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## TWO LIMIT CYCLES FOR A CLASS OF DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS WITH TWO PIECES

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**ABSTRACT.** This paper is a survey on the study of the maximum number of limit cycles of planar continuous and discontinuous piecewise differential systems formed by two linear centers and defined in two pieces separated by

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : x = ly, l \in \mathbb{R} \text{ and } y \geq 0\} \\ \cup \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\}.$$

We restrict our attention to the crossing limit cycles, i.e. to the limit cycles having exactly two or four points on  $\Sigma$ . We prove that such discontinuous piecewise linear differential systems can have 1 or 2 limit cycles. The limit cycles having two intersection points with  $\Sigma$  can reach the maximum number 2. The limit cycles having four intersection points with  $\Sigma$  are at most 1, and if it exists, the systems could simultaneously have 1 limit cycle intersecting  $\Sigma$  in three points.

**Key words:** Discontinuous piecewise linear differential systems, linear centers, first integrals, limit cycles.

### 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

One of the most challenging problems in the qualitative theory of planar ordinary differential equations is the second part of the classical 16th Hilbert problem: the determination of an upper bound for the number of limit cycles (and their relative positions) for the class of polynomial vector fields of degree  $n$ . This problem remains unsolved if  $n \geq 2$ . The case  $n = 1$ , that is for the class of planar linear vector

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fields the problem has a trivial answer. However, this problem presents a surprising richness when adapted to the class of the planar piecewise linear systems.

The study of the piecewise linear differential systems goes back to Andronov, Vitt and Khaikin [1], and nowadays such systems still continue to receive the attention of many researchers. These differential systems are widely used to model processes appearing in electronics, mechanics, economy, etc..., see for instance the books of di Bernardo [2] and Simpson [16], the survey of Makarenkov and Lamb [11], as well as hundreds of references quoted in these last three works.

A periodic orbit of a differential system which *is isolated in the set of all periodic orbits of the system is a limit cycle*. There are two types of limit cycles in the planar discontinuous piecewise linear differential systems, the crossing and sliding ones. The “*sliding limit cycles*” contain some arc of the lines of discontinuity that separate the different linear differential systems (more precise definition can be found in [14]). The “*crossing limit cycles*” only contain isolated points of the lines of discontinuity.

Discontinuous piecewise linear systems with two regions separated by a straight line have received a lot of attention during the last years, see for instance [3, 5, 6, 7, 8, 9, 12] among other papers. In [5], the authors conjectured that piecewise linear systems with two regions separated by a straight line could have at most two crossing limit cycles. Later on in [6], the authors provided numerical evidence on the existence of three crossing limit cycles, which was analytically proved in [9]. Sufficient condition on piecewise linear system implying the existence of at most 3 crossing limit cycles can be found in [3, 8, 12]. As far as we know, there are no examples of piecewise linear vector fields separated by a straight line with more than 3 crossing limit cycles. In fact, although there is no proof, it is common sense that 3 is very likely the upper bound in this case. It is worthwhile to mention that the shape of the discontinuity set plays an important role in the number of crossing limit cycles. Indeed, if the discontinuity set is not a straight line, then one may find an arbitrary number of crossing limit cycles (see [13]).

Here, our objective is to study the number of limit cycles, which can exhibit the planar discontinuous piecewise linear differential systems separated by two pieces of straight lines such that both linear differential systems are formed by centers. In [8], it is proved that : A discontinuous piecewise linear differential system separated by one straight line formed by two linear centers has no limit cycles.

In this work we study the crossing limit cycles of discontinuous piecewise differential systems separated by

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = ly, l \in \mathbb{R} \text{ and } y \geq 0\},$$

and formed by two linear centers. The two components of  $\mathbb{R}^2 \setminus \Sigma$  are

$$\begin{aligned} S_1 &= \{(x, y) \in \mathbb{R}^2 : x > ly \text{ and } y > 0\}, \\ S_2 &= \{(x, y) \in \mathbb{R}^2 : x \leq ly \text{ and } y > 0\} \cup \{(x, y) \in \mathbb{R}^2 : y < 0, x \in \mathbb{R}\}, \end{aligned}$$

or convenience, using the notations :  $\Sigma = \Sigma_1 \cup \Sigma_2$  where

$$\Sigma_1 = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\}, \quad \Sigma_2 = \{(x, y) \in \mathbb{R}^2 : x = ly \text{ and } y \geq 0\},$$

We observe that we have three types of crossing limit cycles, namely, *crossing limit cycles of type 1* which intersect in a unique point each branch of the set  $\Sigma$ , *crossing limit cycles of type 2* which intersect the set  $\Sigma$  in four points (intersect

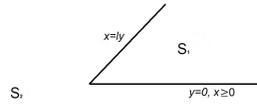


FIG. 1. The regions if  $l > 0$ .

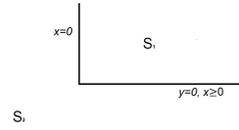


FIG. 2. The regions if  $l = 0$

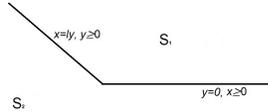


FIG. 3. The regions if  $l < 0$

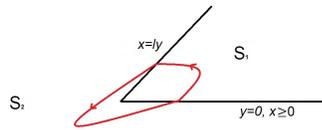


FIG. 4. Crossing limit cycle of type 1.

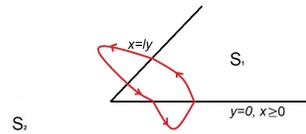


FIG. 5. Crossing limit cycle of type 2.

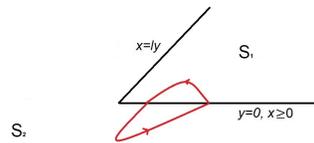
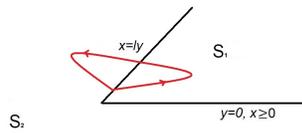


FIG. 6. Crossing limit cycles of type 3

only two branches of in exactly two points in each branch) and *crossing limit cycles of type 3* which intersect in two points only one branch of the set  $\Sigma$  :

According to the uniqueness Theorem for the solutions of an ordinary differential equation with a given initial value, then the solutions of any piecewise linear differential system connecting the points  $(x_i, 0)$  and  $(ly_i, y_i)$  cannot intersect. Then if any discontinuous piecewise linear system in the plane with two pieces separated by  $\Sigma$  has two crossing limit cycles, these two crossing limit cycles should be some of Figure 7.

We note that if discontinuous piecewise linear differential system with two pieces separated by the set  $\Sigma$  and formed by two arbitrary linear centers the cases of Figure 7 (A), (C), (D), (E), (I), (J), (M), (O) and (S) are not possible because in

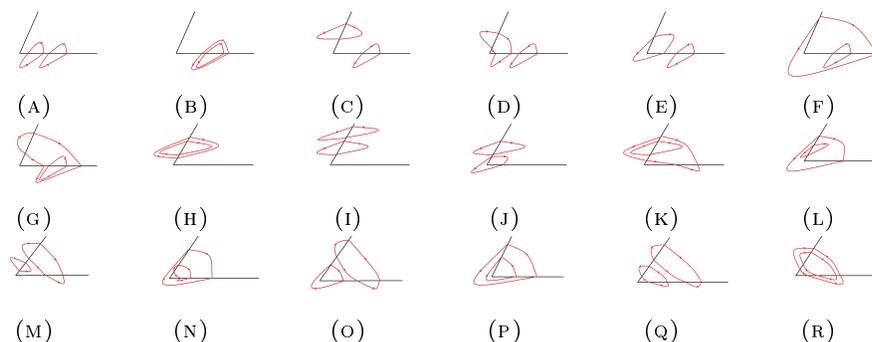


FIG. 7. Possible cases of two crossing limit cycles of any discontinuous piecewise linear system in the plane with two pieces separated by  $\Sigma$ .

these cases the pieces of the ellipses of linear differential centers in the regions  $S_1$  and  $S_2$ , would not be nested which contradicts that the linear differential systems in each of these regions are linear centers.

Here we do not consider the case where crossing limit cycles intersecting only one branch of  $\Sigma$  in two points (type 3), because in [10, 8] it was proved that discontinuous piecewise linear differential systems separated by a straight line have no crossing limit cycles, therefore the cases of Figure 7 (F), (G), (H), (K), and (L) are not possible because the inner limit cycle that intersecting only one branch of  $\Sigma$  in two points do not exist.

In this subsection we give the upper bound of crossing limit cycles of planar discontinuous piecewise linear differential centers and separated by the set  $\Sigma$ . We consider only the crossing limit cycles that intersect each branch of  $\Sigma$  in one point (type 1) and the crossing limit cycles intersecting the set  $\Sigma$  in four points (type 2).

The following normal form for the discontinuous piecewise linear differential systems in  $\mathbb{R}^2$  separated by the set  $\Sigma$  when both linear differential systems have a center will help us to prove our main result, the Theorem which follows the next proposition.

**Proposition 1.** *After a linear change of variables and a rescaling of the independent variable any discontinuous piecewise linear differential systems in  $\mathbb{R}^2$  separated by the set  $\Sigma$  when both linear differential systems have a center can be written as*

$$(1) \quad \begin{aligned} \dot{x} &= -bx - \frac{4b^2+v^2}{4a}y + \alpha, & \dot{y} &= ax + by + d & \text{in } S_1, \\ \dot{x} &= -Bx - \frac{4B^2+w^2}{4A}y + C, & \dot{y} &= Ax + By + D & \text{in } S_2, \end{aligned}$$

with  $a \neq 0, A \neq 0, v > 0$  and  $w > 0$ .

*Proof.* The linear differential system in the region  $S_1$  is

$$\dot{x} = \beta x + \gamma y + \alpha, \quad \dot{y} = ax + by + d,$$

assuming that it has a center. Since the eigenvalues of this system are

$$\frac{1}{2} \left( b + \beta \pm \sqrt{(b - \beta)^2 + 4a\gamma} \right)$$

in order to have a center we must have that  $b + \beta = 0$  and  $(b - \beta)^2 + 4a\gamma = -w^2$  for some  $w > 0$  and  $a\gamma < 0$ , or equivalently  $\beta = -b$ ,  $\gamma = -\frac{1}{4a}(4b^2 + w^2)$ . For the linear differential system in the region  $S_2$  the proof is similar to the proof of linear centre in  $S_1$ . This completes the proof of the Proposition 1.  $\square$

Our main results are the following:

**Theorem 1.** *For a planar discontinuous piecewise linear differential centers with two pieces separated by the set  $\Sigma$ , the following statements hold:*

- a) *The maximum number of crossing limit cycles intersecting in a unique point with each of the two branches of  $\Sigma$  in one point (type 1) is two.*
- b) *The maximum number of crossing limit cycles intersecting the set  $\Sigma$  in four points (type 2) is one.*

Theorem 1 is proved in Section 2.

The next propositions shows that the upper bound for the maximum number of crossing limit cycles provided in Theorem 1 is reached.

**Proposition 2.** *The following statements hold.*

- i) *There are discontinuous piecewise linear differential system separated by  $\Sigma$  formed by two linear centers, having exactly one crossing limit cycle of the type 1, see Figures 24, 25, 26.*
- ii) *There are discontinuous piecewise linear differential system separated by  $\Sigma$  formed by two linear centers, having exactly two crossing limit cycle of the type 1, see Figures 27, 28, 29.*
- iii) *There are discontinuous piecewise linear differential system separated by  $\Sigma$  formed by two linear centers, having exactly one crossing limit cycle of the type 2, see Figures 30, 31, 32.*

This Proposition will be proved in section 2.

Here we study the maximum number of crossing limit cycles of a discontinuous piecewise linear differential system separated by  $\Sigma$  formed by two linear centers that intersect the set  $\Sigma$  in two and in four points simultaneously, and we consider only the existence and the number of crossing limit cycles of the type 1 and type 2 simultaneously.

Our main result is the following:

**Theorem 2.** *A planar discontinuous piecewise linear differential system separated by  $\Sigma$  formed by two linear centers, can have at most one limit cycle intersecting the set  $\Sigma$  in exactly four points (type 2) and at most 1 limit cycle intersecting each branch of  $\Sigma$  in one point (type 1). Moreover this upper bound is reached. See Figure 41.*

Theorem 2 is proved in Section 3.

## 2. PROOF OF THEOREM 1

**2.1. Proof of statement (a) of Theorem 1.** Assume that we have a discontinuous piecewise linear differential system separated by the set  $\Sigma$  and formed by two centers. By Proposition 1, we can write such a discontinuous piecewise linear differential systems as

$$(2) \quad \begin{aligned} \dot{x} &= -bx - \frac{4b^2 + v^2}{4a}y + \alpha, & \dot{y} &= ax + by + d & \text{in } S_1, \\ \dot{x} &= -Bx - \frac{4B^2 + w^2}{4A}y + C, & \dot{y} &= Ax + By + D & \text{in } S_2, \end{aligned}$$

with  $A \neq 0, a \neq 0, v > 0$  and  $w > 0$ . The first integrals of the tow linear differential systems (2) in  $S_1$  and  $S_2$  are

$$(3) \quad \begin{aligned} H_1(x, y) &= 4(ax + by)^2 + 8a(dx - \alpha y) + v^2y^2, \\ H_2(x, y) &= 4(Ax + By)^2 + 8A(Dx - Cy) + w^2y^2, \end{aligned}$$

respectively. Suppose that this discontinuous piecewise differential system has some limit cycles each one intersecting each branch of  $\Sigma$  in one point, namely  $(x_1, 0)$  with  $x_1 > 0$  and  $(ly_1, y_1)$  with  $y_1 > 0$ . Then, the first integrals  $H_1$ , and  $H_2$  must satisfy the following two equations

$$(4) \quad \begin{aligned} H_1(x_1, 0) - H_1(ly_1, y_1) &= 0, \\ H_2(ly_1, y_1) - H_2(x_1, 0) &= 0, \end{aligned}$$

or equivalently

$$(5) \quad \begin{aligned} 4a^2x_1^2 + 8adx_1 - (4(b + al)^2 + v^2)y_1^2 + 8a(\alpha - dl)y_1 &= 0, \\ 4A^2x_1^2 + 8ADx_1 - (4(B + Al)^2 + w^2)y_1^2 + 8A(C - lD)y_1 &= 0. \end{aligned}$$

We recall that Bezout Theorem (see for instance [15]) states that if a polynomial system of equations has finitely many solutions, then the number of its solutions is at most the product of the degrees of the polynomials which appear in the system. Then by Bezout Theorem system (5) has at most 4 solutions  $(x_i, y_i), i = 1, 2, 3, 4$ . So, the discontinuous piecewise linear differential system (2) can have at most 4 limit cycles.

Notice that, the polynomial system (5) has the solution  $(0, 0)$ , which, cannot contribute a limit cycle. So, in this case, system (5) can have eventually three real solutions,  $(x_i, y_i)$  for  $i = 1, 2, 3$  producing three limit cycles for the discontinuous piecewise linear differential system (2). According to the Uniqueness Theorem for the solutions of an ordinary differential equation with a given initial value, then the solutions of the piecewise linear differential system (2) connecting the points  $(x_i, 0)$  and  $(ly_i, y_i)$  cannot intersect. So the polynomial system (5) can have eventually three real solutions,  $(x_i, y_i)$  for  $i = 1, 2, 3$ , producing three limit cycles for the discontinuous piecewise linear differential system (2), it is necessary that  $x_1, x_2$  and  $x_3$  have the same order as that of  $y_1, y_2$  and  $y_3$ . For instance

$$(6) \quad 0 < x_1 < x_2 < x_3 \quad \text{and} \quad 0 < y_1 < y_2 < y_3,$$

to prove the system (5) cannot have 3 solutions  $(x_i, y_i), i = 1, 2, 3$  satisfying the orders of (6), we write the first equation of (5) in

$$(7) \quad (2ax_1 + 2d)^2 - \left( \sqrt{4(b + al)^2 + v^2}y_1 - \frac{4a(\alpha - dl)}{\sqrt{4(b + al)^2 + v^2}} \right)^2 = 4d^2 - \frac{16(a(\alpha - dl))^2}{4(b + al)^2 + v^2},$$

which is a hyperbola if  $8la^2d\alpha - 4a^2\alpha^2 + 8labd^2 + 4b^2d^2 + d^2v^2 \neq 0$ , denoted by  $\mathcal{H}$ . Moreover the hyperbola  $\mathcal{H}$  has the two asymptotes:

$$\begin{aligned} L_{\mathcal{H}}^1 : 2ax_1 + \sqrt{4(b + al)^2 + v^2}y_1 - \frac{4a(\alpha - dl)}{\sqrt{4(b + al)^2 + v^2}} + 2d &= 0, \\ L_{\mathcal{H}}^2 : 2ax_1 - \sqrt{4(b + al)^2 + v^2}y_1 + \frac{4a(\alpha - dl)}{\sqrt{4(b + al)^2 + v^2}} + 2d &= 0, \end{aligned}$$

which intersect at  $I_{\mathcal{H}} = \left(-\frac{d}{a}, \frac{4a(\alpha-dl)}{4(b+al)^2+v^2}\right)$ . We notice that the second equation of (5) can be write as

$$(8) \quad (2Ax_1 + 2D)^2 - \left(\sqrt{4(B + Al)^2 + w^2}y_1 - \frac{4A(C-lD)}{\sqrt{4(B+Al)^2+w^2}}\right)^2 = 4D^2 - \frac{16A^2(C-lD)^2}{4(B+Al)^2+w^2},$$

which is a hyperbola if  $8lA^2CD - 4A^2C^2 + 8lABD^2 + 4B^2D^2 + w^2D^2 \neq 0$ , denoted by  $\mathcal{H}^*$ . Moreover, the hyperbola  $\mathcal{H}^*$  has the two asymptotes

$$L_{\mathcal{H}^*}^1 : 2Ax_1 + \sqrt{4(B + Al)^2 + w^2}y_1 - \frac{4A(C-lD)}{\sqrt{4(B+Al)^2+w^2}} + 2D = 0,$$

$$L_{\mathcal{H}^*}^2 : 2Ax_1 - \sqrt{4(B + Al)^2 + w^2}y_1 + \frac{4A(C-lD)}{\sqrt{4(B+Al)^2+w^2}} + 2D = 0,$$

which intersect at  $I_{\mathcal{H}^*} = \left(-\frac{D}{A}, \frac{4A(C-lD)}{4(B+Al)^2+w^2}\right)$ .

To study the maximum number of limit cycles of system (2) intersecting each branch of  $\Sigma$  in one point is equivalent to find the maximum number of intersection points  $P_i$ 's of the hyperbolas  $\mathcal{H}$  in (7) with  $\mathcal{H}^*$  in (8), whose coordinates satisfy (6).

Denote by  $P_i = (x_i, y_i)$  the intersection points of  $\mathcal{H}$  with  $\mathcal{H}^*$ . Under condition (6), hereafter, we write

$$(9) \quad P_0 = (0, 0) \prec P_1 \prec P_2 \prec \dots$$

We further assume without loss of generality that the hyperbola  $\mathcal{H}$  is in left–right way, i.e., its two branches face, respectively, the left and right sides, and they are denoted, respectively, by  $\mathcal{H}_L$  and  $\mathcal{H}_r$ .

*Case 1:* if  $P_i$  are located on the right branch  $\mathcal{H}_r$  of  $\mathcal{H}$ .

– In case that the hyperbola  $\mathcal{H}^*$  is of left–right type, i.e., the two branches of  $\mathcal{H}^*$  face, respectively, left and right, and are denoted by  $\mathcal{H}_L^*$  and  $\mathcal{H}_r^*$ , respectively, there are at most two intersection points  $P_i = (x_i, y_i), i = 1, 2$ , of  $\mathcal{H}$  with  $\mathcal{H}^*$ , which satisfy (9), one on  $\mathcal{H}_L^*$  and another on  $\mathcal{H}_r^*$ . Note that in this case and in order that there are exactly tow intersection points  $P_i = (x_i, y_i), i = 1, 2$  which satisfy (9), should be the following conditions holds:

- i) The straight line  $L_{\mathcal{H}^*}^2$  must have positive slope larger than that of  $L_{\mathcal{H}}^2$ .
- ii)  $I_{\mathcal{H}} \prec I_{\mathcal{H}^*}$  ( $I_{\mathcal{H}^*}$  is located on the right hand side of  $\mathcal{H}_r$  or above the  $\mathcal{H}_r$ ).
- iii)  $P_0 = (0, 0) \in \mathcal{H}_r \cap \mathcal{H}_L^*$ . See the figures 8 and 9.

– In case that the hyperbola  $\mathcal{H}^*$  is of upper–down type i.e., the two branches of  $\mathcal{H}^*$  face, respectively, upper and down, and are denoted by  $\mathcal{H}_u^*$  and  $\mathcal{H}_d^*$ , respectively. there are also at most two intersection points  $P_i = (x_i, y_i), i = 1, 2$ , of  $\mathcal{H}$  with  $\mathcal{H}^*$ , which satisfy (9). These point are located on the upper branch  $\mathcal{H}_u^*$  of  $\mathcal{H}^*$  if  $I_{\mathcal{H}^*}$  is located on the right hand side of  $\mathcal{H}_r$  or are located on the down branch  $\mathcal{H}_d^*$  of  $\mathcal{H}^*$  if  $I_{\mathcal{H}^*}$  is located above the  $\mathcal{H}_r$ . In order that there exist exactly two intersection points  $P_i = (x_i, y_i), i = 1, 2$  which satisfy (9), these points must satisfy the following conditions:

- i) The straight line  $L_{\mathcal{H}^*}^2$  must have positive slope larger than that of  $L_{\mathcal{H}}^2$ .
- ii)  $I_{\mathcal{H}} \prec I_{\mathcal{H}^*}$  and  $I_{\mathcal{H}^*}$  is located on the right hand side of  $\mathcal{H}_r$  or above the  $\mathcal{H}_r$ ,
- iii)  $P_0 = (0, 0) \in \mathcal{H}_r \cap \mathcal{H}_d^*$ . See the figures 10 and 11.

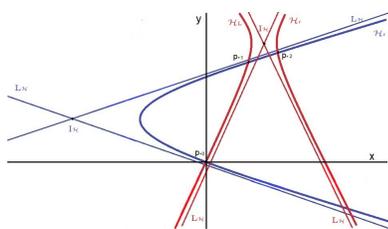


FIG. 8. The tow intersection points in the case  $P_i$  are located on the right branch  $H_r$  of  $H$  and  $H^*$  is of left-right type with  $I_{H^*}$  is located above the  $H_r$ .

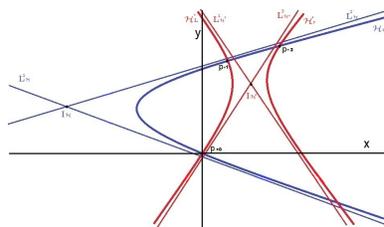


FIG. 9. The tow intersection points in the case  $P_i$  are located on the right branch  $H_r$  of  $H$  and  $H^*$  is of left-right type with  $I_{H^*}$  is located on the right hand side of  $H_r$ .

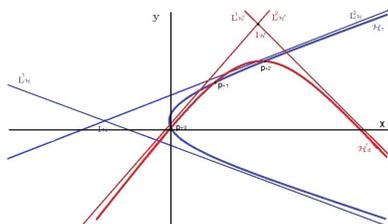


FIG. 10. The tow intersection points in the case  $P_i$  are located on the right branch  $H_r$  of  $H$  and  $H^*$  is of upper-down type with  $I_{H^*}$  is located above the  $H_r$ .

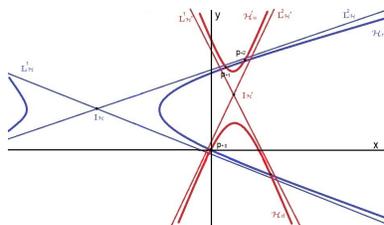


FIG. 11. The tow intersection points in the case  $P_i$  are located on the right branch  $H_r$  of  $H$  and  $H^*$  is of upper-down type with  $I_{H^*}$  is located on the right hand side of  $H_r$ .

Case 2:  $P_i$  are located on the left branch  $\mathcal{H}_L$  of  $\mathcal{H}$ .

In case that the hyperbola  $\mathcal{H}^*$  is of left-right type there are at most two intersection points  $P_i = (x_i, y_i), i = 1, 2$ , of  $\mathcal{H}$  with  $\mathcal{H}^*$ , which satisfy (9). Moreover these two points are located on the right branch  $\mathcal{H}_r^*$  of  $\mathcal{H}^*$  and exists if the following conditions holds :

- i) the straight line  $L_{\mathcal{H}^*}^2$  must have positive slope larger than that of  $L_{\mathcal{H}}^2$ .
- ii)  $I_{\mathcal{H}^*} < I_{\mathcal{H}}$  and  $I_{\mathcal{H}^*}$  is located on the left hand side of  $\mathcal{H}_L$  or under the  $\mathcal{H}_L$ .
- iii)  $P_0 = (0, 0) \in \mathcal{H}_L \cap \mathcal{H}_L^*$ . See the figures 12 and 13.

In case that the hyperbola  $\mathcal{H}^*$  is of upper-down type, there are also at most two intersection points  $P_i = (x_i, y_i), i = 1, 2$ , of  $\mathcal{H}$  with  $\mathcal{H}^*$ , which satisfy (9); one on  $\mathcal{H}_u^*$  and another on  $\mathcal{H}_d^*$  or the two pints on  $\mathcal{H}_u^*$ . Notice that in order that there exist

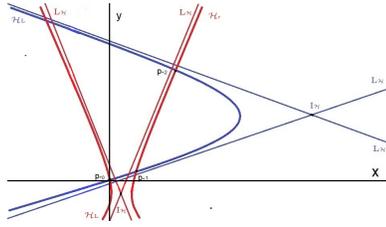


FIG. 12. The tow intersection points in the case  $P_i$  are located on the left branch  $H_L$  of  $H$  and  $H^*$  is of left-right type with  $I_{H^*}$  is located under  $H_r$ .

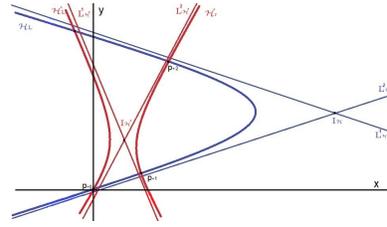


FIG. 13. The tow intersection points in the case  $P_i$  are located on the left branch  $H_L$  of  $H$  and  $H^*$  is of left-right type with  $I_{H^*}$  is located on the left hand side of  $H_r$ .

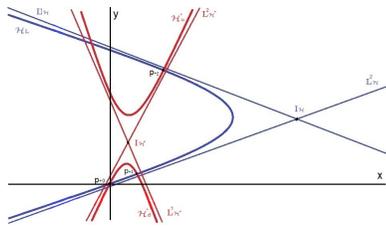


FIG. 14. The tow intersection points in the case  $P_i$  are located on the left branch  $H_L$  of  $H$  and  $H^*$  is of upper-down type with  $I_{H^*}$  is located on the left hand side of  $H_L$ .

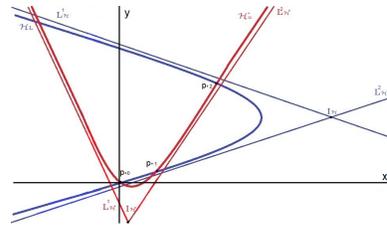


FIG. 15. The tow intersection points in the case  $P_i$  are located on the left branch  $H_L$  of  $H$  and  $H^*$  is of upper-down type with  $I_{H^*}$  is located under  $H_L$ .

two intersection points  $P_i = (x_i, y_i), i = 1, 2$  one on  $\mathcal{H}_u^*$  and another on  $\mathcal{H}_d^*$  (resp there exists two intersection points  $P_i = (x_i, y_i), i = 1, 2$  on  $\mathcal{H}_u^*$ ) which satisfy (9), should be the following conditions holds:

- i) The straight line  $L_{\mathcal{H}^*}^2$  must have positive slope larger than that of  $L_{\mathcal{H}}^2$  (resp The straight line  $L_{\mathcal{H}^*}^2$  must have positive slope larger than that of  $L_{\mathcal{H}}^2$ ).
- ii)  $I_{\mathcal{H}^*} < I_{\mathcal{H}}$  and  $I_{\mathcal{H}^*}$  is located on the left hand side of  $\mathcal{H}_L$  (resp  $I_{\mathcal{H}^*}$  is located under the  $\mathcal{H}_L$ ),
- iii)  $P_0 = (0, 0) \in \mathcal{H}_L \cap \mathcal{H}_d^*$ . See the figure 14 (resp  $P_0 = (0, 0) \in \mathcal{H}_L \cap \mathcal{H}_u^*$ . See the figure 15).

*Case 3:*  $P_i$  are located on the two branches  $\mathcal{H}_L$  and  $\mathcal{H}_r$  of  $\mathcal{H}$ .

In case that the hyperbola  $\mathcal{H}^*$  is of left-right type and  $P_i$  are also located on the two branches  $\mathcal{H}_L^*$  and  $\mathcal{H}_r^*$  of  $\mathcal{H}^*$  there are at most two intersection points

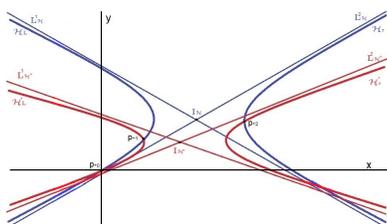


FIG. 16. The tow intersection points in the case  $P_i$  are located on the two branches  $H_L$  and  $H_r$  of  $H$  and  $H^*$  is of left-right type with  $I_H^* < I_H$

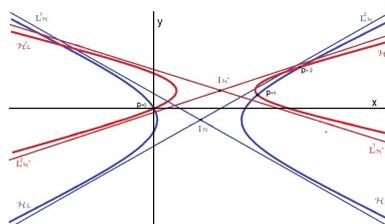


FIG. 17. The tow intersection points in the case  $P_i$  are located on the two branches  $H_L$  and  $H_r$  of  $H$  and  $H^*$  is of left-right type with  $I_H < I_H^*$

$P_i = (x_i, y_i), i = 1, 2$ , of  $\mathcal{H}$  with  $\mathcal{H}^*$ , which satisfy (9), one in  $\mathcal{H}_L^* \cap \mathcal{H}_L$  and another in  $\mathcal{H}_r^* \cap \mathcal{H}_r$  or the two pints on  $\mathcal{H}_r^* \cap \mathcal{H}_r$ . In order that there exist exactly two intersection points  $P_i = (x_i, y_i), i = 1, 2$  which satisfy (9), the following conditions must be holds:

- i) the straight line  $L_{\mathcal{H}^*}^2$  must have positive slope smaller than that of  $L_{\mathcal{H}}^2$ .
- ii)  $I_{\mathcal{H}^*}$  is located in between  $\mathcal{H}_L$  and  $\mathcal{H}_r$ .
- iii)  $P_0 = (0, 0) \in \mathcal{H}_L \cap \mathcal{H}_L^*$ . See the figures 16 and 17.

– In case that the hyperbola  $\mathcal{H}^*$  is of upper-down type and  $P_i$  are also located on the two branches  $\mathcal{H}_u^*$  and  $\mathcal{H}_d^*$  of  $\mathcal{H}^*$ ; there are at most two intersection points  $P_i = (x_i, y_i), i = 1, 2$ , of  $\mathcal{H}$  with  $\mathcal{H}^*$ , which satisfy (9), one in  $\mathcal{H}_u^* \cap \mathcal{H}_L$  and another in  $\mathcal{H}_d^* \cap \mathcal{H}_L$  or one in  $\mathcal{H}_u^* \cap \mathcal{H}_L$  and another on  $\mathcal{H}_u^* \cap \mathcal{H}_r$ . In order that there exist exactly two intersection points  $P_i = (x_i, y_i), i = 1, 2$  which satisfy (9), the following conditions must be holds:

- i) the straight line  $L_{\mathcal{H}^*}^2$  must have positive slope smaller than that of  $L_{\mathcal{H}}^2$ ,
- ii)  $I_{\mathcal{H}^*}$  is located in between  $\mathcal{H}_L$  and  $\mathcal{H}_r$ ,
- iii)  $P_0 = (0, 0) \in \mathcal{H}_L \cap \mathcal{H}_d^*$ . See the figure 18 and 19.

–The case if the hyperbola  $\mathcal{H}^*$  is of left-right type and  $P_i$  are located on the left branch  $\mathcal{H}_L^*$  of  $\mathcal{H}^*$  or  $P_i$  are located on the right branch  $\mathcal{H}_r^*$  of  $\mathcal{H}^*$  similar to the case 1 (if  $P_i$  are located on the right branch  $\mathcal{H}_r$  of  $\mathcal{H}$  and the hyperbola  $\mathcal{H}^*$  is of left-right type), enough just replace  $\mathcal{H}^*$  by  $\mathcal{H}$ . See the figure 20 and 21.

In case that the hyperbola  $\mathcal{H}^*$  is of upper-down and  $P_i$  are located on the upper branch  $\mathcal{H}_u^*$  of  $\mathcal{H}^*$  there are at most two intersection points  $P_i = (x_i, y_i), i = 1, 2$ , of  $\mathcal{H}$  with  $\mathcal{H}^*$ , which satisfy (9), these two points are located on the right branch  $\mathcal{H}_r$ . In order that there exist two intersection points  $P_i = (x_i, y_i), i = 1, 2$  which satisfy (9), these points must satisfy following conditions:

- i) the straight line  $L_{\mathcal{H}^*}^2$  must have positive slope larger than that of  $L_{\mathcal{H}}^2$ ,
- ii)  $I_{\mathcal{H}^*} < I_{\mathcal{H}}$  and  $I_{\mathcal{H}^*}$  is located in between  $\mathcal{H}_L$  and  $\mathcal{H}_r$ ,
- iii)  $P_0 = (0, 0) \in \mathcal{H}_L \cap \mathcal{H}_u^*$ .

In case that the hyperbola  $\mathcal{H}^*$  is of upper-down and  $P_i$  are located on the down branch  $\mathcal{H}_d^*$  of  $\mathcal{H}^*$  there are at most two intersection points  $P_i = (x_i, y_i), i = 1, 2$ , of  $\mathcal{H}$  with  $\mathcal{H}^*$ , which satisfy (9), one on  $\mathcal{H}_L$  and another on  $\mathcal{H}_r$ . In order that there

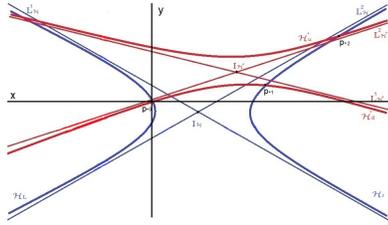


FIG. 18. The tow intersection points in the case  $H^*$  is of upper-down type and  $P_i$  are located on the two branches  $H_L$  and  $H_r$  of  $H$  and are also located on the two branches  $H_u^*$  and  $H_d^*$  with  $I_H \prec I_H^*$

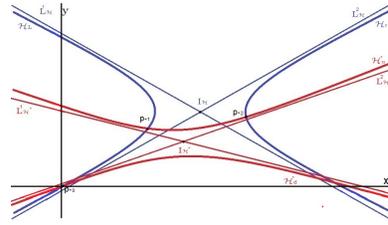


FIG. 19. The tow intersection points in the case  $H^*$  is of upper-down type and  $P_i$  are located on the two branches  $H_L$  and  $H_r$  of  $H$  and are also located on the two branches  $H_u^*$  and  $H_d^*$  with  $I_H \succ I_H^*$

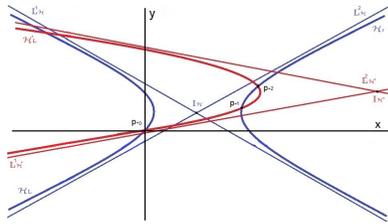


FIG. 20. The tow intersection points in the case  $P_i$  are located on the two branches  $H_L$  and  $H_r$  of  $H$  and  $H^*$  is of left-right type with  $I_{H^*}$  is located on the right hand side of  $H_L$

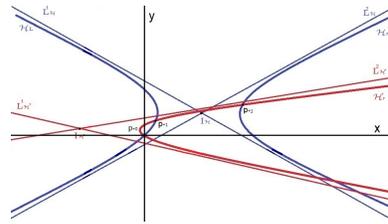


FIG. 21. The tow intersection points in the case  $P_i$  are located on the two branches  $H_L$  and  $H_r$  of  $H$  and  $H^*$  is of left-right type with  $I_{H^*}$  is located on the left hand side of  $H_L$

exist two intersection points  $P_i = (x_i, y_i), i = 1, 2$  which satisfy (9), the following conditions must be holds:

- i) the straight line  $L_{\mathcal{H}^*}^2$  must have positive slope larger than that of  $L_{\mathcal{H}}^2$ ,
- ii)  $I_{\mathcal{H}} \prec I_{\mathcal{H}^*}$  and  $I_{\mathcal{H}^*}$  is located in between  $\mathcal{H}_L$  and  $\mathcal{H}_R$ ,
- iii)  $P_0 = (0, 0) \in \mathcal{H}_L \cap \mathcal{H}_d^*$ . See the figure 22 and 23.

Consequently, system (5) cannot have 3 solutions  $(x_i, y_i), i = 1, 2, 3$  satisfying the orders of (6). Hence, the discontinuous piecewise linear differential system (2) cannot have 3 limit cycles. This completes the proof of statement (a) of Theorem 1.

**2.2. Proof of statement (b) of Theorem 1.** Suppose that we have an arbitrary discontinuous piecewise linear differential system separated by the set  $\Sigma$  formed by the two linear centers given in (2). These linear centers in  $S_1$ , and  $S_2$  have the first

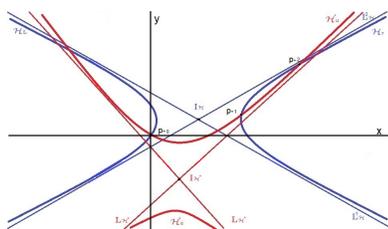


FIG. 22. The tow intersection points in the case  $H^*$  is of upper-down type and  $P_i$  are located on the two branches  $H_L$  and  $H_r$  of  $H$  and on the upper branch  $H_u^*$  of  $H^*$ .

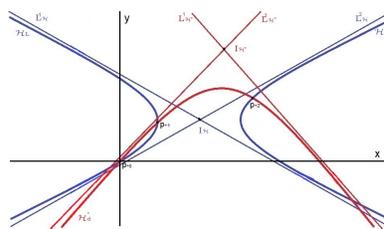


FIG. 23. The tow intersection points in the case  $H^*$  is of upper-down type and  $P_i$  are located on the two branches  $H_L$  and  $H_r$  of  $H$  and on the down branch  $H_d^*$  of  $H^*$ .

integrals  $H_1$ , and  $H_3$  described in (3). Assume that this discontinuous piecewise differential system has some limit cycles intersecting in four points of the set  $\Sigma$ . We can assume that these four points are  $(x_{1i}, 0), (x_{2i}, 0), (ly_{1i}, y_{1i})$ , and  $(ly_{2i}, y_{2i})$  with  $0 < x_{1i} < x_{2i}$  and  $0 < y_{1i} < y_{2i}$ . In order that these four points correspond to the intersection points of a limit cycle with the set  $\Sigma$ , they must satisfy:

$$\begin{aligned}
 H_1(x_{11}, 0) - H_1(ly_{11}, y_{11}) &= 0, \\
 H_1(x_{21}, 0) - H_1(ly_{21}, y_{21}) &= 0, \\
 H_2(ly_{21}, y_{21}) - H_2(ly_{11}, y_{11}) &= 0, \\
 H_2(x_{11}, 0) - H_2(x_{21}, 0) &= 0,
 \end{aligned}
 \tag{10}$$

or equivalently

$$\begin{aligned}
 4a^2x_{11}^2 + 8adx_{11} - \left(4(b+al)^2 + v^2\right)y_{11}^2 + 8a(\alpha - dl)y_{11} &= 0, \\
 4a^2x_{21}^2 + 8adx_{21} - \left(4(b+al)^2 + v^2\right)y_{21}^2 + 8a(\alpha - dl)y_{21} &= 0, \\
 (y_{11} - y_{21})(8AC - 8AID - (4A^2l^2 + 4B^2 + w^2 + 8ABl)(y_{11} + y_{21})) &= 0, \\
 4A(x_{11} - x_{21})(2D + Ax_{11} + Ax_{21}) &= 0.
 \end{aligned}
 \tag{11}$$

Since  $x_{11} < x_{21}$  and  $y_{11} < y_{21}$  the previous system is equivalent to the system

$$\begin{aligned}
 4a^2x_{11}^2 + 8adx_{11} - \left(4(b+al)^2 + v^2\right)y_{11}^2 + 8a(\alpha - dl)y_{11} &= 0, \\
 4a^2x_{21}^2 + 8adx_{21} - \left(4(b+al)^2 + v^2\right)y_{21}^2 + 8a(\alpha - dl)y_{21} &= 0, \\
 (8AC - 8AID - (4A^2l^2 + 4B^2 + w^2 + 8ABl)(y_{11} + y_{21})) &= 0, \\
 (2D + Ax_{11} + Ax_{21}) &= 0,
 \end{aligned}
 \tag{12}$$

from the third and fourth equations of (12) we isolated  $y_{21}$  and  $x_{21}$  we obtain

$$x_{21} = -\frac{1}{A}(2D + Ax_{11}), \quad y_{21} = \frac{8A(C-ID)}{4(B+Al)^2 + w^2} - y_{11}.
 \tag{13}$$

Substituting (13) into the second equation of (12), we get

$$\frac{4a^2(2D+Ax_{11})^2}{A^2} - \frac{8ad(2D+Ax_{11})}{A} - \left(4(b+al)^2 + v^2\right) \left(\frac{8AC-8AID}{4(B+Al)^2+w^2} - y_{11}\right)^2 + 8a(\alpha - dl) \left(\frac{8AC-8AID}{4(B+Al)^2+w^2} - y_{11}\right) = 0.$$

Summing up this last equation and the first equation of (12), we get

$$\frac{(a(dl-\alpha)(4(B+Al)^2+w^2)+A(4(b+al)^2+v^2)(C-lD))(4A(lD-C)+(4(B+Al)^2+w^2)y_{11})}{(4(B+Al)^2+w^2)^2} - a\frac{Ad-aD}{A^2}(D+Ax_{11}) - = 0.$$

Now, solving (12) reduces to solve

$$4a^2x_{11}^2 + 8adx_{11} - \left(4(b+al)^2 + v^2\right) y_{11}^2 + 8a(\alpha - dl) y_{11} = 0, \tag{14}$$

$$\frac{(a(dl-\alpha)(4(B+Al)^2+w^2)+A(4(b+al)^2+v^2)(C-lD))(4A(lD-C)+(4(B+Al)^2+w^2)y_{11})}{(4(B+Al)^2+w^2)^2} - a\frac{(Ad-aD)(D+Ax_{11})}{A^2} = 0.$$

If  $a(Ad - aD) = a(dl - \alpha) \left(4(B + Al)^2 + w^2\right) + A \left(4(b + al)^2 + v^2\right) (C - lD) = 0$ , then the second equation of (14) is trivial, and the first equation of (14) has a continuum family of solutions, and so (12) has a continuum solution. Consequently, system (2) has no limit cycles.

If  $a(Ad - aD) \neq 0$  and

$$a(dl - \alpha) \left(4(B + Al)^2 + w^2\right) + A \left(4(b + al)^2 + v^2\right) (C - lD) \neq 0,$$

we notice that the first equation of (14) can be write as (7) which is the hyperbola  $\mathcal{H}$  if  $8la^2d\alpha - 4a^2\alpha^2 + 8labd^2 + 4b^2d^2 + d^2v^2 \neq 0$ , and the second equation of (14) is a straight line, denoted by  $\mathcal{L}$ . Clearly, we can chose the values of the parameters of system (2) such that the straight line  $\mathcal{L}$  can intersect the hyperbola  $\mathcal{H}$  in either zero point or one point or two points whose coordinates have only positive entries. If  $\mathcal{L}$  does not intersect  $\mathcal{H}$ , then (14) has no solution, and system (2) has no limit cycle.

If  $\mathcal{L}$  intersects  $\mathcal{H}$  in a unique point or in a point multiple two, again, system (2) has no limit cycle.

If the intersection points are two, we denote them by  $(x_{11}, y_{11})$  and  $(x_{21}, y_{21})$  with  $0 < x_{11} < x_{21}$  and  $0 < y_{11} < y_{21}$ . This implies that  $(x_{11}, 0)$  and  $(x_{21}, 0)$  ( $(ly_{11}, y_{11})$  and  $(ly_{21}, y_{21})$ ) are the four intersection points of the limit cycle (if exist) with the branch of  $\Sigma$ . This completes the proof of statement (b) of Theorem 1.

### 2.3. Proof of Proposition 2.

2.3.1. *Proof of statement (i) of Proposition 2.* The proof of statement (i) of Proposition 2 is provide by the following examples

**Example 1 (Case  $l > 0$ ).** Consider the discontinuous piecewise linear differential system with two pieces separated by the set  $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = \frac{1}{2}y \text{ and } y \geq 0\}$ , defined by

$$\tag{15} \begin{aligned} \dot{x} &= (4 - 4\sqrt{11})x + (4\sqrt{11} - 26)y + 8, & \dot{y} &= 8x + (4\sqrt{11} - 4)y + 16 & \text{in } S_1, \\ \dot{x} &= -(8x + 10y - 8), & \dot{y} &= 8x + 8y + 8 & \text{in } S_2, \end{aligned}$$

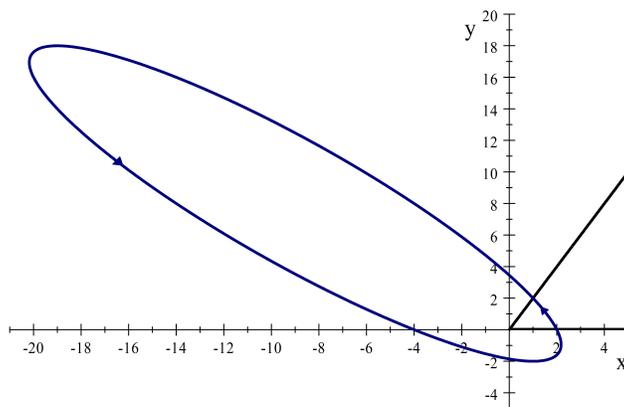


FIG. 24. The limit cycle of the discontinuous piecewise linear differential system (15).

It is easy to see that this system has the first integrals

$$\begin{aligned}
 H_1(x, y) &= 4 \left( x + \frac{\sqrt{11}-1}{2} y \right)^2 + 8(2x - y) + y^2, \\
 H_2(x, y) &= 4(x + y)^2 + 8(x - y) + y^2,
 \end{aligned}$$

in  $S_1$ , and  $S_2$ , respectively. The eigenvalues of the matrices of the two linear differential systems (15) are  $\pm 4i$ . So the two systems have a linear center. We shall use the notation and the expressions of the proof of statement (ii) of Theorem 1. System (4) with  $l = \frac{1}{2}$  can be written as the

$$\begin{aligned}
 8x_1 + 4y_1 + 4x_1^2 - 10y_1^2 &= 0, \\
 16x_1 + 4x_1^2 - 12y_1^2 &= 0.
 \end{aligned}$$

Taking into account that  $0 < x_1$  and  $0 < y_1$  the unique solution  $(x_1, y_1)$  of the previous system is  $(2, 2)$ . Straightforward computations show that : the implicit form of the solution of the first linear differential system of (15) passing through the crossing points  $(1, 2)$  and  $(2, 0)$  is  $H_1(x, y) = 48$  and the implicit form of the solution of the second linear differential system of (15) passing through the crossing points  $(1, 2)$  and  $(2, 0)$  is  $H_2(x, y) = 32$ . Moreover, the orbit arc in  $S_1$  starting from  $(2, 0)$  satisfies  $\dot{x}|_{(2,0)} < 0$  and  $\dot{y}|_{(2,0)} > 0$ , so it runs in counterclockwise. The orbit arc in  $S_2$  starting from  $(1, 2)$  satisfies  $\dot{x}|_{(1,2)} < 0$  and  $\dot{y}|_{(1,2)} > 0$ , and so it runs in counterclockwise. Drawing the orbit

$$\Gamma = \{(x, y) \in S_1 : H_1(x, y) = 48\} \cup \{(x, y) \in S_2 : H_2(x, y) = 32\},$$

we obtain the limit cycle of Figure 24, which is traveled in counterclockwise sense.

**Example 2 (Case  $l = 0$ ).** Consider the discontinuous piecewise linear differential system with two pieces separated by the set  $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ and } y \geq 0\}$ , defined by

$$(16) \quad \begin{aligned}
 \dot{x} &= -(8x + 10y - \frac{25}{3}), & \dot{y} &= 8x + 8y + 16 & \text{in } S_1, \\
 \dot{x} &= -(2y + 1), & \dot{y} &= 8x + 2 & \text{in } S_2,
 \end{aligned}$$

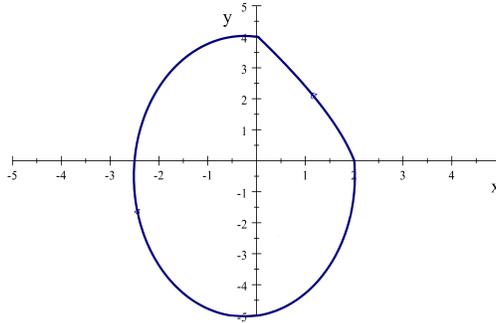


FIG. 25. The limit cycle of the discontinuous piecewise linear differential system (16).

It is easy to see that its corresponding Hamiltonian functions are

$$(17) \quad H_1(x, y) = 4(x + y)^2 + 8\left(2x - \frac{25}{24}y\right) + y^2,$$

$$(18) \quad H_2(x, y) = 4x^2 - (-2x - y) + y^2,$$

in  $S_1$ , and  $S_2$ , respectively. The eigenvalues of the matrices of the two linear differential systems (16) are  $\pm 4i$ . So the two systems have a linear center. System (4) with  $l = 0$  can be written as the

$$\begin{aligned} 4x_1^2 + 16x_1 - 5y_1^2 + \frac{25}{3}y_1 &= 0, \\ 4x_1^2 + 2x_1 - y_1^2 - y_1 &= 0. \end{aligned}$$

The unique solution  $(x_1, y_1)$  of this last system satisfying the necessary conditions  $x_1 > 0$  and  $y_1 > 0$  is  $(x_1, y_1) = \left(\frac{49}{24}, \frac{49}{12}\right)$ . Straightforward computations show that the solution passing through the crossing points  $(0, y_1)$  and  $(x_1, 0)$  correspond to

$$\Gamma = \{(x, y) \in S_1 : H_1(x, y) = 48\} \cup \{(x, y) \in S_2 : H_2(x, y) = 20\}.$$

On the other hand, the orbit arc in  $S_1$  starting from  $\left(\frac{49}{24}, 0\right)$  satisfies  $\dot{x}|_{\left(\frac{49}{24}, 0\right)} < 0$  and  $\dot{y}|_{\left(\frac{49}{24}, 0\right)} > 0$ , so it runs in counterclockwise. The orbit arc in  $S_2$  starting from  $\left(0, \frac{49}{12}\right)$  satisfies  $\dot{x}|_{\left(0, \frac{49}{12}\right)} < 0$  and  $\dot{y}|_{\left(0, \frac{49}{12}\right)} > 0$ , and so it runs in counterclockwise. Drawing the orbit  $\Gamma$  we obtain the limit cycle of figure 25.

**Example 3 (Case  $l < 0$ ).** Consider the discontinuous piecewise linear differential system with two pieces separated by the set  $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = -y \text{ and } y \geq 0\}$ , defined by

$$(19) \quad \begin{aligned} \dot{x} &= -\left(4x + \frac{17}{2}y + 1\right), & \dot{y} &= 2x + 4y + 2 & \text{in } S_1, \\ \dot{x} &= x - \frac{5}{4}y + \frac{5}{2}, & \dot{y} &= x - y + 1 & \text{in } S_2, \end{aligned}$$

It is easy to see that its corresponding Hamiltonian functions are

$$H_1(x, y) = 4(x + 2y)^2 + 8\left(x + \frac{1}{2}y\right) + y^2,$$

$$H_2(x, y) = 4(x - y)^2 + 8\left(x - \frac{5}{2}y\right) + y^2,$$

in  $S_1$ , and  $S_2$ , respectively. The eigenvalues of the matrices of the two linear differential systems (19) are  $\pm i$  and  $\pm \frac{1}{2}i$ , respectively. So the two systems have a linear

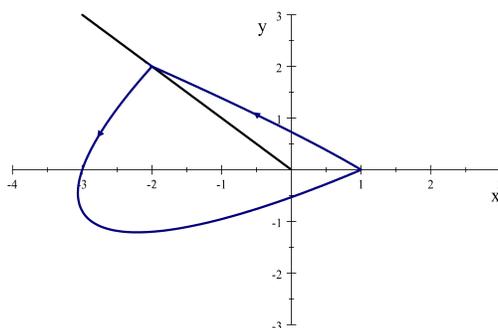


FIG. 26. The limit cycle of the discontinuous piecewise linear differential system (19).

center. System (4) with  $l = -1$  can be written as the

$$\begin{aligned} 4x_1^2 + 8x_1 - 5y_1^2 + 4y_1 &= 0, \\ 4x_1^2 + 8x_1 - 17y_1^2 + 28y_1 &= 0. \end{aligned}$$

The unique solution  $(x_1, y_1)$  of this last system satisfying the necessary conditions  $x_1 > 0$  and  $y_1 > 0$  is  $(x_1, y_1) = (1, 2)$ . Straightforward computations show that the solution passing through the crossing points  $(x_1, 0)$ , and  $(ly_1, y_1)$  correspond to

$$\Gamma = \{(x, y) \in S_1 : H_1(x, y) = 12\} \cup \{(x, y) \in S_2 : H_2(x, y) = 12\}.$$

The orbits arc in  $Q_1$  and  $Q_2$  starting from  $(1, 0)$  and  $(-2, 2)$  satisfies  $\dot{x}|_{(1,0)} < 0$ ,  $\dot{y}|_{(1,0)} > 0$  and  $\dot{x}|_{(-2,2)} < 0$  and  $\dot{y}|_{(-2,2)} < 0$ , so it runs in counterclockwise. Drawing the orbit  $\Gamma$  we obtain the limit cycle of figure 26.

2.3.2. Proof of statement (ii) of proposition 2. proof of statement (ii) of Proposition 2 is provide by the following examples.

**Example 4 (Case  $l > 0$ ).** Consider the discontinuous piecewise linear differential system with two pieces separated by the set  $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = y \text{ and } y \geq 0\}$  defined by

$$(20) \quad \begin{aligned} \dot{x} &= -16x - 34y + 16, & \dot{y} &= 8x + 16y + 64 & \text{in } S_1, \\ \dot{x} &= -8x - 10y + 8, & \dot{y} &= 8x + 8y + 8 & \text{in } S_2, \end{aligned}$$

It is obvious that The two linear differential systems of (20) have the following first integrals

$$\begin{aligned} H_1(x, y) &= 4(x + 2y)^2 + 8(8x - 2y) + y^2, \\ H_2(x, y) &= 4(x + y)^2 + 8(x - y) + y^2, \end{aligned}$$

in  $S_1$  and  $S_2$ , respectively. Since  $\pm 4i$  are the eigenvalues of the matrices of the two linear differential systems of (16), these systems have their equilibria as centers. Then, for the discontinuous piecewise linear differential system (20), system (4) with  $l = 1$  becomes:

$$\begin{aligned} 4x_i^2 + 8x_i - 17y_i^2 &= 0, \\ 4x_i^2 + 64x_i - 37y_i^2 - 48y_i &= 0. \end{aligned}$$

Taking into account that we are only interested in the solutions  $(x_i, y_i)$ ,  $i = 1, 2$  satisfying  $x_2 > x_1 > 0$ , and  $y_2 > y_1 > 0$ , the unique two solutions of the previous

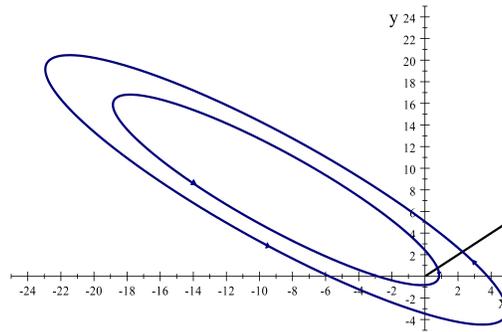


FIG. 27. The two limit cycles of the discontinuous piecewise linear differential system (20).

system are  $(x_1, y_1) = (0.85359, 0.75705)$ ,  $(x_2, y_2) = (3.7825, 2.2686)$ . Straightforward computations show that the solution passing through the crossing points  $(x_i, 0)$ , and  $(\frac{1}{2}y_i, y_i)$ ,  $i = 1, 2$  correspond to

$$\begin{aligned} \Gamma &= \Gamma_1 \cup \Gamma_2 \\ &= \{(x, y) \in S_1 : H_1(x, y) = 57.544\} \cup \{(x, y) \in S_2 : H_2(x, y) = 9.7432\}, \\ \Gamma_2 &= \Gamma_3 \cup \Gamma_4 \\ &= \{(x, y) \in S_1 : H_1(x, y) = 299.31\} \cup \{(x, y) \in S_3 : H_2(x, y) = 87.489\}. \end{aligned}$$

For the orbit arc  $\Gamma_1 = \{(x, y) \in S_1 : H_1(x, y) = 9.7432\}$  in  $S_1$  starting from  $(x_1, 0)$  we have used the facts that  $\dot{x}|_{(x_1,0)} > 0, \dot{y}|_{(x_1,0)} > 0$ , so it runs in counterclockwise. For the orbit arc  $\Gamma_2 = \{(x, y) \in S_2 : H_2(x, y) = 57.544\}$  in  $S_2$  starting from  $(y_1, y_1)$  we have used the facts that  $\dot{x}|_{(y_1,y_1)} < 0, \dot{y}|_{(y_1,y_1)} > 0$ , so it runs in counterclockwise. For the orbit arc  $\Gamma_3 = \{(x, y) \in S_1 : H_1(x, y) = 87.489\}$  in  $S_1$  starting from  $(x_2, 0)$  we have used the facts that  $\dot{x}|_{(x_2,0)} < 0, \dot{y}|_{(x_2,0)} > 0$ , so it runs in counterclockwise. For the orbit arc  $\Gamma_4 = \{(x, y) \in S_2 : H_2(x, y) = 299.31\}$  in  $S_2$  starting from  $(y_2, y_2)$  we have used the facts that  $\dot{x}|_{(y_2,y_2)} < 0, \dot{y}|_{(y_2,y_2)} > 0$ , so it runs in counterclockwise. Clearly,  $\Gamma_1 \cup \Gamma_2$ , and  $\Gamma_3 \cup \Gamma_4$  are nested, and  $\Gamma_1 \cup \Gamma_2$  is the inner one and  $\Gamma_3 \cup \Gamma_4$  is the outer one.

**Example 5 (Case  $l = 0$ ).** Consider the discontinuous piecewise linear differential system with two pieces separated by the set  $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ and } y \geq 0\}$ , defined by

$$(21) \quad \begin{aligned} \dot{x} &= -\left(x + \frac{5}{4}y + \frac{31}{16}\right), & \dot{y} &= x + y + \frac{5}{8} & \text{in } S_1, \\ \dot{x} &= -\left(y + \frac{11}{4}\right), & \dot{y} &= x + 1 & \text{in } S_2, \end{aligned}$$

It is easy to see that its corresponding Hamiltonian functions are

$$\begin{aligned} H_1(x, y) &= 4(x + y)^2 + 8\left(\frac{5}{8}x + \frac{31}{16}y\right) + y^2, \\ H_2(x, y) &= \left(y + \frac{11}{4}\right)^2 + (x + 1)^2, \end{aligned}$$

in  $S_1$ , and  $S_2$ , respectively. The eigenvalues of the matrices of the two linear differential systems (21) are  $\pm \frac{1}{2}i$  and  $\pm i$ , respectively. So the two systems have a linear center. System (4) with  $l = 0$  can be written as the

$$\begin{aligned} 4x_i^2 + 5x_i - 5y_i^2 - \frac{31}{2}y_i &= 0, \\ x_i^2 + 2x_i - y_i^2 - \frac{11}{2}y_i &= 0. \end{aligned}$$

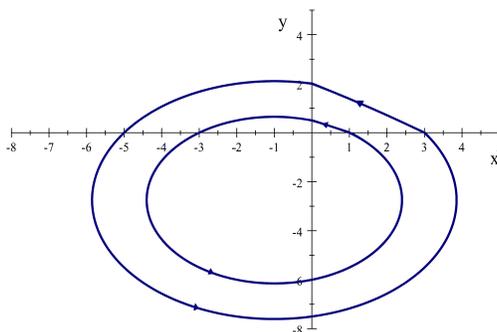


FIG. 28. The two limit cycles of the discontinuous piecewise linear differential system (21).

Taking into account that we are only interested in the solutions  $(x_i, y_i)$  satisfying  $x_i > 0$  and  $y_i > 0$ , the unique two solutions of the previous system are  $(x_1, y_1) = (1, \frac{1}{2})$  and  $(x_2, y_2) = (3, 2)$ . The orbit passing through the crossing points  $(x_1, 0)$ , and  $(0, y_1)$  correspond to

$$\Gamma_1 = \{(x, y) \in S_1 : H_1(x, y) = 9\} \cup \{(x, y) \in S_2 : H_2(x, y) = \frac{185}{16}\},$$

and the solution passing through the crossing points  $(x_2, 0)$ , and  $(0, y_2)$  correspond to

$$\Gamma_2 = \{(x, y) \in S_1 : H_1(x, y) = 51\} \cup \{(x, y) \in S_2 : H_2(x, y) = \frac{377}{16}\}.$$

Clearly,  $\Gamma_1$ , and  $\Gamma_2$  are nested, and  $\Gamma_1$  is the inner one and  $\Gamma_2$  is the outer one. Notice that the orbit arc in  $S_1$  starting from  $(1, 0)$  (resp  $(3, 0)$ ) satisfies  $\dot{x}|_{(1,0)} < 0$  and  $\dot{y}|_{(1,0)} > 0$  (resp  $\dot{x}|_{(3,0)} < 0$  and  $\dot{y}|_{(3,0)} > 0$ ), so it runs in counterclockwise. The orbit arc in  $S_2$  starting from  $(0, \frac{1}{2})$  (resp  $(0, 2)$ ) satisfies  $\dot{x}|_{(0, \frac{1}{2})} < 0$  and  $\dot{y}|_{(0, \frac{1}{2})} > 0$  (resp  $\dot{x}|_{(0,2)} < 0$  and  $\dot{y}|_{(0,2)} > 0$ ), and so it runs in counterclockwise. Drawing the two orbits  $\Gamma_1$ , and  $\Gamma_2$  we obtain the two limit cycles of figure 28 which are traveled in counterclockwise sense.

**Example 6 (Case  $l < 0$ ).** Consider the discontinuous piecewise linear differential system with two pieces separated by the set  $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = -\frac{1}{3}y \text{ and } y \geq 0\}$ , defined by

$$(22) \quad \begin{aligned} \dot{x} &= -(x + \frac{5}{4}y - \frac{19}{24}), & \dot{y} &= x + y - \frac{5}{12} & \text{in } S_1, \\ \dot{x} &= -(6y - 1), & \dot{y} &= 6x + 5 & \text{in } S_2, \end{aligned}$$

the first integrals of the two linear differential systems of (22) are

$$\begin{aligned} H_1(x, y) &= 4(x + y)^2 - 8(\frac{5}{12}x + \frac{19}{24}y) + y^2, \\ H_2(x, y) &= (y - \frac{1}{6})^2 + (x + \frac{5}{6})^2, \end{aligned}$$

in  $S_1$ , and  $S_2$ , respectively. The eigenvalues of the matrices of the two linear differential systems (22) are  $\pm \frac{1}{2}i$  and  $\pm 6i$ , respectively. So the two systems have a linear center. System (4) with  $l = -\frac{1}{3}$  can be written as the

$$\begin{aligned} 4x_i^2 - \frac{10}{3}x_i - \frac{25}{9}y_i^2 + \frac{47}{9}y_i &= 0, \\ x_i^2 + \frac{5}{3}x_i - \frac{10}{9}y_i^2 + \frac{8}{9}y_i &= 0. \end{aligned}$$

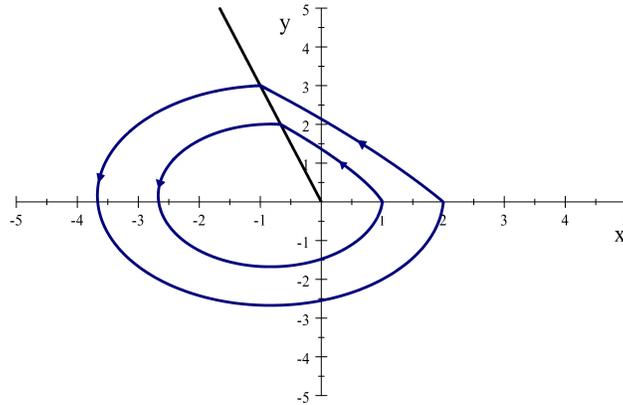


FIG. 29. The two limit cycles of the discontinuous piecewise linear differential system (22).

Taking into account that we are only interested in the solutions  $(x_i, y_i)$  satisfying  $x_i > 0$  and  $y_i > 0$ , the unique two solutions of the previous system are  $(x_1, y_1) = (1, 2)$  and  $(x_2, y_2) = (2, 3)$ . The orbit passing through the crossing points  $(x_1, 0)$ , and  $(0, y_1)$  correspond to

$$\Gamma_1 = \{(x, y) \in S_1 : H_1(x, y) = \frac{2}{3}\} \cup \{(x, y) \in S_2 : H_2(x, y) = \frac{61}{18}\},$$

and the solution passing through the crossing points  $(x_2, 0)$ , and  $(0, y_2)$  correspond to

$$\Gamma_2 = \{(x, y) \in S_1 : H_1(x, y) = \frac{28}{3}\} \cup \{(x, y) \in S_2 : H_2(x, y) = \frac{145}{18}\}.$$

Clearly,  $\Gamma_1$ , and  $\Gamma_2$  are nested, and  $\Gamma_1$  is the inner one and  $\Gamma_2$  is the outer one. Notice that the orbit arc in  $S_1$  starting from  $(1, 0)$  (resp  $(2, 0)$ ) satisfies  $\dot{x}|_{(1,0)} < 0$  and  $\dot{y}|_{(1,0)} > 0$  (resp  $\dot{x}|_{(2,0)} < 0$  and  $\dot{y}|_{(2,0)} > 0$ ), so it runs in counterclockwise. The orbit arc in  $S_2$  starting from  $(-\frac{2}{3}, 2)$  (resp  $(-1, 3)$ ) satisfies  $\dot{x}|_{(-\frac{2}{3}, 2)} < 0$  and  $\dot{y}|_{(-\frac{2}{3}, 2)} > 0$  (resp  $\dot{x}|_{(-1, 3)} < 0$  and  $\dot{y}|_{(-1, 3)} < 0$ ), and so it runs in counterclockwise. Drawing the two orbits  $\Gamma_1$ , and  $\Gamma_2$  we obtain the two limit cycles of figure 29, which are traveled in counterclockwise sense.

The next proposition shows that there are discontinuous piecewise linear differential systems separated by the set  $\Sigma$  with one limit cycle intersecting the set  $\Sigma$  in exactly four points.

**Example 7 (Case  $l < 0$ ).** Consider the discontinuous piecewise linear differential system with two pieces separated by the set  $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = -\frac{1}{4}y \text{ and } y \geq 0\}$  defined by

$$(23) \quad \begin{aligned} \dot{x} &= 8x + 10y - 8, & \dot{y} &= -8x - 8y + 8 & \text{in } S_1, \\ \dot{x} &= 8x + 10y - 32, & \dot{y} &= -8x - 8y + 28 & \text{in } S_2, \end{aligned}$$

the first integrals of the two linear differential systems of (23) are

$$\begin{aligned} H_1(x, y) &= 4(x + y)^2 + 8(-x - y) + y^2, \\ H_2(x, y) &= 4(x + y)^2 + 8\left(-\frac{7}{2}x - 4y\right) + y^2, \end{aligned}$$

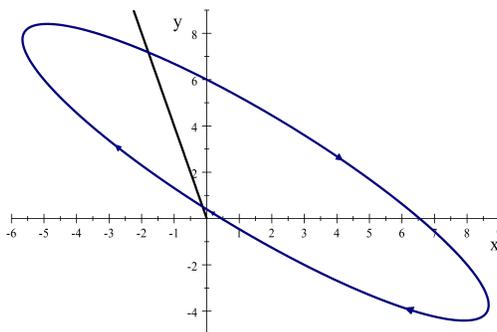


FIG. 30. The limit cycle of the discontinuous piecewise linear differential system (23).

in  $S_1$  and  $S_2$ , respectively. Since the eigenvalues of the matrices of the two linear differential systems of (23) are  $\pm 4i$ , these systems have their equilibria as centers. Then, for the discontinuous piecewise linear differential system (23), system (10), with  $l = -\frac{1}{4}$  becomes

$$\begin{aligned} \frac{1}{4}(-32x_{11} + 24y_{11} + 16x_{11}^2 - 13y_{11}^2) &= 0, \\ \frac{1}{4}(-32x_{21} + 24y_{21} + 16x_{21}^2 - 13y_{21}^2) &= 0, \\ \frac{1}{4}(y_{11} - y_{21})(13y_{11} + 13y_{21} - 100) &= 0, \\ 4(x_{11} - x_{21})(x_{11} + x_{21} - 7) &= 0. \end{aligned}$$

Taking into account that we are only interested in the solutions  $(x_{11}, x_{21}, y_{11}, y_{21})$  satisfying  $0 < x_{11} < x_{21}$  and  $0 < y_{11} < y_{21}$ , the unique solution of the previous system is  $(x_{11}, x_{21}, y_{11}, y_{21}) = (\frac{1}{3}, \frac{20}{3}, \frac{20}{39}, \frac{280}{39})$ . Straightforward computations show that the solution passing through the crossing points  $(\frac{1}{3}, 0)$ ,  $(-\frac{5}{39}, \frac{20}{39})$ ,  $(\frac{20}{3}, 0)$  and  $(-\frac{70}{39}, \frac{280}{39})$  correspond to

$$\begin{aligned} \Gamma &= \{(x, y) \in S_1 : H_1(x, y) = -\frac{20}{9}\} \cup \{(x, y) \in S_2 : H_2(x, y) = -\frac{1400}{117}\}, \\ &\cup \{(x, y) \in S_1 : H_1(x, y) = \frac{1120}{9}\} \cup \{(x, y) \in S_2 : H_2(x, y) = -\frac{80}{9}\}. \end{aligned}$$

For the orbit arc  $\Gamma_1 = \{(x, y) \in S_1 : H_1(x, y) = -\frac{20}{9}\}$  in  $S_1$  starting from  $(\frac{1}{3}, 0)$  we have used the facts that  $\dot{x}|_{(\frac{1}{3}, 0)} < 0, \dot{y}|_{(\frac{1}{3}, 0)} > 0$ , so it runs in clockwise. For the orbit arc  $\Gamma_2 = \{(x, y) \in S_2 : H_2(x, y) = -\frac{1400}{117}\}$  in  $S_2$  starting from  $(-\frac{5}{39}, \frac{20}{39})$  we have used the facts that  $\dot{x}|_{(-\frac{5}{39}, \frac{20}{39})} < 0, \dot{y}|_{(-\frac{5}{39}, \frac{20}{39})} > 0$ , so it runs in clockwise. For the orbit arc  $\Gamma_3 = \{(x, y) \in S_1 : H_1(x, y) = \frac{1120}{9}\}$  in  $S_1$  starting from  $(-\frac{70}{39}, \frac{280}{39})$  we have used the facts that  $\dot{x}|_{(-\frac{70}{39}, \frac{280}{39})} > 0, \dot{y}|_{(-\frac{70}{39}, \frac{280}{39})} < 0$ , so it runs in clockwise. For the orbit arc  $\Gamma_4 = \{(x, y) \in S_2 : H_2(x, y) = -\frac{80}{9}\}$  in  $S_2$  starting from  $(\frac{20}{3}, 0)$  we have used the facts that  $\dot{x}|_{(\frac{20}{3}, 0)} > 0, \dot{y}|_{(\frac{20}{3}, 0)} < 0$ , so it runs in clockwise.

**Example 8 (Case  $l = 0$ ).** Consider the discontinuous piecewise linear differential system with two pieces separated by the set  $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ and } y \geq 0\}$  defined by

$$(24) \quad \begin{aligned} \dot{x} &= -(x + \frac{5}{4}y - 1), & \dot{y} &= x + y - 1 & \text{in } S_1, \\ \dot{x} &= -8x - 10y + 2\sqrt{19} + 7, & \dot{y} &= 8x + 8y - 14 & \text{in } S_2, \end{aligned}$$

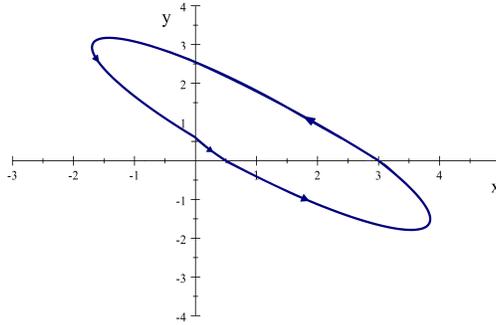


FIG. 31. The limit cycle of the discontinuous piecewise linear differential system (24).

the first integrals of the two linear differential systems of (24) are

$$\begin{aligned} H_1(x, y) &= 4(x+y)^2 - 8(x+y) + y^2, \\ H_2(x, y) &= 4(x+y)^2 - 2\left(7x + \left(\sqrt{19} + \frac{7}{2}\right)y\right) + y^2, \end{aligned}$$

in  $S_1$  and  $S_2$ , respectively. Since the eigenvalues of the matrices of the two linear differential systems of (24) are  $\pm\frac{1}{2}i$  and  $\pm 4i$ , respectively. Then these systems have their equilibria as centers. Then, for the discontinuous piecewise linear differential system (24); the system (10), with  $l = 0$  becomes

$$\begin{aligned} 4x_{11}^2 - 8x_{11} - \left(4(1)^2 + 1\right)y_{11}^2 + 8(1)y_{11} &= 0, \\ 4x_{21}^2 - 8x_{21} - \left(4(1)^2 + 1\right)y_{21}^2 + 8(1)y_{21} &= 0, \\ 2\left(\sqrt{19} + \frac{7}{2}\right) - (4+1)(y_{11} + y_{21}) &= 0, \\ -\frac{7}{2} + x_{11} + x_{21} &= 0. \end{aligned}$$

Taking into account that we are only interested in the solutions  $(x_{11}, x_{21}, y_{11}, y_{21})$  satisfying  $0 < x_{11} < x_{21}$  and  $0 < y_{11} < y_{21}$ , the unique solution of the previous system is  $(x_{11}, x_{21}, y_{11}, y_{21}) = \left(\frac{1}{2}, 3, \frac{3}{5}, \frac{2\sqrt{19}+4}{5}\right)$ . Straightforward computations show that the solution passing through the crossing points  $(0, y_{11})$ ,  $(x_{11}, 0)$ ,  $(0, y_{21})$  and  $(x_{21}, 0)$  correspond to

$$\begin{aligned} \Gamma &= \{(x, y) \in S_1 : H_1(x, y) = -3\} \cup \{(x, y) \in S_2 : H_2(x, y) = -6\}, \\ &\cup \{(x, y) \in S_1 : H_1(x, y) = 12\} \cup \left\{ (x, y) \in S_2 : H_2(x, y) = -\frac{6\sqrt{19}-12}{5} \right\}. \end{aligned}$$

On the other hand, the orbit arc in  $S_1$  starting from  $(3, 0)$  satisfies  $\dot{x}|_{(3,0)} < 0$  and  $\dot{y}|_{(3,0)} > 0$ , so it runs in counterclockwise. The orbit arc in  $S_2$  starting from  $\left(0, \frac{2\sqrt{19}+4}{5}\right)$  satisfies  $\dot{x}|_{(0,y_{21})} < 0$  and  $\dot{y}|_{(0,y_{21})} > 0$ , and so it runs in counterclockwise. The orbit arc in  $S_1$  starting from  $\left(0, \frac{3}{5}\right)$  satisfies  $\dot{x}|_{(0,y_{11})} > 0$  and  $\dot{y}|_{(0,y_{11})} < 0$ , so it runs in counterclockwise, and the orbit arc in  $S_2$  starting from  $\left(\frac{1}{2}, 0\right)$  satisfies  $\dot{x}|_{(\frac{1}{2},0)} > 0$  and  $\dot{y}|_{(\frac{1}{2},0)} < 0$ , so it runs in counterclockwise.

**Example 9 (Case  $l > 0$ ).** Consider the discontinuous piecewise linear differential system with two pieces separated by the set  $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\} \cup$

$\{(x, y) \in \mathbb{R}^2 : x = 2y \text{ and } y \geq 0\}$ , defined by

$$(25) \quad \begin{aligned} \dot{x} &= -\left(x + \frac{5}{4}y - 1\right), & \dot{y} &= x + y - 1 & \text{in } S_1, \\ \dot{x} &= -\left(8x + 10y + \sqrt{33} - 2\sqrt{591} + 40\right), & \dot{y} &= 8x + 8y - 22 & \text{in } S_2, \end{aligned}$$

the first integrals of the two linear differential systems of (25) are

$$\begin{aligned} H_1(x, y) &= 4(x + y)^2 - 8(x + y) + y^2, \\ H_2(x, y) &= 4(x + y)^2 - 8\left(\frac{11}{4}x + \left(\frac{\sqrt{591}}{4} - \frac{\sqrt{33}}{8} - \frac{5}{2}\right)y\right) + y^2, \end{aligned}$$

in  $S_1$  and  $S_2$ , respectively. Since the eigenvalues of the matrices of the two linear differential systems of (25) are  $\pm \frac{1}{2}i$  and  $\pm 4i$ , these systems have their equilibria as centers. Then, for the discontinuous piecewise linear differential system (25), system (10), with  $l = 2$  becomes

$$\begin{aligned} 4x_{11}^2 - 8x_{11} - 37y_{11}^2 + 24y_{11} &= 0, \\ 4x_{21}^2 - 8x_{21} - 37y_{21}^2 + 24y_{21} &= 0, \\ -37y_{21} - 37y_{11} - \sqrt{33} + 2\sqrt{591} + 4 &= 0, \\ x_{11} + x_{21} - \frac{11}{2} &= 0, \end{aligned}$$

taking into account that we are only interested in the solutions  $(x_{11}, x_{21}, y_{11}, y_{21})$  satisfying  $0 < x_{11} < x_{21}$  and  $0 < y_{11} < y_{21}$ , the unique solution of the previous system is  $(x_{11}, x_{21}, y_{11}, y_{21}) = \left(\frac{1}{2}, 5, \frac{12 - \sqrt{33}}{37}, \frac{2\sqrt{591} + 12}{37}\right)$ . Straightforward computations show that the solution passing through the crossing points  $(2y_{11}, y_{11})$ ,  $(x_{11}, 0)$ ,  $(2y_{21}, y_{21})$  and  $(x_{21}, 0)$  correspond to

$$\begin{aligned} \Gamma &= \{(x, y) \in S_1 : H_1(x, y) = -3\} \cup \{(x, y) \in S_2 : H_2(x, y) = -10\} \\ &\quad \cup \{(x, y) \in S_1 : H_1(x, y) = 60\} \\ &\quad \cup \{(x, y) \in S_2 : H_2(x, y) = \frac{2}{37}(\sqrt{33} + 8)(\sqrt{591} + 6)\}. \end{aligned}$$

On the other hand, the orbit arc in  $S_1$  starting from  $(5, 0)$  satisfies  $\dot{x}|_{(5,0)} < 0$  and  $\dot{y}|_{(5,0)} > 0$ , so it runs in counterclockwise. The orbit arc in  $S_2$  starting from  $\left(\frac{4\sqrt{591} + 24}{37}, \frac{2\sqrt{591} + 12}{37}\right)$  satisfies  $\dot{x}|_{(2y_{21}, y_{21})} < 0$  and  $\dot{y}|_{(2y_{21}, y_{21})} > 0$ , and so it runs in counterclockwise. The orbit arc in  $S_1$  starting from  $\left(\frac{24 - 2\sqrt{33}}{37}, \frac{12 - \sqrt{33}}{37}\right)$  satisfies  $\dot{x}|_{(2y_{11}, y_{11})} > 0$  and  $\dot{y}|_{(2y_{11}, y_{11})} < 0$ , so it runs in counterclockwise, and the orbit arc in  $S_2$  starting from  $(\frac{1}{2}, 0)$  satisfies  $\dot{x}|_{(\frac{1}{2}, 0)} < 0$  and  $\dot{y}|_{(\frac{1}{2}, 0)} < 0$ , so it runs in counterclockwise.

### 3. PROOF OF THEOREM 2

Suppose that we have an arbitrary discontinuous piecewise linear differential system separated by the set  $\Sigma$  formed by the two linear centers given in (2). These linear centers in  $S_1$ , and  $S_2$  have the first integrals  $H_1$ , and  $H_2$  described in (3). Here we study the maximum number of crossing limit cycles of planar discontinuous piecewise linear differential system (2) that intersect  $\Sigma$  in two and in four points simultaneously.

For the proof of Theorem 2 we apply the notations given in the proof of Theorem 1. We assume that systems (4) and (10) have two real solutions where each real solution provides one crossing limit cycle with four points on  $\Sigma$  (type 3) and one crossing limit cycle intersecting each branch of  $\Sigma$  in one point (type 2), like in Theorem 1 we proved that discontinuous piecewise linear differential system (2) has at most 1 crossing limit cycle with four points on  $\Sigma$ . We prove that if there

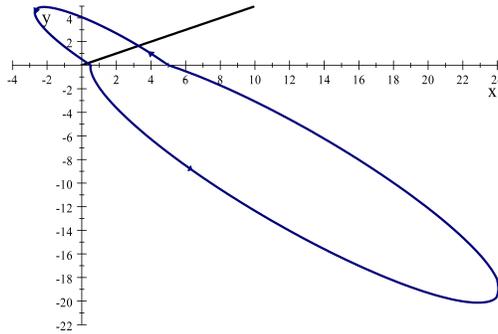


FIG. 32. The limit cycle of the discontinuous piecewise linear differential system (25).

exist one limit cycle intersecting the set  $\Sigma$  in exactly four points  $(x_{11}, 0)$ ,  $(x_{21}, 0)$ ,  $(ly_{11}, y_{11})$  and  $(ly_{21}, y_{21})$  with  $0 < x_{11} < x_{21}$  and  $0 < y_{11} < y_{21}$ . System (2) can have at most one limit cycle (of the type 2) intersecting each branch of  $\Sigma$  in one point, denoted by  $(x_1, 0)$  and  $(ly_1, y_1)$  with  $x_1 > 0$  and  $y_1 > 0$ . We note that the limit cycle intersecting  $\Sigma$  in two points must contain the limit cycle intersecting  $\Sigma$  in four points in its interior (see Figure 7 (N)), this restriction induces the next condition

$$(26) \quad 0 < x_{11} < x_{21} < x_i \text{ and } 0 < y_{11} < y_{21} < y_i.$$

Notice that the systems (10) and (4) can be written as

$$\begin{aligned} \mathcal{H} : & 4(ax_{11} + d)^2 - \left( \sqrt{4(b + al)^2 + v^2}y_{11} - \frac{4a(\alpha - dl)}{\sqrt{4(b + al)^2 + v^2}} \right)^2 = 4d^2 - \frac{16(a(\alpha - dl))^2}{4(b + al)^2 + v^2}, \\ \mathcal{L} : & -\frac{(a(dl - \alpha)(4(B + Al)^2 + w^2) + A(4(b + al)^2 + v^2)(C - lD))(4A(lD - C) + (4(B + Al)^2 + w^2)y_{11})}{(4(B + Al)^2 + w^2)^2} \\ & - \frac{a(Ad - aD)(D + Ax_{11})}{A^2} = 0, \\ & 8AC - 8AlD - (4A^2l^2 + 4B^2 + w^2 + 8ABl)(y_{11} + y_{21}) = 0, \\ & 2D + Ax_{11} + Ax_{21} = 0, \end{aligned}$$

and

$$\begin{aligned} \mathcal{H} : & 4(ax_1 + d)^2 - \left( \sqrt{4(b + al)^2 + v^2}y_1 - \frac{4a(\alpha - dl)}{\sqrt{4(b + al)^2 + v^2}} \right)^2 = 4d^2 - \frac{16(a(\alpha - dl))^2}{4(b + al)^2 + v^2}, \\ \mathcal{H}^* : & 4(Ax_1 + D)^2 - \left( \sqrt{4(B + Al)^2 + w^2}y_1 - \frac{4A(C - lD)}{\sqrt{4(B + Al)^2 + w^2}} \right)^2 \\ & = 4D^2 - \frac{16A^2(C - lD)^2}{4(B + Al)^2 + w^2}. \end{aligned}$$

If the intersection points of  $\mathcal{H}$  and  $\mathcal{L}$  are two, we denote them by  $P_{11} = (x_{11}, y_{11})$  and  $P_{21} = (x_{21}, y_{21})$  with  $0 < x_{11} < x_{21}$  and  $0 < y_{11} < y_{21}$ . Under condition (26), hereafter, we write  $P_0 = (0, 0) \prec P_{11} \prec P_{21}$ . This together with (13) forces

$$0 < x_{11} < -\frac{D}{A}, \quad 0 < y_{11} < \frac{4A(C - lD)}{4(B + Al)^2 + w^2}$$

and, consequently

$$(27) \quad 0 < x_{11} < -\frac{D}{A} < x_{21}, \quad 0 < y_{11} < \frac{4A(C - lD)}{4(B + Al)^2 + w^2} < y_{21}.$$

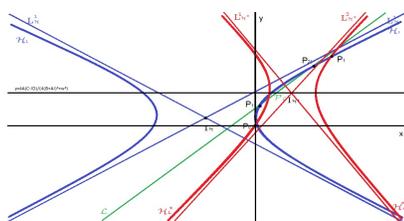


FIG. 33. The intersection points of  $H$  with  $H^*$  in the case  $P_{11}$  and  $P_{21}$  are both located on the right branch  $H_r$  of  $H$  and  $H^*$  is of left-right type

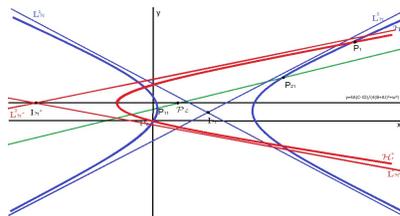


FIG. 34. The intersection points of  $H$  with  $H^*$  in the case  $P_{11}$  and  $P_{21}$  are located, respectively, on the left and right branches  $H_L$  and  $H_r$  of  $H$  and  $H^*$  is of left-right type

To study the maximum number of a limit cycles  $\Gamma_k, k = 1, 2, \dots$  of system (2) intersecting  $\Sigma$  in two points provided the existence of a limit cycle intersecting  $\Sigma$  in four points is equivalent to find the maximum number of intersection points  $P_{1i} = (x_{1i}, y_{1i}), i = 1, 2, \dots$  of the hyperbolas  $\mathcal{H}$  with  $\mathcal{H}^*$ , whose coordinates satisfy (27) and (26) knowing that  $\mathcal{L}$  intersects  $\mathcal{H}$  in two points  $P_{11} = (x_{11}, y_{11})$  and  $P_{21} = (x_{21}, y_{21})$  with  $0 < x_{11} < x_{21}$  and  $0 < y_{11} < y_{21}$ . By (26) we must have

$$(28) \quad P_0 = (0, 0) \prec P_{11} \prec P_{21} \prec P_i, i = 1, 2, \dots$$

Note that  $P_0 = (0, 0) \in \mathcal{H} \cap \mathcal{H}^*$  and  $\mathcal{P}_{\mathcal{L}} = \left(-\frac{D}{A}, \frac{4A(C-ID)}{4(B+AI)^2+w^2}\right)$  is a point of the line  $\mathcal{L}$ . In order that the condition in (27) hold the straight line  $\mathcal{L}$  must have positive slope, we further assume without loss of generality that the hyperbola  $\mathcal{H}$  is in left-right way, then we have three cases possible for the existence two intersection points of  $\mathcal{L}$  with  $\mathcal{H}$ :

- 1:  $\mathcal{P}_{\mathcal{L}}$  is located on the right hand side of  $\mathcal{H}_r$ , and  $\mathcal{L}$  has its slope larger than that of  $L_{\mathcal{H}}^2$ .
- 2:  $\mathcal{P}_{\mathcal{L}}$  is located in between  $\mathcal{H}_L$  and  $\mathcal{H}_r$ , and  $\mathcal{L}$  has its slope smaller than that of  $L_{\mathcal{H}}^2$ .
- 3:  $\mathcal{P}_{\mathcal{L}}$  is located on the left hand side of  $\mathcal{H}_L$ , and  $\mathcal{L}$  has its slope larger than that of  $L_{\mathcal{H}}^2$ .

Since  $I_{\mathcal{H}^*}$  has the same vertical coordinate as that of  $\mathcal{P}_{\mathcal{L}}$ , then in order that there exist at least two  $P_i$  satisfies (28);  $I_{\mathcal{H}^*}$  must be located on the right hand side of  $\mathcal{P}_{\mathcal{L}}$  both above the horizontal line  $y = \frac{4A(C-ID)}{4(B+AI)^2+w^2}$ .

Case 1- If  $P_{11}$  and  $P_{21}$  are both located on the right branch  $\mathcal{H}_r$  of  $\mathcal{H}$  or if  $P_{11}$  and  $P_{21}$  are located, respectively, on the left and right branches  $\mathcal{H}_L$  and  $\mathcal{H}_r$  of  $\mathcal{H}$ .

In case that the hyperbola  $\mathcal{H}^*$  is of left-right type, Since  $P_0 = (0, 0) \in \mathcal{H} \cap \mathcal{H}^*$  and  $P_0 \prec P_{11} \prec P_{21}$  then, there are at most one intersection point  $P_1 = (x_1, y_1)$ , of  $\mathcal{H}$  with  $\mathcal{H}^*$ , which satisfy (28), in this case  $P_1$  on  $\mathcal{H}_r^*$ . See figures 33 and 34.

In case that the hyperbola  $\mathcal{H}^*$  is of upper-down type and since  $P_0 = (0, 0) \in \mathcal{H} \cap \mathcal{H}^*$  and  $P_0 \prec P_{11} \prec P_{21}$ , then there are at most one intersection point  $P_1 = (x_1, y_1)$  of  $\mathcal{H}$  with  $\mathcal{H}^*$ , which satisfy (28), and  $P_1$  on  $\mathcal{H}_u^*$ . See figures 35 and 36.

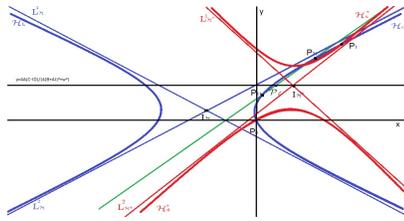


FIG. 35. The intersection points of  $H$  with  $H^*$  in the case  $P_{11}$  and  $P_{21}$  are both located on the right branch  $H_r$  of  $H$  and  $H^*$  is of upper-down type

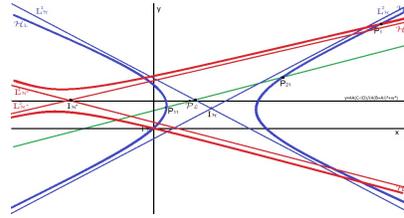


FIG. 36. The intersection points of  $H$  with  $H^*$  in the case  $P_{11}$  and  $P_{21}$  are located, respectively, on the left and right branches  $H_L$  and  $H_r$  of  $H$  with  $H^*$  is of upper-down type

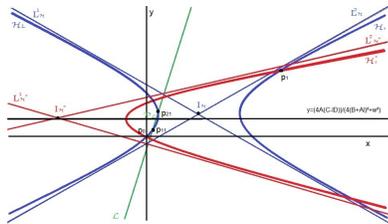


FIG. 37. The intersection points of  $H$  with  $H^*$  in the case  $P_{11}$  and  $P_{21}$  are both located on the right branch  $H_l$  of  $H$  and  $H^*$  is of left-right with  $P_1$  in  $\mathcal{H}_r^* \cap \mathcal{H}_r$

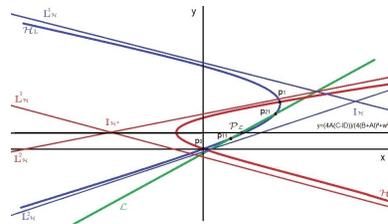


FIG. 38. The intersection points of  $H$  with  $H^*$  in the case  $P_{11}$  and  $P_{21}$  are both located on the right branch  $H_l$  of  $H$  and  $H^*$  is of left-right with  $P_1$  in  $\mathcal{H}_r^* \cap \mathcal{H}_L$

*Case 2-* If  $P_{11}$  and  $P_{21}$  are both located on the left branch  $\mathcal{H}_L$  of  $\mathcal{H}$ . In case that the hyperbola  $\mathcal{H}^*$  is of left-right type, and since  $P_0 = (0, 0) \in \mathcal{H} \cap \mathcal{H}^*$  and  $P_0 \prec P_{11} \prec P_{21}$ , then there are at most one intersection point  $P_1 = (x_1, y_1)$ , of  $\mathcal{H}$  with  $\mathcal{H}^*$ , which satisfy (28), and  $P_1$  in  $\mathcal{H}_r^* \cap \mathcal{H}_L$  or in  $\mathcal{H}_r^* \cap \mathcal{H}_r$ . See figures 37 and 38.

In case that the hyperbola  $\mathcal{H}^*$  is of upper-down type, there are at most two intersection points  $P_0 = (0, 0)$  and  $P_1 = (x_1, y_1)$  of  $\mathcal{H}$  with  $\mathcal{H}^*$ , which satisfy (28), and  $P_1$  in  $\mathcal{H}_u^* \cap \mathcal{H}_L$  or in  $\mathcal{H}_u^* \cap \mathcal{H}_r$ . See figures 39 and 40.

This proves that if there exist one limit cycle intersecting the set  $\Sigma$  in exactly four points (type 2), system (2) can have at most one limit cycle (of the type 1).

Now we verify that this upper bound is reached, for this we present the following example.

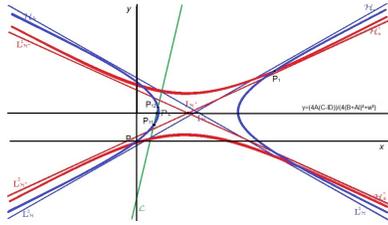


FIG. 39. The intersection points of  $H$  with  $H^*$  in the case  $P_{11}$  and  $P_{21}$  are both located on the right branch  $H_l$  of  $H$  and  $H^*$  is of upper-down with  $P_1$  in  $\mathcal{H}_u^* \cap \mathcal{H}_r$

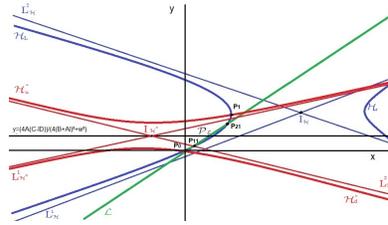


FIG. 40. The intersection points of  $H$  with  $H^*$  in the case  $P_{11}$  and  $P_{21}$  are both located on the right branch  $H_l$  of  $H$  and  $H^*$  is of upper-down with  $P_1$  in  $\mathcal{H}_u^* \cap \mathcal{H}_L$

**Example 10.** Consider the discontinuous piecewise linear differential system with two pieces separated by the set

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } x \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ and } y \geq 0\},$$

defined by

$$(29) \quad \begin{aligned} \dot{x} &= 558x - ky - 36\sqrt{14} - 1119, & \dot{y} &= kx + (16\sqrt{14} - 412)y + 16\sqrt{14} + 828 & \text{in } S_1, \\ \dot{x} &= -x - y + 2, & \dot{y} &= 2x + y - 4 & \text{in } S_2, \end{aligned}$$

where  $k = 93\sqrt{\frac{545-32\sqrt{14}}{31}}$ . The two linear differential systems of (29) have the following first integrals:

$$\begin{aligned} H_1(x, y) &= 4 \left( x - \frac{1}{6} \sqrt{\frac{545-32\sqrt{14}}{31}} y \right)^2 - 8 \left( \left( \frac{2\sqrt{14}}{31} + \frac{373}{186} \right) x + \left( \frac{8\sqrt{14}}{279} + \frac{46}{31} \right) y \right) + y^2, \\ H_2(x, y) &= 4 \left( x + \frac{1}{2} y \right)^2 + 8(-2x - y) + y^2, \end{aligned}$$

in  $S_1$  and  $S_2$ , respectively. Since  $\pm 279i$  and  $\pm i$  are the eigenvalues of the matrices of the two linear differential systems of (29), these systems have their equilibria as centers. For the discontinuous piecewise linear differential system (29), the system (4), with  $l = 0$  becomes

$$(30) \quad \begin{aligned} 4x_1^2 - 8 \left( \frac{2\sqrt{14}}{31} + \frac{373}{186} \right) x_1 - 4 \left( \frac{1}{6} \sqrt{\frac{545-32\sqrt{14}}{31}} y_1 \right)^2 + 8 \left( \frac{8\sqrt{14}}{279} + \frac{46}{31} \right) y_1 - y_1^2 &= 0, \\ 4x_1^2 - 16x_1 - 2y_1^2 + 8y_1 &= 0, \end{aligned}$$

and system (10), with  $l = 0$  becomes

(31)

$$\begin{aligned} 4x_{11}^2 - 8 \left( \frac{2\sqrt{14}}{31} + \frac{373}{186} \right) x_{11} - 4 \left( \frac{1}{6} \sqrt{\frac{545-32\sqrt{14}}{31}} y_{11} \right)^2 + 8 \left( \frac{8\sqrt{14}}{279} + \frac{46}{31} \right) y_{11} - y_{11}^2 &= 0, \\ 4x_{21}^2 - 8 \left( \frac{2\sqrt{14}}{31} + \frac{373}{186} \right) x_{21} - 4 \left( \frac{1}{6} \sqrt{\frac{545-32\sqrt{14}}{31}} y_{21} \right)^2 + 8 \left( \frac{8\sqrt{14}}{279} + \frac{46}{31} \right) y_{21} - y_{21}^2 &= 0, \\ -y_{11}^2 + 8y_{11} + 2y_{21}^2 - 8y_{21} &= 0, \\ 4x_{11}^2 - 16x_{11} - 4x_{21}^2 + 16x_{21} &= 0. \end{aligned}$$

Taking into account that we are only interested in the solutions  $(x_{11}, x_{21}, y_{11}, y_{21})$  satisfying  $0 < x_{11} < x_{21} < x_1$  and  $0 < y_{11} < y_{21} < y_1$ , the unique solution of the system (31) is  $(x_{11}, x_{21}, y_{11}, y_{21}) = (\frac{2}{3}, \frac{10}{3}, 1, 3)$  and the unique solution of the system (30) is  $(x_1, y_1) = (5, \sqrt{14} + 2)$ . Straightforward computations show that the solution passing through the crossing points  $(\frac{2}{3}, 0)$ ,  $(0, 1)$ ,  $(0, 3)$  and  $(\frac{10}{3}, 0)$  correspond to

$$\begin{aligned} \Gamma_1 = \left\{ (x, y) \in S_1 : H_1(x, y) = -\frac{32\sqrt{14}}{93} - \frac{2488}{279} \right\} \cup \left\{ (x, y) \in S_2 : H_2(x, y) = -6 \right\}, \\ \cup \left\{ (x, y) \in S_1 : H_1(x, y) = -\frac{160}{93} \sqrt{14} - \frac{280}{31} \right\} \cup \left\{ (x, y) \in S_2 : H_2(x, y) = -\frac{80}{9} \right\}. \end{aligned}$$

and the solution passing through the crossing points  $(5, 0)$  and  $(0, \sqrt{14} + 2)$  correspond to

$$\Gamma_2 = \left\{ (x, y) \in S_1 : H_1(x, y) = \frac{1840}{93} - \frac{80\sqrt{14}}{31} \right\} \cup \left\{ (x, y) \in S_2 : H_2(x, y) = 20 \right\}.$$

Clearly,  $\Gamma_1$ , and  $\Gamma_2$  are nested, and  $\Gamma_1$  is the inner one and  $\Gamma_2$  is the outer one. Moreover, the orbit arc in  $S_1$  starting from  $(5, 0)$  satisfies  $\dot{x}|_{(5,0)} > 0$  and  $\dot{y}|_{(5,0)} > 0$ , so it runs in counterclockwise. The orbit arc in  $S_2$  starting from  $(0, \sqrt{14} + 2)$  satisfies  $\dot{x}|_{(0, \sqrt{14}+2)} < 0$  and  $\dot{y}|_{(0, \sqrt{14}+2)} > 0$ , and so it runs in counterclockwise, thus  $\Gamma_2$  is traveled in counterclockwise sense. And the orbit arc in  $S_1$  starting from  $(\frac{10}{3}, 0)$  satisfies  $\dot{x}|_{(\frac{10}{3}, 0)} > 0$  and  $\dot{y}|_{(\frac{10}{3}, 0)} > 0$ , so it runs in counterclockwise. The orbit arc in  $S_2$  starting from  $(0, 3)$  satisfies  $\dot{x}|_{(0,3)} < 0$  and  $\dot{y}|_{(0,3)} > 0$ , and so it runs in counterclockwise and the orbit arc in  $S_1$  starting from  $(0, 1)$  satisfies  $\dot{x}|_{(0,1)} < 0$  and  $\dot{y}|_{(0,1)} > 0$ , so it runs in counterclockwise. The orbit arc in  $S_2$  starting from  $(\frac{2}{3}, 0)$  satisfies  $\dot{x}|_{(\frac{2}{3}, 0)} > 0$  and  $\dot{y}|_{(\frac{2}{3}, 0)} < 0$ , and so it runs in counterclockwise, thus  $\Gamma_1$  is traveled in counterclockwise sense. Then the discontinuous piecewise linear differential system (29) has exactly two limit cycles intersecting the set  $\Sigma$  in exactly two points, which are traveled in counterclockwise sense; see them in Figure 41.

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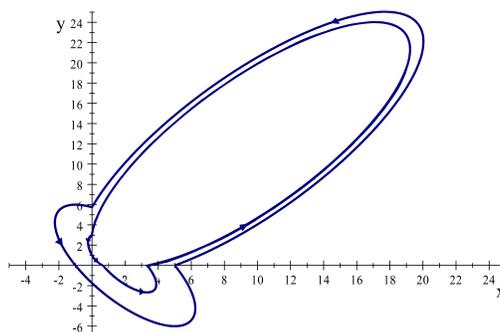


FIG. 41. The two limit cycles of the discontinuous piecewise linear differential system (29).

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