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AN EXTENSION OF FRANKLIN'S THEOREM

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Abstract. Back in 1922, Franklin proved that every 3-polytope with minimum degree 5 has a 5-vertex adjacent to two vertices of degree at most 6, which is tight. This result has been extended and refined in several directions.

It is well-known that each 3-polytope has a vertex of degree at most 5, called minor vertex. A 3-path uvw is an (i, j, k)-path if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$, where $d(x)$ is the degree of a vertex x. A 3-path is minor 3-path if its central vertex is minor.

The purpose of this note is to extend Franklin' Theorem to the 3-polytopes with minimum degree at least 4 by proving that there exist precisely the following ten tight descriptions of minor 3-paths:
$\{(6,5,6),(4,4,9),(6,4,8),(7,4,7)\}$, $\{(6,5,6),(4,4,9),(7,4,8)\}$,
$\{(6,5,6),(6,4,9),(7,4,7)\}$, $\{(6,5,6),(7,4,9)\}$, $\{(6,5,8),(4,4,9),(7,4,7)\}$,
$\{(6,5,9),(7,4,7)\}$, $\{(7,5,7),(4,4,9),(6,4,8)\}$, $\{(7,5,7),(6,4,9)\}$,
$\{(7,5,8),(4,4,9)\}$, and $\{(7,5,9)\}$.

Keywords: planar graph, plane map, 3-polytope, structure properties, tight description, path, weight.

1. Introduction

The *degree* $d(x)$ of a vertex or face x in a plane graph G is the number of its incident edges. A *k-vertex* (*k-face*) is a vertex (face) with degree k, a k^+*-vertex* has degree at least k, etc. The minimum vertex degree of G is $\delta(G)$. We will drop the arguments whenever this does not lead to confusion.

A *k-path* is a path on k vertices. A path uvw is an (i, j, k)*-path* if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$. The *weight* $w(H)$ of a subgraph H of a graph G is the

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degree-sum of the vertices of H in G. By $\mathbf{P}_\delta$ denote the class of 3-polytopes with minimum degree δ; in particular, $\mathbf{P_3}$ is the set of all 3-polytopes.

In 1904, Wernicke [20] proved that if $P_5 \in \mathbf{P_5}$ then P_5 contains a 5-vertex adjacent to a 6^--vertex. This result was strengthened by Franklin [12] in 1922 by proving the existence of a $(6,5,6)$-path in every $P_5 \in \mathbf{P_5}$.

Theorem 1 (Franklin [12])**.** *Every* 3*-polytope in* $\mathbf{P_5}$ *has a* $(6,5,6)$*-path, where no* 6 *can be lowered to* 5*.*

We recall that a description of 3-paths is *tight* if none of its parameters can be strengthened and no term dropped. The tightness of Franklin's description is shown by putting a vertex inside each face of the dodecahedron and joining it to the five boundary vertices.

Franklin's Theorem 1 is fundamental in the structural theory of planar graphs; it has been extended or refined in several directions, see, for example, [1–7, 9–11, 13, 14, 16–19] and surveys [8, 15].

We now mention only a few easily formulated results on $\mathbf{P_5}$, which are the closest to Franklin's Theorem and whose parameters are all sharp.

Borodin [3] proved that there is a 3-face with weight at most 17. Jendrol' and Madaras [14] ensured a 5-vertex that has three neighbors whose weight sums to at most 18 and a 4-path with weight at most 23. Madaras [17] found a 5-path with weight at most 29. We proved [7] that there is a $(5,6,6)$-path.

In 2014, we proved [6] that there exist precisely seven tight descriptions of 3-paths in triangle-free 3-polytopes: $\{(5,3,6),(4,3,7)\}$, $\{(3,5,3),(3,4,4)\}$, $\{(5,3,6),(3,4,3)\}$, $\{(3,5,3),(4,3,4)\}$, $\{(5,3,7)\}$, $\{(3,5,4)\}$, $\{(5,4,6)\}$, which was a result of a new type in the structural theory of plane graphs.

In 1996, Jendrol' [13] gave the following description of 3-paths in $\mathbf{P_3}$: $\{(10,3,10), (7,4,7),(6,5,6),(3,4,15),(3,6,11),(3,8,5),(3,10,3),(4,4,11),(4,5,7),(4,7,5)\}$.

The first tight description of 3-paths in $\mathbf{P_3}$ was obtained in 2013 by Borodin et al. [10]: $\{(3,4,11),(3,7,5),(3,10,4),(3,15,3),(4,4,9),(6,4,8),(7,4,7),(6,5,6)\}$.

Another tight description was given by Borodin, Ivanova and Kostochka [11]: $\{(3,15,3),(3,10,4),(3,8,5),(4,7,4),(5,5,7),(6,5,6),(3,4,11),(4,4,9),(6,4,7)\}$. Also, it is shown in [11] that there exist precisely three tight one-term descriptions of 3-paths in $\mathbf{P_3}$: $\{(5,15,6,)\}$, $\{(5,10,15)\}$, and $\{(10,5,10)\}$.

The problem posed in [11] of describing all tight descriptions of 3-paths in $\mathbf{P_3}$ is still widely open.

The purpose of this note is to extend Franklin's Theorem as follows.

Theorem 2. *There exist precisely the following ten tight descriptions of minor* 3*-paths in* $\mathbf{P_4}$*:*

$(td1)$*:* $\{(6,5,6),(4,4,9),(6,4,8),(7,4,7)\}$,
$(td2)$*:* $\{(6,5,6),(4,4,9),(7,4,8)\}$,
$(td3)$*:* $\{(6,5,6),(6,4,9),(7,4,7)\}$,
$(td4)$*:* $\{(6,5,6),(7,4,9)\}$,
$(td5)$*:* $\{(6,5,8),(4,4,9),(7,4,7)\}$,
$(td6)$*:* $\{(6,5,9),(7,4,7)\}$,
$(td7)$*:* $\{(7,5,7),(4,4,9),(6,4,8)\}$,
$(td8)$*:* $\{(7,5,7),(6,4,9)\}$,
$(td9)$*:* $\{(7,5,8),(4,4,9)\}$,
$(td10)$*:* $\{(7,5,9)\}$.

2. Proving Theorem 2

We first define 3-polytopes H_1–H_4 (see Fig. 1) important for the proof.

Put a 5-vertex into each face of the dodecahedron to obtain H_1, in which every 5-vertex is surrounded by 6-vertices. Thus H_1 has only $(6,5,6)$-paths from those listed in Theorem 2.

Now delete all edges joining two 6-vertices from H_1, and into each its face $wxyz$ with $d(w)=d(y)=3$ and $d(x)=d(z)=5$ put 4-vertices v_1, v_2, where v_1 is adjacent to w,x,y,v_2, while v_2 is adjacent to w,z,y,v_1. The resulting graph H_2 has each 4-vertex adjacent to a 4-vertex and three 9^+-vertices and hence only $(4,4,9)$-paths from the statement of Theorem 2.

To obtain H_3, we start from the octahedron, and for each its face $f=xyz$ first put vertices x', y', and z' on the edges yz, xz, and xy, respectively. Then add vertices v_x, v_y, and v_z joined to each vertex in $\{x,y',z'\}$, $\{x',y',z'\}$, and $\{x',y',z\}$, respectively. Finally, add a vertex v adjacent to each vertex in $\{v_x,z',v_y,x',v_z,y'\}$. Each 4-vertex in the resulting graph has a 6-neighbor and three 8-neighbors, which implies that H_3 has only $(6,4,8)$-paths from Theorem 2.

Take the $(3,4,4,4)$ Archimedean solid, in which every vertex is incident with a 3-face and three 4-faces, and put a 4-vertex into each 4-face. In the graph H_4 obtained every 4-vertex is surrounded by 7^+-vertices, and H_4 has only $(7,4,7)$-paths from those mentioned in Theorem 2.

Figure 1 shows constructing graphs H_1–H_4 from some Platonic and Archimedean solids.

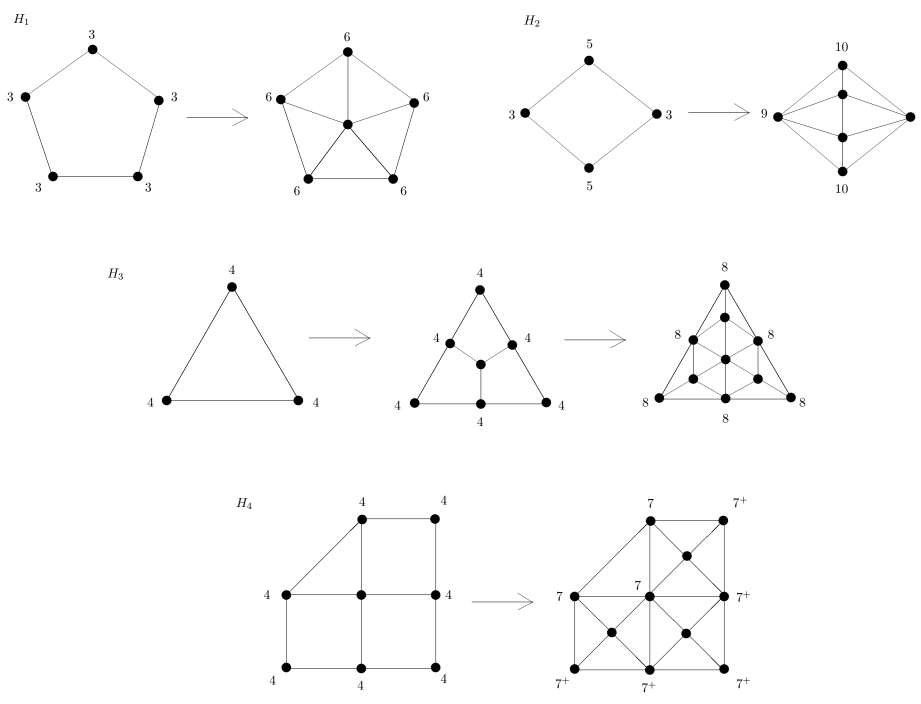

Fig. 1. Graphs H_1–H_4.

Lemma 1. *Each of the sets* $(td1), \dots, (td10)$ *is a description of minor* 3*-paths in* $\mathbf{P_4}$.

Proof. For $(td1)$ this follows from the above mentioned result in [11].

Now note that, by definition, each (i, j, k)-path is also an (i', j', k')-path if $i' \geq i$, $j' \geq j$, and $k' \geq k$. Therefore, for each of the sets $(td2), \dots, (td10)$ it suffices to check that all triplets in each of them together cover all triplets in $(td1)$. For example, the only triplet $(7, 5, 9)$ in $(td10)$ covers each of the triplets in $(td1)$. ■

Lemma 2. *Each of the descriptions* $(td1), \dots, (td10)$ *is tight.*

Proof. The check is based on the properties of H_1–H_4. Namely, each of $(td1), \dots, (td10)$ must contain triplets $(6^+, 5, 6^+)$, $(4^+, 4^+, 9)$, $(6^+, 4^+, 8^+)$ and $(7^+, 4^+, 7^+)$ due to H_1, H_2, H_3 and H_4, respectively.

For example, an attempt to decrease 9 in any of $(td1), \dots, (td10)$ is prevented from by H_2, in which every minor 3-path goes through a 9^+-vertex. Also, we cannot replace the central 5 in any of $(td1), \dots, (td10)$ by 4, since otherwise the thus reduced set of triplets fails to cover H_1. The rest of checking is left to the reader. ■

Lemma 3. *There are no tight descriptions of minor 3-paths in* $\mathbf{P_4}$ *other than* $(td1), \dots, (td10)$.

Proof. Suppose $D = \{(x_1, y_1, z_1), \dots (x_k, y_k, z_k)\}$ is a tight description of 3-paths in $\mathbf{P_5}$. This means that

(1) every $P_4 \in \mathbf{P_4}$ has a (x_i, y_i, z_i)-path for at least one i with $1 \leq i \leq k$, and

(2) if we delete any term (x_i, y_i, z_i) from D or decrease any parameter in D by one without changing the other $3k - 1$ parameters, then the new description is not satisfied by at least one $P_4 \in \mathbf{P_4}$.

Note that, due to its tightness, the description D cannot have triplets (X, Y, Z) and (X', Y', Z') such that $X \leq X'$, $Y \leq Y'$, and $Z \leq Z'$, for $D' = D \setminus \{(X, Y, Z)\}$ is equivalent to D but shorter. Also, all parameters in D should be at least 4 since we deal with $\mathbf{P_4}$. By symmetry, we can assume that $x_i \leq z_i$ whenever $1 \leq i \leq k$.

Note that D must contain a term $(6^+, 5, 6^+)$ to be able to describe H_1. Therefore, our case analysis splits into Cases 1–6.

Case 1. D has a term $(x_1, y_1, z_1) = (7^+, 5, 9^+)$. By Lemma 2, D must coincide with the tight description (td10), so $D = \{(7, 5, 9)\}$.

Case 2. D has a term $(x_1, y_1, z_1) = (6, 5, 9^+)$. Due to H_4, there should be a term $(x_2, y_2, z_2) = (7^+, 4^+, 7^+)$ in D, and hence D coincides with the tight description $\{(6, 5, 9), (7, 4, 7)\}$, that is (td6).

Case 3. D has a term $(x_1, y_1, z_1) = (7^+, 5, 8)$. Due to H_2, there should be a term $(x_2, y_2, z_2) = (4^+, 4^+, 9^+)$, so $D = \{(7, 5, 8), (4, 4, 9)\}$, which is (td9).

Case 4. D has a term $(x_1, y_1, z_1) = (6, 5, 8)$. Now again D must include a term $(x_2, y_2, z_2) = (4^+, 4^+, 9^+)$. If $(x_2, y_2, z_2) = (7^+, 4, 9^+)$, then D is weaker than (td4), which is impossible. If $(x_2, y_2, z_2) = (6^-, 4^+, 9^+)$, then due to H_4 there should exist a term $(x_3, y_3, z_3) = (7^+, 4^+, 7^+)$. Hence $D = \{(6, 5, 8), (4, 4, 9), (7, 4, 7)\}$, which means that we have (td5).

Case 5. D has a term $(x_1, y_1, z_1) = (7, 5, 7)$. Now again D must include a term $(x_2, y_2, z_2) = (4^+, 4^+, 9^+)$. If $(x_2, y_2, z_2) = (6^+, 4^+, 9^+)$, then $D = \{(7, 5, 7), (6, 4, 9)\}$, so D is (td8). Otherwise, that is if $(x_2, y_2, z_2) = (5^-, 4^+, 9^+)$, there should exist $(x_3, y_3, z_3) = (6^+, 4^+, 8^+)$ due to H_3, and we have $D = \{(7,5,7),(4,4,9),(6,4,8)\}$, which is (td7).

Case 6. D has a term $(x_1, y_1, z_1) = (6, 5, 6 \vee 7)$. We shall see from what follows that in fact $(x_1, y_1, z_1) = (6,5,7)$ is impossible. Indeed, on the one hand D is tight, but on the other hand the term $(6,5,7)$ can either be deleted from D if there is also a term $(6,5,6)$ in D, or can be strengthened to $(6,5,6)$ otherwise, since it does not cover any triplets mentioned in Theorem 2. Again we can assume that $(x_2, y_2, z_2) = (4^+, 4^+, 9^+)$.

Subcase 6.1. $(x_2, y_2, z_2) = (7^+, 4, 9^+)$. Now $D = \{(6,5,6),(7,4,9)\}$, so D is (td4).

Subcase 6.2. $(x_2, y_2, z_2) = (6, 4, 9^+)$. Due to H_4, we can assume that there should be $(x_3, y_3, z_3) = (7^+, 4, 7^+)$, which implies that $D = \{(6,5,6),(6,4,9),(7,4,7)\}$, which is (td3).

Subcase 6.3. $(x_2, y_2, z_2) = (5^-, 4, 9^+)$. Note that the first two terms of D do not cover H_3, and thus we should have $(x_3, y_3, z_3) = (6^+, 4, 8^+)$. If in fact $(x_3, y_3, z_3) = (7^+, 4, 8^+)$, then $D = \{(6,5,6),(4,4,9),(7,4,8)\}$, as in $(td2)$. Otherwise, $(x_3, y_3, z_3) = (6, 4, 8^+)$, which implies due to H_4 that D also has a term $(x_4, y_4, z_4) = (7^+, 4, 7^+)$, and is $(td1)$: $\{(6,5,6),(4,4,9),(6,4,8),(7,4,7)\}$. ■

This completes the proof of Theorem 2.

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