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ON EQUILIBRIUM OF A TWO-DIMENSIONAL VISCOELASTIC BODY WITH A THIN TIMOSHENKO INCLUSION

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Abstract. The equilibrium problem for a two-dimensional viscoelastic body containing a thin elastic inclusion modeled as a Timoshenko beam is considered. Cases without delamination as well as the case when the inclusion delaminates from the body forming a crack are studied. Both variational and differential statements of the corresponding problems containing nonlinear boundary conditions, as well as their solvability is justified. The limiting case, as the stiffness parameter of the inclusion tends to infinity is considered and the problem of the thin rigid inclusion is obtained.

Keywords: variational inequality, Timoshenko inclusion, viscoelastic body, thin elastic inclusion, crack, non-penetration conditions, nonlinear boundary conditions.

Mathematical modeling of deformation processes of composite materials calls for well-posed problems where type, physical and geometric properties of present heterogenities, as well as the interaction between a matrix and a filler are taken into account. Several methods for studying two-dimensional problems for composite materials with fibrous filler, where the problem's geometry is modeled as an elastic body with a thin beam type inclusion had been discovered. In recent years a wide array of equilibrium problems for two-dimensional elastic and viscoelastic bodies with thin elastic inclusions, which are usually modeled as either a Bernoulli-Euler beam or a Timoshenko beam, as well as thin rigid inclusions had been considered (see [1]-[13]). One can also find multiple papers on various types of anisotropy of

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elastic beams and coverage of limiting cases, in which rigidness of a beam in one direction tends to infinity, i.e. [14], [15], [16]. When a thin inclusion delaminates from a composite matrix, a crack appears, which stipulates formulating a problem in the area with a cut. In that case it is assumed that a thin inclusion is situated on one of the edges of a cut. In the aforementioned papers a method of establishing inequalities as boundary conditions while treating the edges of a crack as parts of said boundary is used. Since the resulting conditions are nonlinear, variational inequalities must be used when proving their solvability and studying qualitative properties of solutions. General methods of using variational inequalities in nonlinear problems of elasticity theory for non-smooth areas/regions can be found in [17], while various results concerning problems for cracks in viscoelastic bodies are listed in [18], [19].

In this paper we consider the equilibrium problem for a two-dimensional viscoelastic body, containing a thin elastic inclusion, modeled as a Timoshenko beam. Cases with and without delamination of an inclusion are studied. We provide both a variational and a differential statement of a boundary problem and show that they are, in a sense, equivalent. We also prove that aforementioned problems are uniquely solvable and get additional property of smoothness of these solutions with respect to time variable. The limiting case, as the stiffness parameter of the inclusion tends to infinity is considered and it is shown that, in this case, the solutions of the equilibrium problem mentioned above converge to the solutions of the problem of the thin rigid inclusion. Problems of thin and volume inclusions for a viscoelastic body which are also stated using conditions of inequality type can be found in [20]-[22].

1. A thin Timoshenko inclusion without delamination

Problem statement. Assume that a viscoelastic body in a non-deformed state takes up a bounded domain $\Omega \subset R^2$ with a smooth boundary Γ. We will model a thin rectilinear inclusion in a two-dimensional viscoelastic body as a Timoshenko beam, with its axis being defined by the line $\gamma = (0,1) \times \{0\}$, such that $\overline{\gamma} \subset \Omega$. We also introduce the following notation: let $\nu = (0,1)$ be the unit vectors and $s = (1,0)$ a tangent to γ. Assume that γ can be extended until it intersects with Γ, whilst splitting Ω into two subdomains with Lipschitz boundaries (denoted by $\Omega^{\pm}$), such that $meas(\partial\Omega^{\pm} \cap \Gamma) > 0$. Let $\Omega_\gamma = \Omega \setminus \overline{\gamma}$ be a domain with a cut γ (fig. 1). Denote the edges of that cut by γ^+ and γ^-, where $\gamma^{\pm} = \partial\Omega^{\pm} \cap \gamma$. Furthermore, let ν^- and ν^+ be outer normals of $\partial\Omega^{\pm}$ and $\nu = \nu^- = -\nu^+$ on γ. Hence we can see that the boundary of Ω_γ consists of an outer closed curve Γ and two edges of the cut $\gamma^{\pm}$.

We will state the equilibrium problem for a two-dimensional viscoelastic body containing a thin Timoshenko inclusion as follows:

Problem (1). For given functions of external loads $f = (f_1, f_2)$ with domain in $Q = \Omega \times (0,T)$ find

(1) A displacement field $u = (u_1, u_2)$ for all points of a body in a cylinder Q.
(2) A stress tensor $\sigma = \{\sigma_{ij}\}$, $i,j = 1,2$ in a cylinder $Q_\gamma = \Omega_\gamma \times (0,T)$.

Furthermore, on $\gamma \times (0,T)$, find displacements v, w and a rotation angle φ for points of the beam such that the following statements are true:

$$-div\, \sigma(t) = f(t) \quad \text{in} \quad Q_\gamma, \tag{1.1}$$

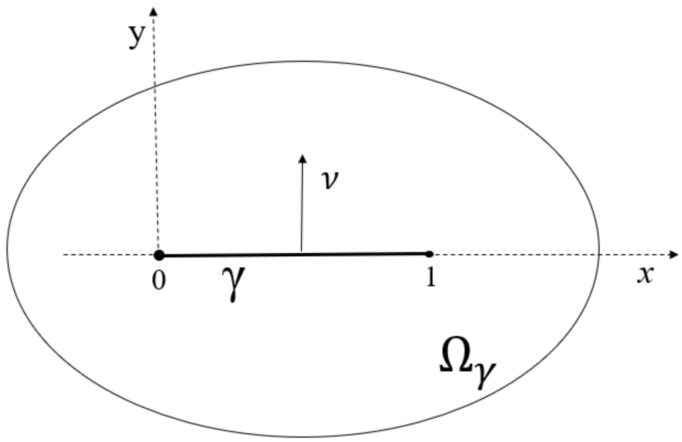

FIGURE 1. A two-dimensional body with an elastic inclusion.

$$\sigma(t) = A\varepsilon(u(t)) + \int_0^t \overline{A}\varepsilon(u(\tau))\,d\tau, \quad \text{in} \quad Q_\gamma, \tag{1.2}$$

$$-w_{xx}(t) = [\sigma_s(t)] \quad \text{on } \gamma \times (0,T), \tag{1.3}$$

$$-\varphi_{xx}(t) + v_x(t) + \varphi(t) = 0 \quad \text{on } \gamma \times (0,T), \tag{1.4}$$

$$-v_{xx}(t) - \varphi_x(t) = [\sigma_\nu(t)] \quad \text{on } \gamma \times (0,T), \tag{1.5}$$

$$u = 0 \quad \text{on } \Gamma \times (0,T), \tag{1.6}$$

$$\varphi(t) + v_x(t) = w_x(t) = \varphi_x(t) = 0 \quad \text{when } x = 0,1;\ t \in (0,T), \tag{1.7}$$

$$v(t) = u_2(t),\ w(t) = u_1(t) \quad \text{on } \gamma \times (0,T). \tag{1.8}$$

Here $div\,\sigma = \sigma_{ij,j}$, and $\xi_{,j} = \frac{\partial \xi}{\partial x_j}$; $\varepsilon_{ij}(u) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$, $i,j = 1,2$, $x_1 = x$, $x_2 = y$; $A\varepsilon(u) = a_{ijkl}\varepsilon_{kl}(u)$, $\overline{A}\varepsilon(u) = \overline{a}_{ijkl}\varepsilon_{kl}(u)$, $i,j,k,l = 1,2$; $\sigma\nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j)$, $\sigma_\nu = \sigma_{ij}\nu_j\nu_i$, $\sigma_s = \sigma\nu \cdot s$; the brackets $[v] = v^+ - v^-$ denote a function jump at the edges of a cut, while superscripts "+" and "−" represent the values of the function on $\gamma^+ \times (0,T)$ and $\gamma^- \times (0,T)$ respectively. The coefficients $a_{ijkl}(x,y)$, $\overline{a}_{ijkl}(x,y) \in L^\infty(\Omega_\gamma)$, $i,j,k,l = 1,2$ are the components of tensors A, $\overline{A}$ which satisfy the following conditions:

$$a_{ijkl} = a_{jikl} = a_{klij},$$

$$a_{ijkl}\xi_{kl}\xi_{ij} \geq c_0|\xi|^2, \quad \forall \xi_{ij} = \xi_{ji},$$

where c_0 is some positive constant. Similar relations also hold for $\overline{A}$. In this text, we implicitly sum over repeated indices.

A function $u = u(x,y,t)$ is defined given $(x,y) \in \Omega$, $t \in (0,T)$, while functions $v = v(x,t)$, $w = w(x,t)$, $\varphi = \varphi(x,t)$ are defined given $0 \leq x \leq 1$, $t \in (0,T)$. Denote by ξ_x and ξ_{xx}, respectively, first and second order derivatives of ξ by x. For simplicity, we did not mention the dependence of functions on spatial variables when introducing the statements (1.1)-(1.8).

Relations (1.1) and (1.2) are, respectively, the equations of equilibrium and state for a viscoelastic body. By (1.3)-(1.5) and (1.7) an equilibrium of a Timoshenko beam with free ends is described, with right sides of equations (1.3), (1.5) considering influence of a viscoelastic environment on the beam. Equation (1.6) represent a fixation of a body over an outer boundary. By (1.8), vertical and horizontal displacements of a beam a similar to normal and tangent displacements of a body on $\gamma \times (0, T)$.

We introduce the following notation

$$\chi = (u, \psi); \quad \psi = (v, w, \varphi)$$

and consider the space:

$$H = \{\chi = (u, \psi) \,|\, u \in H_0^1(\Omega)^2,\, \psi \in H^1(\gamma)^3,\, v = u_\nu,\, w = u_s \text{ on } \gamma\},$$

Let H^* be a dual space of H and consider a linear operator $\Lambda: \; L^2(0,T;H) \to L^2(0,T;H^*)$, which can be written as follows:

$$(\Lambda\chi, \widetilde{\chi}) = \int\limits_0^T \langle \sigma(t), \varepsilon(\widetilde{u}(t))\rangle_\Omega \, dt + \int\limits_0^T \Phi(\psi, \widetilde{\psi})\, dt, \quad \widetilde{\chi} = (\widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{\varphi}) \in L^2(0,T;H).$$

where

$$\Phi(\psi, \widetilde{\psi}) = \langle w_x, \widetilde{w}_x\rangle_\gamma + \langle \varphi_x, \widetilde{\varphi}_x\rangle_\gamma + \langle v_x + \varphi, \widetilde{v}_x + \widetilde{\varphi}\rangle_\gamma,$$

and the brackets $\langle\cdot,\cdot\rangle_D$ denote integration with respect to D.

Definition 1. *A generalised solution to the problem (1.1)-(1.8) for given* $f \in L^2(Q)^2$*;* a_{ijkl}*,* $\overline{a}_{ijkl} \in L^\infty(\Omega)$*,* $i,j,k,l = 1,2$ *is a function* $\chi \in L^2(0,T;H)$*, which satisfies the following equation:*

$$(\Lambda\chi(t), \overline{\chi}(t)) = \int\limits_0^T \langle f(t), \overline{u}(t)\rangle_\Omega \, dt, \quad \forall \overline{\chi} = (\overline{u}, \overline{v}, \overline{w}, \overline{\varphi}) \in L^2(0,T;H). \tag{1.9}$$

We will now prove

Theorem 1. *There exists a unique generalised solution to the problem (1.1)-(1.8).*

Proof. Consider the equation

$$(\Lambda\chi, \chi) = \int\limits_0^T \langle A\varepsilon(u(t)), \varepsilon(u(t))\rangle_\Omega \, dt + \int\limits_0^T \langle \overline{A}\varepsilon(\int\limits_0^t u(\tau)d\tau), \varepsilon(u(t))\rangle_\Omega \, dt$$

$$+ \int\limits_0^T \langle w_x(t), w_x(t)\rangle_\gamma \, dt + \int\limits_0^T \langle \varphi_x(t), \varphi_x(t)\rangle_\gamma \, dt + \int\limits_0^T \langle v_x(t) + \varphi(t), v_x(t) + \varphi(t)\rangle_\gamma \, dt.$$

In a domain Ω the Korn's inequality (see [23]) holds, therefore, we can utilise similar methods to those used in [18], [22] and write

$$\int\limits_0^T \langle A\varepsilon(u(t)), \varepsilon(u(t))\rangle_\Omega \, dt + \int\limits_0^T \langle \overline{A}\varepsilon(\int\limits_0^t u(\tau)d\tau), \varepsilon(u(t))\rangle_\Omega \, dt$$

$$= \int\limits_0^T \langle A\varepsilon(u(t)), \varepsilon(u(t))\rangle_\Omega \, dt + \frac{1}{2}\langle \overline{A}\varepsilon(\int\limits_0^T u(t)dt), \varepsilon(\int\limits_0^T u(t)dt)\rangle_\Omega \, dt \geq$$

$$\geq c_1 \|u\|^2_{L^2(0,T;H_0^1(\Omega)^2)}.$$

Hence, given a sufficiently small $\beta > 0$ it is true that

$$(\Lambda\chi, \chi) \geq \frac{c_1}{2}\|u\|^2_{L^2(0,T;H^1_0(\Omega)^2)}$$

$$+\int\limits_0^T \Phi(\psi,\psi)\,dt + \beta\int\limits_0^T \langle v(t), v(t)\rangle_\gamma\,dt + \beta\int_0^T \langle w(t), w(t)\rangle_\gamma\,dt$$

$$+\bigg(\frac{c_1}{2}\|u\|^2_{L^2(0,T;H^1_0(\Omega)^2)} - \beta\int\limits_0^T \langle v(t), v(t)\rangle_\gamma\,dt - \beta\int\limits_0^T \langle w(t), w(t)\rangle_\gamma\,dt\bigg).$$

From (1.8) and the embedding theorems, we obtain that

$$\|v\|_{L^2(\gamma)} \leq c_2\|u\|_{H^1(\Omega_\gamma)^2}, \quad \|w\|_{L^2(\gamma)} \leq c_3\|u\|_{H^1(\Omega_\gamma)^2}.$$

Therefore,

$$\frac{c_1}{2}\|u\|^2_{L^2(0,T;H^1_0(\Omega)^2)} - \beta\int\limits_0^T \langle v, v\rangle_\gamma\,dt - \beta\int\limits_0^T \langle w, w\rangle_\gamma\,dt \geq 0.$$

From the lemma, proven in [1], we can see that there exists a constant c_4, for which it is true that

$$\beta\Big(\int\limits_0^T \langle v(t), v(t)\rangle_\gamma dt + \int\limits_0^T \langle w(t), w(t)\rangle_\gamma dt\Big) + \\ + \int\limits_0^T \Phi(\psi,\psi)dt \geq c_4\|\psi\|^2_{L^2(0,T;H^1(\gamma)^3)}.$$

From our estimates it follows that there exists $C > 0$, such that

$$(\Lambda\chi, \chi) \geq C\|\chi\|^2_{L^2(0,T;H)}.$$

The last inequality shows that Λ is a coercive operator. Since it is also linear, the same inequality also yields that Λ is monotonous and this, together with the fact that it is semi-continuous [24], shows that this operator is also pseudomonotonous. Therefore, the problem (1.9) is solvable [24]. Since the solution is unique, the theorem is proved. □

Existence of a derivative with respect to time. We will now show that if $f \in H^1(0,T;L^2(\Omega))^2$, then the solution to (1.9) is differentiable with respect to t.

Theorem 2. *If $f \in H^1(0,T;L^2(\Omega))^2$ then the solution to a problem (1.9) has a derivative $\chi_t \in L^2(0,T;H^1(\Omega))^2$ and it is true that*

$$\|\chi_t\|^2_{L^2(0,T;H^1(\Omega))^2} \leq c_5\big(\|f_t\|^2_{L^2(Q)^2} + \|f\|^2_{L^2(Q)^2}\big).$$

Proof. We will rewrite (1.9) as follows:

$$\int\limits_0^T \langle A\varepsilon(u(t)), \varepsilon(\overline{u}(t))\rangle_\Omega\,dt + \int\limits_0^T \langle\overline{A}\varepsilon(\int\limits_0^t u(\tau)d\tau), \varepsilon(\overline{u}(t))\rangle_\Omega\,dt$$

$$+\int_0^T \langle w_x(t), \overline{w}_x(t)\rangle_\gamma\, dt + \int_0^T \langle \varphi_x(t), \overline{\varphi}_x(t)\rangle_\gamma\, dt + \int_0^T \langle v_x(t)+\varphi(t), \overline{v}_x(t)+\overline{\varphi}(t)\rangle_\gamma\, dt$$

$$= \int_0^T \langle f(t), \overline{u}(t)\rangle_\Omega\, dt, \quad \forall \overline{\chi} \in L^2(0,T;H). \tag{1.10}$$

Pick an arbitrary $\alpha > 0$ and fix $t \in (0,T)$, such that $(t-\alpha, t+\alpha) \in (0,T)$. Consider a test function of the form

$$\overline{\chi}(\theta) = \begin{cases} \widetilde{\chi} - \chi(\theta), & \theta \in (t-\alpha, t+\alpha), \\ 0, & \theta \notin (t-\alpha, t+\alpha), \end{cases}$$

where $\widetilde{\chi} = (\widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{\varphi}) \in H$ is a fixed element. By substituting $\overline{\chi}(\theta)$ into (1.10) and dividing by 2α, we obtain

$$\frac{1}{2\alpha}\Big(\int_{t-\alpha}^{t+\alpha} \langle A\varepsilon(u(\theta)), \varepsilon(\widetilde{u} - u(\theta))\rangle_\Omega\, d\theta + \int_{t-\alpha}^{t+\alpha} \langle \overline{A}\,\varepsilon(\int_0^\theta u(\tau)\, d\tau), \varepsilon(\widetilde{u} - u(\theta))\ \rangle_\Omega\, d\theta$$

$$+ \int_{t-\alpha}^{t+\alpha} \Phi(\psi(\theta), \widetilde{\psi} - \psi(\theta))\, d\theta \Big) = \frac{1}{2\alpha}\int_{t-\alpha}^{t+\alpha} \langle f(\theta), \widetilde{u} - u(\theta)\ \rangle_\Omega\, d\theta.$$

By passing to the limit as $\alpha \to 0$, we obtain that, given a fixed $t \in (0,T)$, the following is true

$$\langle A\varepsilon(u(t)), \varepsilon(\widetilde{u} - u(t))\rangle_\Omega + \langle \overline{A}\,\varepsilon(\int_0^t u(\tau)\, d\tau), \varepsilon(\widetilde{u} - u(t))\ \rangle_\Omega$$

$$+\Phi(\psi(t), \widetilde{\psi} - \psi(t)) = \langle f(t), \widetilde{u} - u(t)\ \rangle_\Omega, \quad \widetilde{\chi} \in H. \tag{1.11}$$

We will first consider a test function $\widetilde{\chi} = \chi(t+h)$ in (1.11) and then compute (1.11) at $t+h$ and take $\chi(t)$ as $\widetilde{\chi}$. We sum these equations and divide the result by h^2. Then, denoting

$$d_h v(t) = \frac{v(t+h) - v(t)}{h}, \quad d_h^\tau v(t) = \frac{1}{h}\int_t^{t+h} v(\tau) d\tau, \quad h > 0,$$

we will obtain

$$\langle A\varepsilon(d_h u(t)),\, \varepsilon(d_h u(t))\rangle_\Omega + \Phi(d_h\psi(t), d_h\psi(t))$$
$$= \langle d_h f(t), d_h u(t)\ \rangle_\Omega - \langle \overline{A}\varepsilon(d_h^\tau u(t)),\, \varepsilon(d_h u(t))\ \rangle_\Omega. \tag{1.12}$$

By using Korn's inequality in the left side of the last equation, we can, for a sufficiently small $\alpha > 0$, get the following estimate:

$$\langle A\varepsilon(d_h u(t)),\, \varepsilon(d_h u(t))\rangle_\Omega + \Phi(d_h\psi(t), d_h\psi(t))$$
$$\geq \frac{c_6}{2}\|d_h u(t)\|^2_{H^1(\Omega)^2} + \Phi(d_h\psi(t), d_h\psi(t))$$
$$+\alpha\Big(\langle d_h w(t), d_h w(t)\rangle_\gamma + \langle d_h v(t), d_h v(t)\rangle_\gamma \Big)$$
$$+\Big(\frac{c_6}{2}\|d_h u(t)\|^2_{H^1(\Omega)^2} - \alpha\langle d_h w(t), d_h w(t)\rangle_\gamma - \alpha\langle d_h v(t), d_h v(t)\rangle_\gamma \Big) \geq$$

$$\geq c_7 \bigg(\|d_h u(t)\|^2_{H^1(\Omega)^2} + \|d_h \psi(t)\|^2_{H^1(\gamma)^3} \bigg).$$

From the estimate, given $\lambda > 0$, we obtain from (1.12) that

$$c_7 \|d_h \chi(t)\|^2_H \leq \frac{1}{\lambda} \|d_h f(t)\|^2_{L^2(\Omega)^2} + \lambda \|d_h u(t)\|^2_{L^2(\Omega)^2} + \frac{1}{\lambda} \|d_h^\tau u(t)\|^2_{L^2(\Omega)^2} + \lambda \|d_h u(t)\|^2_{L^2(\Omega)^2}.$$

Therefore, for a sufficiently small $\lambda > 0$, there exists $c_8 > 0$, such that

$$\|d_h \chi(t)\|^2_H \leq c_8 \left(\|d_h f(t)\|^2_{L^2(\Omega)^2} + \|d_h^\tau u(t)\|^2_{L^2(\Omega)^2} \right). \tag{1.13}$$

Note that, for a smooth function $v(x,t)$, it is true that

$$\int_0^{T-h} \|d_h^\tau v(t)\|^2_{L^2(\Omega)} dt \leq \int_0^T \|v(t)\|^2_{L^2(\Omega)} dt, \tag{1.14}$$

therefore, by integrating (1.13) over t from 0 to $T-h$ and using (1.14), we get

$$\int_0^{T-h} \|d_h \chi(t)\|^2_H \, dt \leq c_8 \left(\int_0^{T-h} \|d_h f(t)\|^2_{L^2(\Omega)^2} \, dt + \int_0^T \|u(t)\|^2_{L^2(\Omega)^2} \, dt \right). \tag{1.15}$$

It can be shown that

$$\int_0^{T-h} \|d_h f(t)\|^2_{L^2(\Omega)^2} \, dt = \int_0^{T-h} \|d_h^\tau f_t(t)\|^2_{L^2(\Omega)^2} \, dt.$$

Since $f_t(t) \in L^2(Q)^2$, we can write (1.14) for functions $v = f_t$:

$$\int_0^{T-h} \|d_h^\tau f_t(t)\|^2_{L^2(\Omega)} \, dt \leq \int_0^T \|f_t(t)\|^2_{L^2(\Omega)} \, dt.$$

It then follows from (1.15) that

$$\int_0^{T-h} \|d_h \chi(t)\|^2_H \, dt \leq c_8 \left(\int_0^T \|f_t(t)\|^2_{L^2(\Omega)^2} \, dt + \int_0^T \|u(t)\|^2_{L^2(\Omega)^2} \, dt \right).$$

Consider a sufficiently small h_0, such that $h_0 \geq h$. Then it is true that

$$\int_0^{T-h_0} \|d_h \chi(t)\|^2_H \, dt \leq c_8 \left(\int_0^T \|f_t(t)\|^2_{L^2(\Omega)^2} \, dt + \int_0^T \|u(t)\|^2_{L^2(\Omega)^2} \, dt \right).$$

By passing to the limit as $h \to 0$ in the last inequality, we get the following:

$$\int_0^{T-h_0} \|\chi_t\|^2_H \, dt \leq c_8 \left(\int_0^T \|f_t\|^2_{L^2(\Omega)^2} \, dt + \int_0^T \|u\|^2_{H^1(\Omega)^2} \, dt \right).$$

Since $h_0 \geq 0$ is arbitrary, we can see that

$$\|\chi_t\|^2_{L^2(0,T;H^1(\Omega))^2} \leq c_8 \left(\|f_t\|^2_{L^2(Q)^2} + \|u\|^2_{L^2(0,T;H^1(\Omega))^2} \right).$$

Hence a derivative χ_t exists: moreover, by considering a test function $\overline{\chi} = \chi(t)$ in (1.10), we will get that

$$(\Lambda\chi, \chi) = \int_0^T \langle f(t),\, u(t)\,\rangle_\Omega \, dt.$$

Therefore, since Λ is coercive, we obtain the following

$$\|\chi\|^2_{L^2(0,T;H)} \le c_9 \|f\|^2_{L^2(Q)^2}.$$

From this, the statement of the theorem follows. □

2. A delaminated thin Timoshenko inclusion

In this section we assume that a thin inclusion delaminates from a viscoelastic matrix, forming a crack along the line γ, therefore a viscoelastic body takes up a domain, Ω_γ, i.e. a domain with a cut, with an elastic inclusion situated on one of the edges of the cut. We will set conditions for displacements $u(x,t)$ on the edges. They will look as follows:

$$[u]\nu \ge 0 \quad \text{on} \quad \gamma \times (0,T).$$

These conditions will ensure that surfaces $\gamma^{\pm} \times (0,T)$ will not penetrate each other [17].

We will state the equilibrium problem for a two-dimensional viscoelastic body, taking up the domain Ω_γ and containing a delaminated thin Timoshenko inclusion with the form specified by γ as follows:

Problem (2). For given functions of external loads $f = (f_1, f_2)$ with domain in Q find a displacement field $u = (u_1, u_2)$ of body points and a stress tensor $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$ in a cylinder Q_γ. Furthermore, on $\gamma \times (0,T)$, find displacements v, w and a rotation angle φ for points of the beam such that the following statements are true:

$$-div\,\sigma(t) = f(t) \quad \text{in} \quad Q_\gamma, \tag{2.1}$$

$$\sigma(t) = A\varepsilon(u(t)) + \int_0^t \overline{A}\varepsilon(u(\tau))\,d\tau, \quad \text{in} \quad Q_\gamma, \tag{2.2}$$

$$-w_{xx}(t) = [\sigma_s(t)] \quad \text{on } \gamma \times (0,T), \tag{2.3}$$

$$-\varphi_{xx}(t) + v_x(t) + \varphi(t) = 0 \quad \text{on } \gamma \times (0,T), \tag{2.4}$$

$$-v_{xx}(t) - \varphi_x(t) = [\sigma_\nu(t)] \quad \text{on } \gamma \times (0,T), \tag{2.5}$$

$$u = 0 \quad \text{on } \Gamma \times (0,T), \tag{2.6}$$

$$\varphi(t) + v_x(t) = w_x(t) = \varphi_x(t) = 0 \quad \text{when } x = 0, 1;\ t \in (0,T), \tag{2.7}$$

$$[u_\nu(t)] \ge 0, \quad v(t) = u_2^-(t),\ w(t) = u_1^-(t) \quad \text{on } \gamma \times (0,T), \tag{2.8}$$

$$\sigma_\nu^+(t) \le 0,\ \sigma_s^+(t) = 0,\ \sigma_\nu^+(t)[u_\nu(t)] = 0 \quad \text{on } \gamma \times (0,T). \tag{2.9}$$

Non-penetration conditions stated in (2.8) together with relations stated in (2.9) are commonly used as boundary conditions for nonlinear crack problems [17], [18].

Consider the spaces:

$$H^{1,0}(\Omega_\gamma) = \{v \in H^1(\Omega_\gamma) \mid v = 0 \text{ on } \Gamma\}, \quad H_\gamma = H^{1,0}(\Omega_\gamma)^2 \times H^1(\gamma)^3,$$

as well as a convex closed set of functions

$$K = \{\chi \in H_\gamma \mid [u_\nu] \ge 0,\, v = u_\nu^-,\, w = u_s^- \text{ on } \gamma\}.$$

We will use a set of possible displacements in the following form

$$\mathscr{K} = \{\chi \in L^2(0,T;H_\gamma) \mid \chi(t) \in K \text{ when } t \in (0,T)\}.$$

By H_γ^* denote a dual space of H_γ and consider a linear operator $\Lambda_\gamma : L^2(0,T;H_\gamma) \to L^2(0,T;H_\gamma^*)$, of the form

$$(\Lambda_\gamma \chi, \widetilde{\chi}) = \int_0^T \langle \sigma(t), \varepsilon(\widetilde{u}(t))\rangle_{\Omega_\gamma}\, dt + \int_0^T \Phi(\psi, \widetilde{\psi})\, dt, \quad \widetilde{\chi} \in L^2(0,T;H_\gamma).$$

Definition 2. *A generalised solution to the problem (2.1)-(2.9) for given* $f \in L^2(Q)^2$*;* a_{ijkl}*,* $\overline{a}_{ijkl} \in L^\infty(\Omega)$*,* $i,j,k,l = 1,2$ *is a function* $\chi \in \mathscr{K}$ *which satisfies the following variational inequality*

$$(\Lambda_\gamma \chi(t), \overline{\chi}(t) - \chi(t)) \geq \int_0^T \langle f(t), \overline{u}(t) - u(t)\rangle_{\Omega_\gamma}\, dt, \quad \forall \overline{\chi} \in \mathscr{K}. \tag{2.10}$$

The following statement holds:

Theorem 3. *There exists a unique generalised solution to the problem (2.1)-(2.9).*

Proof. Since Korn's inequality also holds in Ω_γ, the proof is similar to that of Theorem 1.1. □

Theorem 4. *The problem (2.1)-(2.9) is equivalent to the variational inequality (2.10), given sufficient smoothness of solutions.*

Proof. Assume all relations (2.1)-(2.9) hold. Consider $\overline{\chi} \in \mathscr{K}$ and multiply both sides of (2.1) by $\overline{u} - u$ and multiply both sides of equations (2.3)-(2.5) by $\overline{w} - w$, $\overline{\varphi} - \varphi$, $\overline{v} - v$ respectively. Integrate the first one of the resulting equations with respect to Q_γ and others with respect to $\gamma \times (0,T)$. Now, summing them, obtain

$$-\int_0^T \langle div\, \sigma(t), \overline{u}(t) - u(t)\rangle_{\Omega_\gamma} dt - \int_0^T \langle w_{xx}(t), \overline{w}(t) - w(t)\rangle_\gamma dt$$

$$+ \int_0^T \langle -\varphi_{xx}(t) + v_x(t) + \varphi(t), \overline{\varphi}(t) - \varphi(t)\rangle_\gamma dt - \int_0^T \langle v_{xx}(t) + \varphi_x(t), \overline{v}(t) - v(t)\rangle_\gamma dt$$

$$= \int_0^T \langle f(t), \overline{u}(t) - u(t)\rangle_{\Omega_\gamma} + \int_0^T \langle [\sigma_s(t)], \overline{w}(t) - w(t)\rangle_\gamma dt + \int_0^T \langle [\sigma_\nu(t)], \overline{v}(t) - v(t)\rangle_\gamma dt.$$

Integrating by parts and considering (2.7), we can write the following

$$-\int_0^T \langle \sigma(t), \varepsilon(\overline{u}(t) - u(t))\rangle_{\Omega_\gamma} dt + \int_0^T \Phi(\psi(t), \overline{\psi}(t) - \psi(t)) dt$$

$$= \int_0^T \langle f(t), \overline{u}(t) - u(t)\rangle_{\Omega_\gamma} dt + L,$$

where

$$L = -\int_0^T [\langle \sigma(t)\nu, \overline{u}(t) - u(t)\rangle_\gamma]\, dt$$

$$+\int_0^T \langle [\sigma_s(t)], \overline{w}(t) - w(t)\rangle_\gamma\, dt + \int_0^T \langle [\sigma_\nu(t)], \overline{v}(t) - v(t)\rangle_\gamma\, dt.$$

To prove that the variational inequality (2.10) holds, it now suffices to show that $L \geq 0$. This can be done by rewriting L while using the second and the third conditions stated in (2.8), the second condition stated in (2.9) and properties of elements $\overline{\chi} \in \mathscr{K}$. This way we obtain the following:

$$L = -\int_0^T \langle \sigma_\nu^+(t), \overline{u}_\nu^+(t) - u_\nu^+(t)\rangle_\gamma dt + \int_0^T \langle \sigma_\nu^+(t), \overline{v}(t) - v(t)\rangle_\gamma\, dt$$

$$= -\int_0^T \langle \sigma_\nu^+(t), [\overline{u}_\nu(t)]\rangle_\gamma\, dt + \int_0^T \langle \sigma_\nu^+(t), [u_\nu(t)]\rangle_\gamma\, dt.$$

Since in the last equation the first term is nonnegative (because of the first condition from (2.9) and properties of functions $\overline{\chi} \in \mathscr{K}$), and the second term is equal to zero (because of the third condition of (2.9)), we have that $L \geq 0$.

We will now prove the converse. Assume that variational inequality (2.10) holds. Rewrite it as follows:

$$\chi \in \mathscr{K},\ \int_0^T \langle \sigma(t), \varepsilon(\overline{u}(t) - u(t))\rangle_{\Omega_\gamma} dt + \int_0^T \langle w_x(t), \overline{w}_x(t) - w_x(t)\rangle_\gamma\, dt$$

$$+\int_0^T \langle \varphi_x(t), \overline{\varphi}_x(t) - \varphi_x(t)\rangle_\gamma\, dt + \int_0^T \langle v_x(t) + \varphi(t), \overline{v}_x(t) + \overline{\varphi}(t) - v_x(t) - \varphi(t)\rangle_\gamma dt$$

$$\geq \int_0^T \langle f(t), \overline{u}(t) - u(t)\rangle_{\Omega_\gamma} dt. \tag{2.11}$$

We can use the similar technique to the one we used to obtain (1.11) and produce the statement, equivalent to (2.11):

$$\chi \in \mathscr{K},\ \langle \sigma(t), \varepsilon(\overline{u} - u(t))\rangle_{\Omega_\gamma} \langle w_x(t), \overline{w}_x - w_x(t)\rangle_\gamma$$

$$+\langle \varphi_x(t), \overline{\varphi}_x - \varphi_x(t)\rangle_\gamma + \langle v_x(t) + \varphi(t), \overline{v}_x + \overline{\varphi} - v_x(t) - \varphi(t)\rangle_\gamma$$

$$\geq \langle f(t), \overline{u} - u(t)\rangle_{\Omega_\gamma},\ \forall \overline{\chi} \in K,\ t \in (0,T). \tag{2.12}$$

By choosing an arbitrary function $\widetilde{u} \in C_0^\infty(Q_\gamma)$ and substituting a test function $\overline{\chi} = \chi \pm (\widetilde{u}, 0, 0, 0)$ into (2.12) we can get that (2.1) holds in Q_γ. Next, we consider an arbitrary point $x^0 \in \gamma$ and denote by D some neighbourhood of x^0, such that ∂D intersects γ in two points. Let $D^\pm = \Omega^\pm \cap D$. By substituting test functions $\overline{\chi} = \chi + (\widetilde{u}, 0, 0, 0)$ (where $\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2)$, $\widetilde{u}_\nu \geq 0$ on $\gamma \times (0,T)$, $supp\, \widetilde{u} \subset \overline{D}^+$) into (2.12), we obtain that

$$\langle \sigma(t), \varepsilon(\widetilde{u})\rangle_{D^+} \geq \langle f(t), \widetilde{u}\rangle_{D^+}.$$

From there, we get that

$$\langle \sigma(t)\nu, \widetilde{u} \rangle_{\gamma^+} \leq 0.$$

Since $\widetilde{u}_s$ is arbitrary, it follows that $\sigma_s^+ = 0$, $\sigma_\nu^+ \leq 0$, $\sigma_\nu^+[u_\nu] = 0$ on γ. Now consider a test function $\overline{\chi} = \chi \pm \widetilde{\chi}$, where $\widetilde{\chi} = (\widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{\varphi}) \in K$, $[\widetilde{u}_\nu] = 0$ on γ, and substitute it into (2.12). We obtain that

$$\langle \sigma(t), \varepsilon(\widetilde{u}) \rangle_{\Omega_\gamma} + \Phi(\psi(t), \widetilde{\psi}) = \langle f(t), \widetilde{u} \rangle_{\Omega_\gamma}.$$

From integration by parts, while considering (2.1) and (2.9) it follows that

$$\langle [\sigma(t)\nu], \widetilde{u} \rangle_\gamma - \langle w_{xx}(t), \widetilde{w} \rangle_\gamma - \langle \varphi_{xx}(t), \widetilde{\varphi} \rangle_\gamma - \langle v_{xx}(t) + \varphi_x(t), \widetilde{v} \rangle_\gamma$$
$$+ \langle v_x(t) + \varphi(t), \widetilde{\varphi} \rangle_\gamma + w_x(t)\widetilde{w}_x|_0^1 + \varphi_x(t)\widetilde{\varphi}|_0^1 + (v_x(t) + \varphi(t))\widetilde{v}|_0^1 = 0.$$

By using the properties of functions in K and obtained conditions (2.9), we can rewrite the last equation as follows:

$$\langle [\sigma_\nu(t)] - v_x x(t) - \varphi_x(t), \widetilde{v} \rangle_\gamma - \langle \sigma_s^-(t), \widetilde{u}_s^- \rangle_\gamma - \langle w_{xx}(t), \widetilde{w} \rangle_\gamma - \langle \varphi_{xx}(t), \widetilde{\varphi} \rangle_\gamma$$
$$+ \langle v_x(t) + \varphi(t), \widetilde{\varphi} \rangle_\gamma + w_x(t)\widetilde{w}_x|_0^1 + \varphi_x(t)\widetilde{\varphi}|_0^1 + (v_x(t) + \varphi(t))\widetilde{v}|_0^1 = 0. \qquad (2.13)$$

Let $\widetilde{w}_x = \widetilde{v} = \widetilde{\varphi} = 0$ when $x = 0, 1$. Then

$$[\sigma_\nu(t)] - v_x x(t) - \varphi_x(t) = 0, \ [\sigma_s(t)] + w_{xx}(t) = 0, \ -\varphi_{xx}(t) + v_x(t) + \varphi(t) = 0 \text{ on } \gamma.$$

From the relations we obtained and (2.13), it follows that

$$w_x = \varphi_x = v_x + \varphi = 0 \text{ when } x = 0, 1, \ t \in (0, T).$$

Hence we have derived all the statements (2.1)-(2.9) from variational inequality (2.10) □

3. Passage to the limit with respect to a stiffness parameter

When stating Problem (2), we considered the coefficients, denoting stiffness of the material in the equations, modeling the equilibrium of the beam, to be equal to 1. In this section, we will state the problem with the stiffness parameter $\delta > 0$ and consider the limit case, as it tends to infinity.

Problem (3). For given functions of external loads $f = (f_1, f_2)$ with domain in Q, find a displacement field $u^\delta = (u_1^\delta, u_2^\delta)$ of body points and a stress tensor $\sigma^\delta = \{\sigma_{ij}^\delta\}$, $i, j = 1, 2$ in a cylinder Q_γ. Furthermore, on $\gamma \times (0, T)$, find displacements v^δ, w^δ and a rotation angle φ^δ for points of the beam, such that the following statements are true:

$$-div\, \sigma^\delta(t) = f(t) \quad \text{in} \quad Q_\gamma, \qquad (3.1)$$

$$\sigma^\delta(t) = A\varepsilon(u^\delta(t)) + \int_0^t \overline{A}\varepsilon(u^\delta(\tau))\, d\tau, \quad \text{in} \quad Q_\gamma, \qquad (3.2)$$

$$-\delta w_{xx}^\delta(t) = [\sigma_s^\delta(t)] \quad \text{on } \gamma \times (0, T), \qquad (3.3)$$

$$-\delta\varphi_{xx}^\delta(t) + \delta v_x^\delta(t) + \delta\varphi^\delta(t) = 0 \quad \text{on } \gamma \times (0, T), \qquad (3.4)$$

$$-\delta v_{xx}^\delta(t) - \delta\varphi_x^\delta(t) = [\sigma_\nu^\delta(t)] \quad \text{on } \gamma \times (0, T), \qquad (3.5)$$

$$u^\delta = 0 \quad \text{on } \Gamma \times (0, T), \qquad (3.6)$$

$$\varphi^\delta(t) + v_x^\delta(t) = w_x^\delta(t) = \varphi_x^\delta(t) = 0 \quad \text{when } x = 0, 1; \ t \in (0, T), \qquad (3.7)$$

$$[u_\nu^\delta(t)] \geq 0, \quad v^\delta(t) = u_2^{\delta -}(t), \ w^\delta(t) = u_1^{\delta -}(t) \quad \text{on } \gamma \times (0, T), \qquad (3.8)$$

$$\sigma_\nu^{\delta +}(t) \leq 0, \ \sigma_s^{\delta +}(t) = 0, \ \sigma_\nu^{\delta +}(t)[u_\nu^\delta(t)] = 0 \quad \text{on } \gamma \times (0, T). \qquad (3.9)$$

Consider a linear operator $\Lambda_\gamma^\delta : \ L^2(0,T;H_\gamma) \to L^2(0,T;H_\gamma^*)$, of the form

$$(\Lambda_\gamma^\delta \chi, \widetilde{\chi}) = \int_0^T \langle \sigma(t), \varepsilon(\widetilde{u}(t))\rangle_{\Omega_\gamma}\, dt + \delta \int_0^T \Phi(\psi, \widetilde{\psi})\, dt, \quad \widetilde{\chi} \in L^2(0,T;H_\gamma).$$

Definition 3. *A generalised solution to the problem (3.1)-(3.9) for given* $f \in L^2(Q)^2$*;* a_{ijkl}*,* $\overline{a}_{ijkl} \in L^\infty(\Omega)$*,* $i,j,k,l = 1,2$*;* $\delta > 0$ *is a function*

$$\chi^\delta = (u^\delta, v^\delta, w^\delta, \varphi^\delta) \in \mathscr{K},$$

which satisfies the following variational equation:

$$(\Lambda_\gamma^\delta \chi^\delta(t), \overline{\chi}(t) - \chi^\delta(t)) \geq \int_0^T \langle f(t), \overline{u}(t) - u^\delta(t)\rangle_{\Omega_\gamma}\, dt, \quad \forall \overline{\chi} \in \mathscr{K}. \tag{3.10}$$

It can be seen that for every $\delta > 0$ the problem (3.10) has a unique generalised solution. Differential statement (3.1)-(3.9) is also equivalent to (3.10), given sufficient smoothness of solutions. In this section, we will consider the passing to the limit, as δ tends to infinity.

Substitute $\overline{\chi} = 0$ and $\overline{\chi} = 2\chi^\delta$ into (3.10). Then we get that

$$\int_0^T \langle \sigma^\delta(t), \varepsilon(u^\delta(t))\rangle_{\Omega_\gamma} dt + \delta \int_0^T \Phi(\psi^\delta(t), \psi^\delta(t)) dt = \int_0^T \langle f(t), u^\delta(t)\rangle_{\Omega_\gamma} dt. \tag{3.11}$$

The following conditions

$$\|v^\delta\|_{L^2(\gamma)} \leq c_{10} \|u^\delta\|_{H^1(\Omega_\gamma)^2}, \quad \|w^\delta\|_{L^2(\gamma)} \leq c_{11} \|u^\delta\|_{H^1(\Omega_\gamma)^2}$$

and the Korn's inequality imply that, given sufficiently small $\beta > 0$, the following is true:

$$\frac{1}{2} \int_0^T \langle \sigma^\delta(t), \varepsilon(u^\delta(t))\rangle_{\Omega_\gamma} dt - \beta\Big(\int_0^T \langle w^\delta(t), w^\delta(t)\rangle_\gamma dt + \int_0^T \langle v^\delta(t), v^\delta(t)\rangle_\gamma dt\Big) \geq 0.$$

Hence, from (3.11), we get that

$$\frac{1}{2} \int_0^T \langle \sigma^\delta(t), \varepsilon(u^\delta(t))\rangle_{\Omega_\gamma} dt - \beta\Big(\int_0^T \langle w^\delta(t), w^\delta(t)\rangle_\gamma dt + \int_0^T \langle v^\delta(t), v^\delta(t)\rangle_\gamma dt\Big)$$

$$= -\frac{1}{2} \int_0^T \langle \sigma^\delta(t), \varepsilon(u^\delta(t))\rangle_{\Omega_\gamma} dt - \delta \int_0^T \Phi(\psi^\delta(t), \psi^\delta(t)) dt + \int_0^T \langle f(t), u^\delta(t)\rangle_{\Omega_\gamma} dt$$

$$-\beta\Big(\int_0^T \langle w^\delta(t), w^\delta(t)\rangle_\gamma dt + \int_0^T \langle v^\delta(t), v^\delta(t)\rangle_\gamma dt\Big) \geq 0.$$

The last inequailty implies the following

$$\frac{1}{2} \int_0^T \langle \sigma^\delta(t), \varepsilon(u^\delta(t))\rangle_{\Omega_\gamma} dt + \delta \int_0^T \Phi(\psi^\delta(t), \psi^\delta(t)) dt$$

$$+\beta\Big(\int_0^T \langle w^\delta(t), w^\delta(t)\rangle_\gamma dt + \int_0^T \langle v^\delta(t), v^\delta(t)\rangle_\gamma dt\Big) \le \int_0^T \langle f(t), u^\delta(t)\rangle_{\Omega_\gamma} dt.$$

By Korn's inequality and lemma from [1], we get that for all $\delta \le \beta$,

$$\|u^\delta\|^2_{L^2(0,T;H^{1,0}(\Omega_\gamma))^2} + \delta\|\psi^\delta\|^2_{L^2(0,T;H^1(\gamma))} \le c_{12}\|u^\delta\|_{L^2(0,T;H^{1,0}(\Omega_\gamma))^2}.$$

It follows that

$$\|u^\delta\|_{L^2(0,T;H^{1,0}(\Omega_\gamma))^2} \le c_{12}. \tag{3.12}$$

From (3.11), we obtain that, given $\delta \ge \delta_0$

$$\|\psi^\delta\|_{L^2(0,T;H^1(\gamma))^3} \le c_{13}, \tag{3.13}$$

$$\delta \int_0^T \Phi(\psi^\delta(t), \psi^\delta(t)) dt \le c_{14}. \tag{3.14}$$

From the estimates (3.12)-(3.14), we get the convergence as $\delta \to +\infty$

$$u^\delta \to u \text{ weakly in } L^2(0,T;H^{1,0}(\Omega_\gamma))^2, \tag{3.15}$$

$$\psi^\delta \to \psi \text{ weakly in } L^2(0,T;H^1(\gamma))^3, \quad \psi = (v,w,\varphi), \tag{3.16}$$

Moreover,

$$w_x = \varphi_x = v_x + \varphi = 0 \text{ on } \gamma \times (0,T). \tag{3.17}$$

In particular, there exist d, g^1, $g^2 \in L^2(0,T)$, such that

$$w(x,t) = d(t),\ \varphi(x,t) = g^1(t),\ v(x,t) = -g^1(t)x + g^2(t) \text{ on } \gamma \times (0,T). \tag{3.18}$$

We will now consider the limit case and show that it, in fact, induces the problem of thin rigid inclusion. We introduce the following spaces:

$$R(\gamma) = \{\rho = (\rho_1,\rho_2)\,|\,\rho_1 = b_1, \rho_2 = -b_2 x + b_3,\ x \in \gamma,\ b_i \in \mathbb{R},\ i = 1,2,3\},$$

$$R_\gamma = \{\rho = (\rho_1,\rho_2)\,|\,\rho_1 = b_1(t),\ \rho_2 = -b_2(t)x + b_3(t) \text{ on } \gamma \times (0,T);$$
$$b_i \in L^2(0,T), i = 1,2,3\},$$

and the following sets:

$$K_r = \{v \in H^{1,0}(\Omega_\gamma)^2\,|\,[v_\nu] \ge 0 \text{ on } \gamma;\ v|_{\gamma^-} \in R(\gamma)\},$$

$$\mathscr{K}_r = \{v \in L^2(0,T;H^{1,0}(\Omega_\gamma)^2)\,|\,v(t) \in K_r,\ t \in (0,T);\ v|_{\gamma^- \times (0,T)} \in R_\gamma\}.$$

Hence, since for the limit element u the conditions $[u_\nu] \ge 0$, $u_\nu^- = v$, $u_s^- = w$ on $\gamma \times (0,T)$ and also (3.18), it follows that $u \in \mathscr{K}_r$. Consider an arbitrary element $\widetilde{u} \in \mathscr{K}_r$. Then $\overline{\chi} = (\tilde{u}, \tilde{\psi}) = (\widetilde{u}, \widetilde{u}_2^-, \widetilde{u}_1^-, -(\widetilde{u}_2^-)_x) \in \mathscr{K}$. After substituting $\overline{\chi}$ into (3.10), we obtain that

$$\chi^\delta \in \mathscr{K} : \int_0^T \langle \sigma^\delta(t), \varepsilon(\widetilde{u}(t) - u^\delta(t))\rangle_{\Omega_\gamma} dt + \delta \int_0^T \Phi(\psi^\delta(t), \widetilde{\psi} - \psi^\delta(t)) dt \ge$$

$$\ge \int_0^T \langle f(t), \widetilde{u} - u^\delta(t)\rangle_{\Omega_\gamma} dt,\ \tilde{u} \in \mathscr{K}_r.$$

From the choice of $\overline{\chi}$, it follows that $\Phi(\psi^\delta, \tilde{\psi}) = 0$. Then the last inequality yields that

$$\int_0^T \langle \sigma^\delta(t), \varepsilon(\widetilde{u}(t) - u^\delta(t))\rangle_{\Omega_\gamma} dt - \int_0^T \langle f(t), \widetilde{u}(t) - u^\delta(t)\rangle_{\Omega_\gamma} dt \geq 0.$$

Considering the limit case, we get that

$$u \in \mathscr{K}_r : \int_0^T \langle A\varepsilon(u(t)), \varepsilon(\widetilde{u}(t) - u(t))\rangle_{\Omega_\gamma} dt + \int_0^T \langle \overline{A}\varepsilon(\int_0^t u(\tau)d\tau), \varepsilon(\widetilde{u}(t) - u(t))\rangle_{\Omega_\gamma} dt \geq$$

$$\geq \int_0^T \langle f(t), \widetilde{u}(t) - u(t)\rangle_{\Omega_\gamma} dt, \;\; \widetilde{u} \in \mathscr{K}_r. \tag{3.19}$$

Prolem (3.19) can be stated in an equivalent differential form.

Problem (4). In the cylinder Q_γ, find functions u, functions σ_{ij}, $i, j = 1, 2$ and $\rho^0 \in R_\gamma$, such that the following statements are true.

$$-div\, \sigma(t) = f(t) \quad \text{in} \quad Q_\gamma,$$

$$\sigma(t) = A\varepsilon(u(t)) + \int_0^t \overline{A}\varepsilon(u(\tau))\, d\tau, \quad \text{in} \quad Q_\gamma,$$

$$u = 0 \quad \text{on } \Gamma \times (0, T),$$

$$[u(t)]\nu \geq 0, \;\; u^-(t) = \rho^0(t), \;\; \sigma_\nu^+(t) \leq 0, \;\; \sigma_s^+(t) = 0 \quad \text{on } \gamma \times (0, T),$$

$$\sigma_\nu^+(t)[u(t)]\nu = 0 \quad \text{on } \gamma \times (0, T),$$

$$\int_\gamma [\sigma(t)\nu]\overline{\rho}\, dx = 0, \quad \overline{\rho} \in R(\gamma), \;\; t \in (0, T).$$

Problem (4) describes an equilibrium of a two-dimensional viscoelastic body with a delaminated thin rigid inclusion.

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