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FACTORIZATION OF SPECIAL HARMONIC POLYNOMIALS OF THREE VARIABLES

V.M. GICHEV

ABSTRACT. We consider homogeneous harmonic polynomials of real variables x, y, z that are eigenfunctions of the rotations about the axis z. They have the form $(x \pm yi)^n p(x, y, z)$, where p is a rotation invariant polynomial. Let $\mathfrak{R}_m$ be the family of the homogeneous rotation invariant polynomials p of degree m such that p is reducible over the rationals and $(x + yi)^n p$ is harmonic for some $n \in \mathbb{N}$. We describe $\mathfrak{R}_m$ for $m \leq 5$ and prove that $\mathfrak{R}_6$ and $\mathfrak{R}_7$ are finite.

Keywords: Legendre functions, harmonic polynomials, factorization.

1. INTRODUCTION

The space $\mathcal{H}$ of all complex valued harmonic polynomials on $\mathbb{R}^3$ admits a linear basis of the form

$$f(x, y, z) = (x \pm yi)^k p(x, y, z),$$

where p is a homogeneous polynomial which depends only on $x^2 + y^2$ and z. The polynomials f are harmonic eigenfunctions of the circle group T_o of rotations about the axis z. The polynomials p are T_o-invariant and, moreover, they are closely related to the associated Legendre functions P_n^m as well as to the Laplace eigenfunctions Y_n^m. If $k = 0$, then a change of variables and normalization converts p into the Legendre polynomial P_n. In this paper, we consider the problem of reducibility over the field $\mathbb{Q}$ for the polynomials p of small degrees. This is evidently connected with the problem of existence of common zeroes of a couple of harmonic polynomials and a description of their nontrivial factors.

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1.1. **Brief history.** Algebraic and arithmetic properties of classical orthogonal polynomials are still not well understood. Maybe, Tchebyshev's polynomials could be considered as an exception. In 1890, Stieltjes in a letter to Hermite (see [14, Letter 275]) formulated two questions: 1) are there two distinct polynomials P_n and P_m that have a nontrivial common root? 2) are $P_n(x)$ for even n and $\frac{P_n(x)}{x}$ for odd n irreducible? The existence of a common root implies the reducibility of at least one of the polynomials. Hence the affirmative answer to the first question implies the negative answer to the second one. However, the second question remains unanswered yet. In several special situations, the irreducibility of P_n was proved by Holt [4]–[6], Ille [7], McCoart [11], Melnikov [12], and Wahab [15], [16]. It follows from their results that P_n is irreducible for all $n \leq 500$ with a few possible exceptions.

In [3], Armitage proved that an entire harmonic function vanishes on the round cone $x_1^2 = \alpha^2(x_1^2 + \cdots + x_n^2)$ in $\mathbb{R}^n$, where $0 < \alpha < 1$, if and only if α is a root of an ultraspherical polynomial or of some its derivative.

The complete characterization of the quadratic divisors of real harmonic polynomials on $\mathbb{R}^n$ in terms of polynomial solutions to some Fuchsian ordinary differential equations was obtained by Agranovsky and Krasnov in the paper [2]. In the earlier paper [1], Agranovsky proved that a product of linear forms is harmonic if and only if the family of its zero hyperplanes forms a Coxeter system. The investigation was motivated by problems of mathematical physics and PDE, particularly the injectivity of the Radon transform and the stationary points of the wave equation.

Let $\mathcal{H}(p)$ denote the set of harmonic polynomials which the polynomial p divides. The papers [1] and [2] contain many open questions. Most of them are still open. The major thread of the questions may be stated as follows: when $0 < \dim \mathcal{H}(p) < \infty$? and what happens if this is not true? A satisfactory answer is known only in the case of the product of linear forms vanishing without multiplicities on hyperplanes from a Coxeter system. Then $\dim \mathcal{H}(p) = \infty$ and the asymptotic behavior of dimensions of the homogeneous components $\mathcal{H}_n(p)$ of $\mathcal{H}(p)$ as $n \to \infty$ can be described. This was done in [1].

The case of a quadratic polynomial p is much more difficult. The polynomial $p_3(x, y, z) = x^2 + y^2 - 2z^2$ seems to be simplest in a sense. It is harmonic, irreducible and z-axial, i.e., it depends only on the distance to the axis z. However, it is not known yet if $\dim \mathcal{H}(p_3) < \infty$. This was questioned in [1] and once more in the recent paper [10] by Mangoubi and Weller Weiser. The main results of [10] are rather algebraic and cannot be stated without preliminaries. There is also a more definite question in the paper [1] which is still open: does $\mathcal{H}(p_3)$ contain a polynomial of degree greater than 5? Note that $\mathcal{H}_5(p_3)$ contains the real and imaginary parts of $(x + yi)^2 z\, p_3(x, y, z)$.

On $\mathbb{R}^4$, for the similar polynomial $p_4(x, y, z, u) = x^2 + y^2 + z^2 - 3u^2$ we have $\dim \mathcal{H}(p_4) = \infty$ according to [8, Example 4.3]. The papers [8], [9] by Logunov and Malinnikova contain the fundamental results concerning zero sets of harmonic functions. For instant, they prove that the ratio of harmonic functions with the same zero set extends to a real analytic function and satisfies the Harnack inequalities.

1.2. **Notation.** The harmonic polynomial of the type

$$(x + yi)^k p(x, y, z),$$

where p is homogeneous of degree d and T_o-invariant, is unique up to a multiplicative constant that depends only on k and d. It is convenient to deal with complex valued polynomials while the divisibility problem is more interesting in the case of real ones. In the above setting we may assume p real without loss of generality. Hence the real part of the above polynomial is the product of p and $\Re(x+yi)^k$ and the same is true for the imaginary part. Due to the homogeneity and T_o-invariance, p actually depends of r^2 and the variable $t=\frac{x^2+y^2}{z^2}$. The corresponding polynomial $\tilde{p}(t)$ has only real roots. It is easy to prove these facts. For the polynomials that we consider in this article they follow from a straightforward computation. Thus the assumption that f is complex is not restrictive. We assume in the sequel that the coefficient at the highest power of x in p (equivalently, at the highest power of y or of x^2+y^2) is equal to 1. This condition together with the numbers k, d uniquely defines p.

To avoid cumbersome indices, we use the following notation for these polynomials:

$$f_{n,d}(x,y,z)=(x+yi)^{n-\left[\frac{d}{2}\right]}\,p_{n,d}(x,y,z),$$

where the brackets stand for the integer part of a number. Thus $\deg f_{n,d}$ is either $n+d$ or $n+d+1$ for d even and d odd, respectively.

Unless the contrary is stated explicitly "reducibility" means "reducibility over $\mathbb{Q}$". Set

$$\mathfrak{R}_d=\{f_{n,d}:\,\Delta f_{n,d}=0\ \text{ and }\ p_{n,d}\ \text{ is reducible}\}.$$

Let $t=\frac{x^2+y^2}{z^2}$ and put

$$\tilde{p}_{n,d}(t)=\frac{p_{n,d}(x,y,z)}{z^d}.$$

Then $f_{n,d}\in\mathfrak{R}_d$ if and only if the polynomial $\tilde{p}_{n,d}$ is reducible. If $d\le 7$, then $\deg\tilde{p}_{n,d}\le 3$. Hence for $d\le 7$ we have the equivalence

$$f_{n,d}\in\mathfrak{R}_d \quad\Longleftrightarrow\quad \tilde{p}_{n,d}\ \text{ has a rational root.}$$

For short, we denote

$$\zeta=x+yi.$$

1.3. **Methods and results.** The case $d\le 3$ is trivial: the polynomials ζ^n and $\zeta^n z$ are always harmonic, for $d=2,3$ we have $\Delta f_{n,d}=0$ if and only if there is $n\in\mathbb{N}$ such that

$$f_{n,2}=\zeta^{n-1}(\zeta\bar{\zeta}-2nz^2),$$
$$f_{n,3}=\zeta^{n-1}z\left(\zeta\bar{\zeta}-\tfrac{2}{3}nz^2\right),$$

respectively. Thus we may assume $4\le d\le 7$ in the sequel.

There is one-to-one correspondence between the solutions $u,v\in\mathbb{N}\setminus\{1\}$ to the Pell equation $u^2-6v^2=1$ and $\mathfrak{R}_4$. For $\mathfrak{R}_5$ there are three similar series of solutions to the Pell type equation $u^2-10v^2=9$. The methods are elementary: the polynomials $\tilde{t}_d$ are quadratic and the rationality of a root is equivalent to some Diophantine equation since n is integer.

In the case $d=6$ we prove that $n=Ku^3$ and $2n+1=Lv^3$, where $u,v\in\mathbb{N}$ and K,L runs over a finite subset of $\mathbb{N}$. Hence u,v satisfy the Diophantine equation $Lv^3-2Ku^3=1$. If $d=7$, then a similar assertion holds for n, $2n+3$, and the equation $Lv^3-2Ku^3=3$. The sets $\mathfrak{R}_6$ and $\mathfrak{R}_7$ are finite because every Diophantine equation above has a finite number of solutions due to Thue–Siegel–Roth theorem on Diophantine approximation.

1.4. **Reminder on the Pell equation.** This is the Diophantine equation $u^2 - Dv^2 = 1$, where $D \in \mathbb{N}$ is not a square. Let Q be the quadratic form on the left, $\phi = (u, v) \in \mathbb{Z}^2$,

$$\mathfrak{S} = \{\phi \in \mathbb{N} \times \mathbb{Z} : Q(\phi) = 1\},$$
$$\mathfrak{S}^+ = \mathfrak{S} \cap (\mathbb{N} \times \mathbb{N}).$$

The solution $\phi_1 = (u_1, v_1) \in \mathfrak{S}^+$ such that $u_1 = \min\{u : \phi \in \mathfrak{S}^+\}$ is called fundamental. The subgroup Γ of $\mathrm{SO}(Q, \mathbb{Z})$ generated by

$$M = \begin{pmatrix} u_1 & Dv_1 \\ v_1 & u_1 \end{pmatrix}$$

preserves the lattice $\mathbb{Z}^2$, the quadratic form Q, and consequently the set $\mathfrak{S}$. Moreover, it leaves invariant the branch B of the hyperbola $Q = 1$ which contains $\phi_0 = (1, 0)$. The arc of B with endpoints ϕ_0, ϕ_1 is a fundamental domain of Γ in B since it contains no solution except for its endpoints by definition of ϕ_1. Thus Γ is transitive on $\mathfrak{S}$ and, moreover,

$$\mathfrak{S}^+ = \{M^k \phi_0 : k \in \mathbb{N}\}.$$

Put $\phi_k = M^k \phi_0 = (u_k, v_k)$. Any linear combination X_k of the sequences u_k, v_k is subject to the formula

$$X_k = C_1 \lambda_1^k + C_2 \lambda_2^k,$$

where λ_1, λ_2 are the eigenvalues of M and C_1, C_2 are constant, and consequently it satisfies the recurrence relation

$$X_{k+1} - 2u_1 X_k + X_{k-1} = 0.$$

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2. Reducibility of $p_{n,4}$ and the Pell equation

2.1. **The Diophantine equation.** Let the polynomial

$$f_{n,4} = \zeta^{n-2}(\zeta\bar{\zeta} - A_n z^2)(\zeta\bar{\zeta} - B_n z^2)$$

of degree $n + 2$ be harmonic. Computing $\Delta f_{n,4}$, we get the equalities

$$\begin{cases} A_n + B_n = 4n, \\ A_n B_n = \frac{4}{3}\, n(n-1). \end{cases}$$

Therefore, $\tilde{p}_{n,4}(t) = t^2 - 4nt + \frac{4}{3} n(n-1)$ and

$$\begin{cases} A_n = 2\Big(n - \sqrt{\frac{2n^2+n}{3}}\Big), \\ B_n = 2\Big(n + \sqrt{\frac{2n^2+n}{3}}\Big). \end{cases} \tag{1}$$

Hence $0 < A_n \le B_n$. Since $n \in \mathbb{N}$, either $m = \sqrt{\frac{2n^2+n}{3}}$ is integer or both A_n and B_n are irrational. Thus we have the Diophantine equation

$$2n^2 + n = 3m^2 \tag{2}$$

with positive integer n, m. There are two evident solutions: $n = m = 0$ and $n = m = 1$. They correspond to the harmonic polynomials $\bar{\zeta}^2$ and $\bar{\zeta}(\zeta\bar{\zeta} - 4z^2)$ relating to the cases $d = 0$ and $d = 2$, respectively. We do not take them in account. Thus we assume

$$n > 1.$$

Then the converse is also true: *for every solution* $(n, m) \in (\mathbb{N} \setminus \{1\}) \times \mathbb{N}$ *to (2) we have* $\deg f_{n,4} = n + 2$ *and* $\Delta f_{n,4} = 0$.

2.2. **Description of $\mathfrak{R}_4$.** Let the sequence α_k be defined by the recurrence relation

$$X_{k+1} = 10X_k - X_{k-1} \tag{3}$$

and the initial data $\alpha_0 = \alpha_1 = 1$. Set

$$\begin{aligned} a_k &= \tfrac{1}{2}(\alpha_k - 1), \\ n_k &= \tfrac{1}{8}(\alpha_k + \alpha_{k+1} - 2). \end{aligned} \tag{4}$$

Note that $n_k = \frac{1}{4}(a_k + a_{k+1})$. We shall prove that $a_k = A_{n_k}$ in the notation of (1). These numbers are integer. There is a short table at the end of the section.

Theorem 1. *The family* $\mathfrak{R}_4$ *consists of the polynomials*

$$f_{n_k,4}(x, y, z) = (x + yi)^{n_k - 2}(x^2 + y^2 - a_k z^2)(x^2 + y^2 - a_{k+1} z^2),$$

where a_k *and* n_k *are as above and* $k > 1$.

Corollary 1. *The consecutive polynomials* $f_{n_k,4}$ *and* $f_{n_{k+1},4}$ *have a common quadratic irreducible factor.*

Proof. The form $x^2 + y^2 - a_{k+1} z^2$ is non-degenerate if $a_{k+1} > 0$. Hence it cannot be a product of two linear forms. □

Proof of the theorem. Setting

$$\begin{aligned} u &= 4n + 1, \\ v &= 2m, \end{aligned} \tag{5}$$

in (2) we get the Pell equation

$$u^2 - 6v^2 = 1. \tag{6}$$

Its fundamental solution ϕ_1 is equal to $(5, 2)$. We get the recurrence relations

$$\begin{cases} u_{k+1} = 5u_k + 12v_k, \\ v_{k+1} = 2u_k + 5v_k \end{cases}$$

for solutions to (6) and (3) for the linear combinations of u_k and v_k. In particular, $\phi_2 = (49, 20)$. This provides the initial data for (3). It follows from (3) that

$$\begin{aligned} n_k &= \tfrac{1}{4}(u_k - 1), \\ m_k &= \tfrac{1}{2}v_k \end{aligned} \tag{7}$$

are integer. Hence the formulas (5) define one-to-one correspondence between $\mathfrak{S}^+$ and the set of solutions to (2) in positive integers. By (1),

$$\begin{aligned} A_{n_k} &= 2(n_k - m_k), \\ B_{n_k} &= 2(n_k + m_k) \end{aligned} \tag{8}$$

and the polynomials $f_{n_k,4}$ are harmonic. Thus we have to prove that

$$\begin{aligned} a_k &= A_{n_k}, \\ a_{k+1} &= B_{n_k}. \end{aligned} \tag{9}$$

The initial data for u_k, v_k are $u_0 = 1$, $u_1 = 5$ and $v_0 = 0$, $v_1 = 2$. Hence $\alpha_0 = u_0 - 2v_0$ and $\alpha_1 = u_1 - 2v_1$. Due to (3) this implies

$$\alpha_k = u_k - 2v_k.$$

for all k. The sequence α_{k+1} also satisfies (3) and the initial conditions $\alpha_1 = u_0 + 2v_0 = 1$, $\alpha_2 = u_1 + 2v_1 = 9$. Therefore,

$$\alpha_{k+1} = u_k + 2v_k.$$

Hence $\alpha_k + \alpha_{k+1} = 2u_k$ and $\alpha_{k+1} - \alpha_k = 4v_k$. By (7),

$$n_k = \tfrac{1}{8}(\alpha_k + \alpha_{k+1} - 2),$$
$$m_k = \tfrac{1}{8}(\alpha_{k+1} - \alpha_k).$$

It follows that

$$A_{n_k} = 2(n_k - m_k) = \tfrac{1}{2}(\alpha_k - 1),$$
$$B_{n_k} = 2(n_k + m_k) = \tfrac{1}{2}(\alpha_{k+1} - 1).$$

This proves (9). □

2.3. **Table and formulas.** Here are the first six terms of the above quantities. We drop the cases $k = 0$ and $k = 1$ since they do not relate to harmonic polynomials of the degree $n_k + 2$.

k	2	3	4	5	6	7
u_k	49	485	4801	47525	470449	4656965
$v_k.$	20	198	1960	19402	192060	1901198
n_k	12	121	1200	11881	117612	1164241
$m_k.$	10	99	980	9701	96030	950599
a_k	4	44	440	4360	43164	427284

The matrix M has eigenvalues $5 \pm 2\sqrt{6}$. Here are the explicit formulas for u_k and v_k:

$$u_k = \frac{(5+2\sqrt{6})^k + (5-2\sqrt{6})^k}{2},$$
$$v_k = \frac{(5+2\sqrt{6})^k - (5-2\sqrt{6})^k}{2\sqrt{6}}.$$

The formulas for their generating functions follow:

$$G_u(t) = \frac{1-5t}{t^2-10t+1},$$
$$G_v(t) = \frac{2t}{t^2-10t+1}.$$

For α_k and m_k we have the functions $G_u - 2G_v$ and $\frac{1}{2}G_v$, respectively. Due to (7) and (4),

$$G_n(t) = \tfrac{1}{4}(G_u(t) - \tfrac{1}{1-t}) = \frac{t(1+t)}{(1-t)(1-10t+t^2)},$$
$$G_a(t) = \tfrac{1}{2}(G_u(t) - 2G_v(t) - \tfrac{1}{1-t}) = \frac{4t^2}{(1-t)(1-10t+t^2)}.$$

3. Reducibility of $p_{n,5}$ and a Pell type equation

In this section we consider the harmonic polynomials of the type

$$f_{n,5} = \zeta^{n-2} z(\zeta\bar{\zeta} - A_n z^2)(\zeta\bar{\zeta} - B_n z^2). \tag{10}$$

with rational A_n, B_n. The same method works with some complications.

3.1. **The Diophantine equation.** It is convenient to formulate precisely the reduction to the Diophantine equation.

Lemma 1. *Suppose $A_n, B_n \in \mathbb{Q}$. The polynomial $f_{n,5}$ is harmonic of degree $n+3$ if and only if*

$$\begin{cases} A_n = \frac{2}{3}(n-m), \\ B_n = \frac{2}{3}(n+m), \end{cases} \tag{11}$$

where $n, m \in \mathbb{N} \setminus \{1\}$ satisfy the Diophantine equation

$$3n + 2n^2 = 5m^2. \tag{12}$$

Proof. Computing $\Delta f_{n,5}$, we get

$$\begin{cases} 3(A_n + B_n) = 4n, \\ 5A_nB_n = (n-1)(A_n + B_n). \end{cases}$$

Hence $\tilde{p}_{n,5}(t) = t^2 - \frac{4}{3}nt + \frac{4}{15}n(n-1)$. Assuming $A_n \leq B_n$, we write the solution as

$$A_n = \frac{2}{3}\left(n - \sqrt{\tfrac{3n+2n^2}{5}}\right),$$
$$B_n = \frac{2}{3}\left(n + \sqrt{\tfrac{3n+2n^2}{5}}\right).$$

We assume $n \geq 2$ since $f_{0,5} = \bar{\zeta}^2 z$ and $f_{1,5} = \bar{\zeta} z(\zeta\bar{\zeta} - \frac{4}{3}z^2)$. Set

$$m = \sqrt{\frac{3n + 2n^2}{5}}.$$

Clearly, $m > 1$ and m can be either integer or irrational. The assumption that A_n or B_n is rational implies that $m \in \mathbb{Z}$ and n, m satisfy (12). □

3.2. **The Pell type equation.** For u, v defined as

$$\begin{cases} u = 4n + 3, \\ v = 2m, \end{cases} \tag{13}$$

the equation (12) implies the equality

$$u^2 - 10v^2 = 9. \tag{14}$$

Every solution to the Pell equation

$$u^2 - 10v^2 = 1 \tag{15}$$

after multiplication by 3 satisfies (14). The fundamental solution to (15) is $(19, 6)$. Thus

$$M = \begin{pmatrix} 19 & 60 \\ 6 & 19 \end{pmatrix}.$$

We consider the solutions (u, v) to (14) with $u > 0$ and use the notation of subsection 1.4 of Introduction. Set

$$T_0 = \{(7, -2),\ (3, 0),\ (7, 2)\}.$$

Lemma 2. *The family $\mathfrak{S}$ of the solutions to the Diophantine equation (14) lying in the right halfplane is the union of the orbits of the vectors of the triple T_0 under the action of Γ.*

Proof. The hyperbola $u^2 - 10u^2 = 9$ contains the family of solutions to (14). It is clear that its branch B lying in the right halfplane contains the above mentioned vectors, the group Γ generated by M preserves B and the lattice $\mathbb{Z}^2$ and consequently their orbits. Thus we have to prove that the orbits are disjoint and their union exhaust the family of solutions.

The equality $M(7,-2)^{\intercal} = (13,4)^{\intercal}$ shows that the arc in B with endpoints $(7,-2)$ and $(13,4)$ is the fundamental domain for the action of Γ on B which is equivalent to the shift $t \to t+1$ in $\mathbb{R}$. The straightforward computation shows that $9 + 10v^2$ for $v = -2, \dots, 3$ is the square of some $u \in \mathbb{N}$ only in the cases $(u,v) \in T_0$. This proves the lemma. □

In the next lemma we characterize the solutions relating to (12) among all solutions to (14). Let $\mathfrak{S}^+$ denote the family of solutions to (14) which correspond to that of (12) via (13) with integer $n \geq 2$ and set

$$T_k = M^{2k} T_0.$$

We have

$$M^2 = \begin{pmatrix} 721 & 2280 \\ 228 & 721 \end{pmatrix},$$

$$T_1 = \{(487,154),\ (2163,684),\ (9607,3038)\}.$$

Lemma 3. *The set $\mathfrak{S}^+$ is the union of the triples T_k, $k \in \mathbb{N}$.*

Proof. We note some properties on the action of Γ. The recurrence relation defined by M implies

$$\begin{cases} u_{k+1} = -u_k, \\ v_{k+1} = 2u_k - v_k \end{cases} \qquad \text{mod } 4.$$

It is obvious that

- if u_1 is odd and v_1 is even, then the numbers u_k of the type $4n+3$ and $4n+1$ alternate and v_k remains even,
- the first components of vectors in T_0 are of the type $4n+3$ and the second ones are even,
- $T_0 \cap \mathfrak{S}^+ = \varnothing$ since $n \geq 2$ implies $4n+3 \geq 11$.

Using Lemma 1, it is easy to check the inclusion $T_1 \subseteq \mathfrak{S}^+$. Positivity of the entries of M^2 implies $T_k \subseteq \mathfrak{S}^+$ for all $k \in \mathbb{N}$. The inverse inclusion follows from Lemma 2. The remaining assertion is evident. □

3.3. **Description of $\mathfrak{R}_5$.** In the theorem below, we use the indices k, ϵ for the entries of $\mathfrak{S}^+$, where k stands for the number of the triple and ϵ indicates the orbit of Γ. The index ϵ takes values $-1, 0, 1$ in the order of the representatives of the orbits in the triple T_0. For example, $v_{k,1}$ denotes the second component of the third vector in the triple T_k. Note that the indices agree with the natural order on both components.

Theorem 2. *Let the polynomial $f_{n,5}$ defined by (10) be harmonic and let A_n or B_n be rational. Then there are $k \in \mathbb{N}$ and $\epsilon \in \{-1,0,1\}$ such that*

$$n = \tfrac{1}{4}(u_{k,\epsilon} - 3) \tag{16}$$

and, if $A_n \le B_n$*,*

$$A_n = \tfrac{1}{6}(u_{k,\epsilon} - 2v_{k,\epsilon} - 3), \quad B_n = \tfrac{1}{6}(u_{k,\epsilon} + 2v_{k,\epsilon} - 3), \tag{17}$$

Conversely, the above formulas and (10) define a harmonic polynomial for every $k \in \mathbb{N}$ *and* $\epsilon \in \{-1, 0, 1\}$*.*

Proof. The family $\mathfrak{R}_5$ is described implicitly in Lemma 3. Thus we have to assign to the indices k, ϵ the numbers n, A_n and B_n. The formulas (16) and (17) do it. The formula (16) and the equality $m = \frac{1}{2}v_{k,\epsilon}$ follow from (13). Substituting this to (11) we get (17). It follows from Lemma 2, Lemma 1, and Lemma 3, that the assumptions of the theorem hold if and only if $f_{n,5}$ is harmonic. □

The table below contains the first four entries of the three series of the numbers $n = n_{k,\epsilon}$ defined by (16). The indices $k = 2, .., 5$ and $\epsilon = -1, 0, 1$ enumerate columns and rows, respectively.

121	175561	253159921	365056431601
540	779760	1124414460	1621404872640
2401	3463321	4994107561	7201499640721

Up to the end of this section we change the notation for the polynomials $p_{n,5} \in \mathfrak{R}_5$ and the numbers A_n, B_n replacing n with the corresponding k, ϵ in accordance with Theorem 2 and dropping 5. Thus

$$p_{k,\epsilon} = p_{n_{k,\epsilon},5}.$$

The matrix M^2 defines the following recurrence relation for the sequences $u_{k,s}, v_{k,s}$:

$$X_{k+1} = 1442X_k - X_{k-1}. \tag{18}$$

The initial data for all series are contained in the triples T_0 and T_1 that are written above Lemma 2 and Lemma 3, respectively.

Every k correspond six numbers $A_{k,\epsilon}, B_{k,\epsilon}$. In fact, there are only four distinct of them by the following theorem.

Theorem 3. *For any* $k \in \mathbb{N}$

(i) $A_{k,0} = B_{k,-1}$ *and* $B_{k,0} = A_{k,1}$*,*

(ii) *the above numbers are integer, the remaining* $A_{k,-1}$ *and* $B_{k,1}$ *are not integer.*

Proof. Due to (17), (i) is equivalent to the equalities

$$u_{k,-1} + 2v_{k,-1} = u_{k,0} - 2v_{k,0},$$
$$u_{k,0} + 2v_{k,0} = u_{k,1} - 2v_{k,1}.$$

Both left and right parts of these equalities are subject to (18). Hence it is sufficient to check them for $k = 0$ and $k = 1$. This is an easy calculation which proves (i).

According to (17), (13), and (11) the recurrence relations for $u_{k,\epsilon}, v_{k,\epsilon}$ defined by M^2 may be rewritten for $A_{k,\epsilon}, B_{k,\epsilon}$ as follows:

$$\begin{cases} A_{k+1,\epsilon} = -77A_{k,\epsilon} + 342B_{k,\epsilon} + 132, \\ B_{k+1,\epsilon} = -342A_{k,\epsilon} + 1519B_{k,\epsilon} + 588. \end{cases} \tag{19}$$

By (17), the initial data relating to T_0 are $(\frac{4}{3}, 0, 0)$ for $A_{0,\epsilon}$ and $(0, 0, \frac{4}{3})$ for $B_{0,\epsilon}$, where $\epsilon = -1, 0, 1$, respectively. Since

$$-77 = 1519 = 1 \mod 3$$

and other coefficients on the right are divisible by 3, the fractional parts of $A_{k,-1}$ and $B_{k,1}$ equals $\frac{1}{3}$ for all $k \in \mathbb{N}$ while the two others $A_{k,\epsilon}, B_{k,\epsilon}$ are integer. Thus (ii) is true. □

Corollary 2. *In every triple $T_k = (p_{k,-1}, p_{k,0}, p_{k,1})$ the consecutive polynomials have a common quadratic irreducible factor. Polynomials from different triples and $p_{k,-1}, p_{k,1}$ have no common factor.* □

Corollary 3. *A quadratic polynomial $x^2 + y^2 - Az^2$ divides at most two distinct $p_{k,\epsilon}$. This happens if and only if it divides one of them and A is integer.* □

3.4. **Tables and formulas.** According to (13), for $n_{k,\epsilon}, m_{k,\epsilon}$ we have

$$\begin{cases} n_{k+1,\epsilon} = 721 n_{k,\epsilon} + 1140 m_{k,\epsilon} + 540, \\ m_{k+1,\epsilon} = 456 n_{k,\epsilon} + 721 m_{k,\epsilon} + 342. \end{cases} \tag{20}$$

The initial data relating to T_0 are

$$\begin{array}{llll} (u_{0,\epsilon}, v_{0,\epsilon}) & (7,-2), & (3,0), & (7,2), \\ (n_{0,\epsilon}, m_{0,\epsilon}) & (1,-1), & (0,0) & (1,1), \\ (A_{0,\epsilon}, B_{0,\epsilon}) & \left(\frac{4}{3},0\right), & \left(0,0\right) & \left(0,\frac{4}{3}\right). \end{array} \tag{21}$$

The linear parts of the affine transformations on the right of (20) and (19) are conjugate to M^2. Hence we get the sequences subject to (18) removing the origin to the fixed points of these transformations which are equal to $(-\frac{3}{4}, 0)$ and $(-\frac{1}{2}, -\frac{1}{2})$, respectively. Let $X_k(x_0, x_1)$ be the sequence satisfying (18) with the initial data $X_0(x_0, x_1) = x_0$, $X_1(x_0, x_1) = x_1$. Then

$$\begin{array}{lcl} n_{k,\epsilon} & = & X_k(n_{0,\epsilon} + \frac{3}{4}, n_{1,\epsilon} + \frac{3}{4}) - \frac{3}{4}, \\ A_{k,\epsilon} & = & X_k(A_{0,\epsilon} + \frac{1}{2}, A_{1,\epsilon} + \frac{1}{2}) - \frac{1}{2}, \end{array} \tag{22}$$

$m_{k,\epsilon} = X_k(m_{0,\epsilon}, m_{1\epsilon})$, and $B_{k,\epsilon}$ is subject to the same formula as $A_{k,\epsilon}$. The data relating to $k = 0$ are given in (21). Here is the similar table for $k = 1$:

$$\begin{array}{llll} (u_{1,\epsilon}, v_{1,\epsilon}) & (487,154), & (2163,684), & (9607,3038) \\ (n_{1,\epsilon}, m_{1,\epsilon}) & (121,77), & (540,342) & (2401,1519), \\ (A_{1,\epsilon}, B_{1,\epsilon}) & \left(\frac{88}{3},132\right), & \left(132,588\right) & \left(588,\frac{7840}{3}\right). \end{array} \tag{23}$$

For the generating function $G_X(t) = \sum_{k=0}^{\infty} X_k t^k$, where $X_k = X_k(x_0, x_1)$ is as above, we have

$$G_X(t) = \frac{x_0 - (1442x_0 - x_1)t}{1 - 1442t + t^2}.$$

Combining it with (22) and using the initial data, we get the formulas for all series:

$$\begin{array}{cccc} G_{n,\epsilon} & \frac{121+958t+t^2}{(1-t)(1-1442t+t^2)} & \frac{540(1+t)}{(1-t)(1-1442t+t^2)} & \frac{2401+1322t+t^2}{(1-t)(1-1442t+t^2)} \\ G_{m,\epsilon} & \frac{77+t}{1-1442t+t^2} & \frac{342}{1-1442t+t^2} & \frac{1519-t}{1-1442t+t^2} \end{array}$$

The generating functions for $A_{k,\epsilon}, B_{k,\epsilon}$ also may be calculated in this way or by the formulas $G_{A,\epsilon} = \frac{2}{3}(G_{n,\epsilon} - G_{m,\epsilon})$, $G_{B,\epsilon} = \frac{2}{3}(G_{n,\epsilon} + G_{m,\epsilon})$ which follow from (11):

$$\begin{array}{cccc} G_{A,\epsilon} & \frac{4(22+517t+t^2)}{3(1-t)(1-1442t+t^2)} & \frac{12(11+49t)}{(1-t)(1-1442t+t^2)} & \frac{12(49+11t)}{(1-t)(1-1442t+t^2)} \\ G_{B,\epsilon} & \frac{12(11+49t)}{(1-t)(1-1442t+t^2)} & \frac{12(49+11t)}{(1-t)(1-1442t+t^2)} & \frac{4(1960-1421t+t^2)}{3(1-t)(1-1442t+t^2)} \end{array}$$

The sequences $A_{k,\epsilon}$ satisfy the inequalities

$$A_{1,-1} < A_{1,0} < A_{1,1} < A_{2,-1} < A_{2,0} < A_{2,1} < \dots .$$

The generating function for the sequence of all A_n enumerated in this order is $G_A = G_{A,-1} + t\,G_{A,0} + t^2\,G_{A,1}$. The function G_B can be defined analogously. They are too awkward to work with but a computer computation shows that the following equality holds:

$$G_A(t) - t\,G_B(t) = \frac{4}{3} + \frac{28(1+t^3)}{1 - 1442t^3 + t^6}$$

It gives a hint of another proof of Theorem 3. The right-hand part characterizes the gaps between the triples.

4. The sets $\mathfrak{R}_6$ and $\mathfrak{R}_7$ are finite

4.1. **The case** $d = 6$. As in the cases $k = 4$ and $k = 5$, we may write $f_{n,6}$ as

$$\zeta^{n-3}(\zeta\bar{\zeta} - A_n z^2)(\zeta\bar{\zeta} - B_n z^2)(\zeta\bar{\zeta} - C_n z^2).$$

If $f_{n,6}$ is reducible, then one of the numbers A_n, B_n, C_n is rational. The equation $\Delta f_{n,6} = 0$ is equiavelent to a system of linear equations with the elementary symmetric polynomials of A_n, B_n, C_n. Then we get the polynomial $p_{n,6}(x, y, z) = z^6 \tilde{p}_{n,6}\left(\frac{x^2+y^2}{z^2}\right)$, where

$$\tilde{p}_{n,6}(t) = t^3 - 6nt^2 + 4n(n-1)t - \frac{8}{15}n(n-1)(n-2)$$

has roots A_n, B_n, C_n. It is more convenient to work with the polynomial $q_{n,6}(t) = \frac{1}{8}\,\tilde{p}_{n,6}\,(2t + 2n)$ that is the canonical form of $\tilde{p}_{n,6}$:

$$q_{n,6}(t) = t^3 - n(2n+1)t - \frac{2}{15}n(2n+1)(4n+1). \tag{24}$$

Let $p, r \in \mathbb{N}$ and p be prime. Recall that $p|m$ indicates that p divides m. Set

$$h(p, r) = \max\{\alpha \in \mathbb{N} \cup \{0\} : p^\alpha | r\}.$$

The definition can be extended onto the rational numbers $r = \frac{m}{k}$ by

$$h(p, r) = h(p, m) - h(p, k).$$

Then $p|r$ means that $h(p, r) > 0$.

Lemma 4. *Suppose that $q_{n,6}$ admits a rational root c. Let p be a prime divisor of $n(2n+1)$. If $p \neq 2, 3, 5$, then $h(p, n(2n+1))$ is divisible by* 3.

Proof. Let $r = \frac{m}{k}$, where m, k are coprime integers. Recall that if r is a root of a polynomial $a_n t^n + \dots + a_0$ with integer coefficients, then $m|a_0$ and $k|a_n$.

The second and the third coefficients of the polynomial $3 \cdot 5^3\,q_{n,6}\left(\frac{k}{5}\right)$ are integer and $n(2n+1)$ is their common factor. Since n, $2n+1$, and $4n+1$ are pairwise coprime p does not divide $4n+1$. Together with the assumption $p \neq 2, 3, 5$ this implies

$$h(p, 15n(2n+1)) = h(p, 2n(2n+1)(4n+1)).$$

Let α denote the above number and set $\beta = h(p, c)$. We have $\alpha > 0$ and consequently $\beta = h(p, c) = \frac{1}{3}h(p, c^3) > 0$. Put

$$\mu = \min\{3\beta, \alpha + \beta, \alpha\}. \tag{25}$$

Dividing (24) with $t = c$ by p^μ we get a sum of three rational numbers such that at least one of them is not divisible by p. We get a contradiction if the others are divisible because the sum equals zero. Hence there are at least two minimal numbers in the triple $3\beta, \alpha + \beta, \alpha$. Then the inequality $\beta > 0$ implies $3\beta = \alpha$. This proves the lemma. □

4.2. **The case $d = 7$.** We perform a preparatory work for the case $d = 7$ before proving the main result. It is similar to that of above. We have

$$\tilde{p}_{n,7}(t) = t^3 - 2nt^2 + \frac{4}{5}n(n-1)t - \frac{8}{105}n(n-1)(n-2)$$

and the following expression for $q_{n,7}(t) = \tilde{p}_{n,7}(t + \frac{2}{3}n)$:

$$q_{n,7}(t) = t^3 - \frac{4}{15}n(2n+3)t - \frac{16}{945}n(2n+3)(4n+3).$$

Lemma 5. *Let $q_{n,7}$ have a rational root c and let p be a prime divisor of $n(2n+3)$. If $p \neq 2, 3, 5, 7$, then $h(p, n(2n+3))$ is divisible by 3.*

Proof. We consider the polynomial $945q_{n.7}$ which has integer coefficients. Note that $945 = 9!! = 3^3 \cdot 5 \cdot 7$. The assumption implies

$$h(p, 2^2 \cdot 3^2 \cdot 7\, n(2n+3)) = h(p, 16n(2n+3)(4n+3)) =: \alpha > 0$$

since the greatest common divisor of every pair in the triple n, $2n+3$, $4n+3$ is either 3 or 1. Set $\beta = h(p, c)$ and let μ be defined by (25). Clearly, $\beta > 0$. As in the case $d = 6$, there are at least two minimal numbers in the triple $3\beta, \alpha+\beta, \alpha$. Hence $\alpha = 3\beta$. This proves the lemma. □

4.3. **Almost all polynomials $p_{n,6}$ and $p_{n,7}$ are irreducible.**

Theorem 4. *The sets $\mathfrak{R}_6$ and $\mathfrak{R}_7$ are finite.*

Proof. Since $\deg \tilde{p}_{n,6} = \deg \tilde{p}_{n,7} = 3$, the reducibility of these polynomials is equivalent to the existence of a rational root. Thus we may apply Lemma 4 and Lemma 5, respectively. Let $d = 6$. Then

$$n(2n+1) = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} p_1^{3\beta_1} p_2^{3\beta_2} \cdots p_k^{3\beta_k},$$

where $p_1, \ldots, p_k$ are primes distinct from $2, 3, 5$. Similar factorization holds for both n and $2n+1$ since they are coprime. It follows that

$$n = Ku^3 \quad \text{and} \quad 2n+1 = Lv^3,$$

where $u, v \in \mathbb{N}$ and $K, L = 2^\alpha 3^\beta 5^\gamma$ with some $\alpha, \beta, \gamma \in \{0, 1, 2\}$. Hence u, v satisfy the Diophantine equation

$$Lv^3 - 2Ku^3 = 1. \tag{26}$$

It is well known that the set of solutions to every such equation is finite due to the celebrated Thue–Siegel–Roth theorem:

- *for any irrational algebraic number a and $\varepsilon > 0$ there exists $C > 0$ such that the inequality $\left|a - \frac{l}{k}\right| > \frac{C}{k^{2+\varepsilon}}$ holds for every $l \in \mathbb{Z}$ and $k \in \mathbb{N}$.*

It obviously implies that for any $C > 0$ the inverse inequality $\left|a - \frac{l}{k}\right| < \frac{C}{k^{2+\varepsilon}}$ may be true only for finite set of $\frac{l}{k}$. For solutions u, v to (26) it is easy to derive the inequalities $0 < a - \frac{u}{v} < \frac{1}{2Ka^2v^3}$, where $a = \sqrt[3]{\frac{L}{2K}}$. Thus the set $\mathfrak{R}_6$ is finite.

The above arguments with minor changes prove that $\mathfrak{R}_7$ is finite. We have to add 7 to the set $\{2,3,5\}$, note that only 3 may be a nontrivial common divisor of n and $2n+3$, replace (26) with the equation $Lv^3 - 2Ku^3 = 3$, and multiply the upper bound for $a - \frac{u}{v}$ by 3. This proves the theorem. □

5. Remarks

5.1. There are more than a hundred Diophantine equations that satisfy the above conditions. Perhaps, a half or more of them have no solution but this needs a careful analysis. The books [17], [13] contain fundamental facts concerting Diophantine equations of third degree. For example, it is known that the equation $u^3 - Dv^3 = 1$, where D is free of cubes, may have at most one solution distinct from the trivial $u = 1$, $v = 0$ (see [17, Ch. VI, §71, Theorem V] or [13, Ch. 24, Theorem 5]). It is worth mentioning that the existence of a solution does not imply the reducibility of the relating polynomial $p_{n,6}$ or $p_{n,7}$.

5.2. Let C be a round cone in $\mathbb{R}^3$ which is not a plane and $\mathcal{H}(C)$ be the space of all harmonic polynomials that vanish on C. There are such cones C in $\mathbb{R}^4$ with infinite dimensional $\mathcal{H}(C)$ (see [8, Example 4.3]). To the best of my knowledge, no such example in $\mathbb{R}^3$ is known and, moreover, the greatest known dimension of $\mathcal{H}(C)$ is 8. It is attained on the round cones that are the zero sets of the common quadratic factors for a couple of the consecutive reducible polynomials $p_{n,d}$, where $d = 4, 5$. For example,

- $x^2 + y^2 - 44z^2$ divides $p_{22,2}$, $p_{66,3}$, $p_{10,4}$, and $p_{99,4}$,
- $x^2 + y^2 - 132z^2$ divides $p_{66,2}$, $p_{198,3}$, $p_{77,5}$, and $p_{540,5}$.

Every $p_{n,d}$ relates to a couple of harmonic functions $f_{n,d}$ and $\bar{f}_{n,d}$ with the same zeros. In the real version, this is the common zeros of their real and imaginary parts. Also, it is not known if there exists a common quadratic divisor for some $p_{n,4}$ and $p_{m,5}$.

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Victor Matveevich Gichev
Sobolev Institute of Mathematics, Omsk Branch,
13, Pevtsova str.,
Omsk, 644099, Russia
Email address: gichev@ofim.oscsbras.ru