du dy, \quad A_2(x) := \int_0^x \int_y^\infty \int_v^\infty [1 - F(u)] du dv dy.$$ 

In order to assess the rate of growth of $A_2(x)$, let us consider the case when the tail $1 - F(x) = R(x)$ of $F$ regularly varies at infinity with index $\alpha \in (-3, -2)$, i.e., when $R(tx)/R(x) \rightarrow t^\alpha$ as $x \rightarrow \infty$, $t > 0$ being fixed. Then

$$Z(x) - \frac{x}{E \mathcal{K}^2} \sim \frac{2A_2(x)}{\sigma^2 E \mathcal{K}^2} \sim \frac{2c^2[1 - F(x)]}{\sigma^2 E \mathcal{K}^2(\alpha + 1)(\alpha + 2)(\alpha + 3)} \quad \text{as } x \rightarrow \infty.$$ 

The last equivalence follows from the properties of regularly varying functions (see [1, § VIII.9, Theorem 1]). Consider the following possibilities

(17) $A_1(x) = o(A_2(x))$, $A_2(x) = o(A_1(x))$, $A_1(x) \sim KA_2(x)$ as $x \rightarrow \infty$,

where $K > 0$. The next results about the asymptotic behavior of the solution $Z_e$ to (1) follow from the above facts and the already proved theorems.

**Theorem 6.** Let the hypotheses of Theorem 4 be satisfied and let $E(X_1^+)^3 < \infty$. Then

$$Z_e(x) - \frac{cx}{E \mathcal{K}_+} \rightarrow \frac{1}{\mu_+} \int_{-\infty}^0 \int_{-y}^\infty f(u) du U_-(dy) + \frac{c E \mathcal{K}^2}{2(E \mathcal{K}^2)^2} \quad \text{as } x \rightarrow \infty.$$ 

**Theorem 7.** Let $F$ be a probability distribution such that condition $(\mathcal{E})$ holds true and let $f$ satisfy the hypotheses of either Theorem 4 or Theorem 5. Suppose that at least one of the functions $A_1(x)$ and $A_2(x)$ tends to infinity as $x \rightarrow \infty$. Then in all three cases of (17) the following relation holds:

$$Z_e(x) = \frac{cx}{E \mathcal{K}_+} \sim \frac{2}{\sigma^2} A_1(x) + \frac{2c}{\sigma^2 E \mathcal{K}_+} A_2(x) \quad \text{as } x \rightarrow \infty.$$ 

**Remark 2.** A probability distribution $F$ is called symmetric if its distribution function $F(x) := F((-\infty, x])$ is equal to $1 - F(-x - 0)$ [15, § 3.1]. In other words, the distribution $F$ of the random variable $X_1$ is symmetric if it coincides with the distribution of $-X_1$. A distribution $F$ is called continuous if its distribution function is continuous. For symmetric distributions, the formulas in Theorems 4–7 become simplified since $E \mathcal{K}_+ = \sigma/\sqrt{2\gamma_0}$ and $\mu_+ = \sigma \sqrt{\gamma_0}/2$, where $\gamma_0$ is equal to $\exp[-\sum_{n=0}^\infty P(S_n = 0)/n]$ (see, for example, [12, Lemma 2]). For continuous distributions $F$ the quantity $\gamma_0$ is equal to one.
The present paper concludes our investigations of the inhomogeneous Wiener-Hopf equation (1) started in papers [16] and [17]. Here we have considered the case when the distribution $F$ generates a random walk of oscillating type whereas the papers [16, 17] deal with cases when the corresponding random walk drifts to $+\infty$ and to $-\infty$, respectively.

If the distribution $F$ is absolutely continuous, i.e., $F(dx) = k(x) \, dx$, then equation (1) is equivalent to the classical Wiener-Hopf equation

$$z(x) = \int_0^\infty k(x-y)z(y) \, dy + f(x), \quad x \geq 0.$$ 

References


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