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THE PROPERTY OF BEING A MODEL COMPLETE THEORY IS PRESERVED BY CARTESIAN EXTENSIONS

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Abstract. Cartesian-quotient extensions of theories constitute a most common class of finitary transformation methods for first-order combinatorics. In this paper, some technical properties of classes of algebraic Cartesian and algebraic Cartesian-quotient interpretations of theories are studied. It is established that any algebraic Cartesian interpretation preserves the property of being a model complete theory; besides, an example of an algebraic Cartesian-quotient interpretation of theories is given, which does not preserve the model-completeness property.

Keywords: first-order logic, incomplete theory, Tarski-Lindenbaum algebra, model-theoretic property, computable isomorphism, Cartesian interpretation, model completeness

High importance in logic has the problem of characterization of the Tarski-Lindenbaum algebra of predicate calculus of a finite rich signature. This problem was initiated by Alfred Tarski in the late 1930th, and the problem was solved by William Hanf in 1975, [1, Th. 3], [2, Th. 23]. Historical background of the Tarski problem can be found in the papers [3, p. 132-134], [4, p. 75-76], [1, p. 587], [5, Sec. 2], [6, p. 357], [7, p. 84-85], and others. As a significant generalization of the Tarski problem, a natural question arises to characterize the structure of the Tarski-Lindenbaum algebra of predicate calculus together with a description of model-theoretic properties of different extensions of this theory. Some advances in this direction are obtained in [8, Th. 6.1], [9, Th. 7.1], and [7, p. 99-102]. A principal result in this direction is announced in [10] together with [11] showing an evident progress

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towards the solution to the generalized Tarski problem. The result is essentially based on Main Theorem in [11] establishing a strong connection between predicate calculi of any two finite rich signatures by a passage via Cartesian extensions; thus, the operation of a Cartesian extension plays the key role in solving generalized Tarski's problem.

Finitary and infinitary first-order combinatorics represents a conceptual basis of investigations on expressive power of predicate logic, [12, Sec. 2]. Cartesian and Cartesian-quotient extensions of theories represent a natural general class of finitary methods of transformations of theories. In this paper, some technical properties of the class of algebraic Cartesian extensions of theories are studied. It is proved that the operation of an algebraic Cartesian extension preserves the property of being a model complete theory. In contrast to this, an example of an algebraic Cartesian-quotient extension of a theory is constructed that does not preserve the property of being a model-complete theory.

0. Preliminaries

We consider theories in first-order predicate logic *with equality* and use general concepts of model theory, algorithm theory, and constructive models found in [13], [14], and [15]. Special concepts of this paper are in accordance with those accepted in [16]. Generally, *incomplete theories* are considered. In the work, the signatures are considered only, which admit Gödel's numberings of the formulas. Such a signature is called *enumerable*.

By $L(T)$, we denote the *Tarski-Lindenbaum algebra* of formulas of theory T without free variables, while $\mathcal{L}(T)$ denotes the Tarski-Lindenbaum algebra $L(T)$ considered together with a Gödel numbering γ; thereby, the concept of a computable isomorphism is applicable to such objects. The following notations are used: $PC(\sigma)$ is predicate calculus of signature σ, i.e., a theory of signature σ defined by an empty set of axioms, $SL(\sigma)$ is the set of all sentences of signature σ, $FL(\sigma)$ is the set of all formulas of signature σ. A finite signature is called *rich*, if it contains at least one n-ary predicate or function symbol for $n \geqslant 2$, or two unary function symbols.

As an $\exists\cap\forall$-formula $\varphi(\bar{x})$ of signature σ, a pair of formulas $(\varphi^e(\bar{x}), \varphi^a(\bar{x}))$ is meant together with the *domain sentence* $DomEA(\varphi(\bar{x})) = (\forall\bar{x})[\varphi^e(\bar{x}) \leftrightarrow \varphi^a(\bar{x})]$, where $\varphi^e(\bar{x})$ is an $\exists$-formula, while $\varphi^a(\bar{x})$ is a $\forall$-formula of signature σ. A formula $\varphi(\bar{x})$ of theory T is said to be $\exists\cap\forall$-presentable in T if $T \vdash DomEA(\varphi(\bar{x}))$. If $\psi(\bar{x})$ is a quantifier-free formula, $DomEA(\psi(\bar{x}))$ is supposed to be a generally true formula. If $\varkappa$ is a finite set (or a sequence) of $\exists\cap\forall$-formulas $\psi_i(\bar{x}_i)$, $i < k$, we denote by $DomEA(\varkappa)$ the conjunction $\bigwedge_{i<k} DomEA(\psi_i(\bar{x}_i))$.

Robinson's criterion, [17], establishes that an arbitrary (in general case, incomplete) theory T is model complete if and only if each formula $\varphi(\bar{x})$ of theory T is $\exists\cap\forall$-presentable in T, equivalently, $\mathfrak{M} \subseteq \mathfrak{M}' \Rightarrow \mathfrak{M} \preccurlyeq \mathfrak{M}'$ is satisfied for all models $\mathfrak{M}$ and $\mathfrak{M}'$ of theory T.

Recall an important definition introduced in [3, Sec. 1]. Given a theory T of signature τ and a theory S of signature σ. Consider a pair of functions $(h, \tilde{h})$, where $h : SL(\tau) \to SL(\sigma)$ is a computable bijection, and $\tilde{h} : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$ is a bijective mapping. This pair $(h, \tilde{h})$ is said to be *Hanf's isomorphism* between the theories, if the following condition is satisfied:

$$\mathfrak{M} \models \varphi \leftrightarrow \tilde{h}(\mathfrak{M}) \models h(\varphi),\ \textit{for all}\ \varphi \in SL(\sigma),\ \mathfrak{M} \in \mathrm{Mod}(T). \tag{0.1}$$

Moreover, it is possible to restrict ourselves with just the case when the mapping $\tilde{h}$ links classes of models of theories T and S of cardinality $\leqslant \gamma$ instead of the classes of all models, where γ is a fixed infinite cardinal number.

Lemma 0.1. [3, Sec. 1] *Let T and S be theories of signatures τ and, respectively, σ. The following statements are equivalent with each other*:

(a) *there is a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(S)$,*

(b) *there is Hanf's isomorphism $(h, \tilde{h})$ between T and S.*

Proof. Immediately. $\square$

Let $\mu : \mathcal{L}(T) \to \mathcal{L}(S)$ be an isomorphism of the Tarski-Lindenbaum algebras of theories T and S. It is a simple fact that μ establishes a one-to-one correspondence between filters in the Boolean algebras $\mathcal{L}(T)$ and $\mathcal{L}(S)$; moreover, ultrafilters in $\mathcal{L}(T)$ will correspond to ultrafilters in $\mathcal{L}(S)$. By construction, filters in the Tarski-Lindenbaum algebras represent theories extending the source theories, while ultrafilters represent complete theories extending the source theories. Based on this observation, we define a natural correspondence between extensions of the theories T and S (including both complete and incomplete ones) by the rules:

(0.2)
$$\begin{aligned} &\texttt{(a)}\quad T' \supseteq T \;\mapsto\; S' \supseteq S,\ \textit{by rule } S' = \mu(T'),\\ &\texttt{(b)}\quad S' \supseteq S \;\mapsto\; T' \supseteq T,\ \textit{by rule } T' = \mu^{-1}(S'). \end{aligned}$$

Thus, if T' is a theory extending T, its full image $S' = \mu(T')$ is a theory extending S, and visa verse, if S' is a theory that is an extension of S, its full preimage $T' = \mu^{-1}(S')$ is a theory extending T. Moreover, the following properties take place:

(0.3) (a) *transitions $T' \mapsto \mu(T')$ and $S' \mapsto \mu^{-1}(S')$ in (0.2) are mutually inverse to each other*;

(b) $(\forall$ *extension* $T' \supseteq T)\left[T'\ \textit{is complete} \Leftrightarrow \mu(T')\ \textit{is complete}\right].$

1. Isostone interpretations

We follow a standard version of the concept of an interpretation of a theory T_0 of signature σ_0 in the domain $U(x)$ of a theory T_1 of signature σ_1, cf. [18, Sec. 4.7]. Interpretation $I : T_0 \rightarrowtail T_1$ is uniquely determined by a mapping i (called the *basic assignment*) from signature symbols of theory T_0 in formulas of theory T_1. The mapping i has to keep (in a sense) quantity of free variables demanding these variables to be restricted in the domain $U(x)$. Each n-ary predicate is mapped in a formula with n free variables, n-ary function in a formula with $n+1$ free variables, and constant in a formula with one free variable. Inductively, the mapping i is expanded upto a transformation $I : FL(\sigma_0) \to FL(\sigma_1)$.

Any interpretation I has to satisfy the following properties for all $\varphi \in SL(\sigma_0)$: (a) $T_1 \vdash (\exists x)U(x)$, (b) $T_0 \vdash \varphi \Rightarrow T_1 \vdash I(\varphi)$. Interpretation I is said to be *faithful* if $T_0 \vdash \varphi \Leftrightarrow T_1 \vdash I(\varphi)$ for all $\varphi \in SL(\sigma_0)$. Interpretation I of theory T_0 in the domain $U(x)$ of theory T_1 is said to be $\exists\cap\forall$-presentable, if both domain formula $U(x)$ and destinations of the basic assignment for I are $\exists\cap\forall$-presentable formulas in T_1 Interpretation I is said to be *effective* if transformation $\varphi \mapsto I(\varphi)$ is defined by a computable function on Gödel numbers.

Let I be an interpretation of a theory T_0 of signature σ_0 in the domain $U(x)$ of a theory T_1. Consider an arbitrary model $\mathfrak{M}$ of theory T_1. Based on interpretation I, it is possible to define all predicates, functions and constants of signature σ_0

in first-order definable set $U(\mathfrak{M})$ obtaining a model $\mathfrak{N} = \langle U(\mathfrak{M}), \sigma_0\rangle$ which is called the *model-kernel* of $\mathfrak{M}$ with respect to the interpretation I, symbolically $\mathfrak{N} = \mathbb{K}_I(\mathfrak{M})$, or briefly $\mathfrak{N} = \mathbb{K}(\mathfrak{M})$, when the interpretation I is defined within the context. Interpretation I is called *model free* if $\mathrm{Mod}(T_0) = \{\mathbb{K}(\mathfrak{M}) \mid \mathfrak{M} \in \mathrm{Mod}(T_1)\}$. Interpretation I is called *isostone* if it is model free, and the following condition is satisfied: $\mathbb{K}(\mathfrak{M}_0) \equiv \mathbb{K}(\mathfrak{M}_1) \;\Rightarrow\; \mathfrak{M}_0 \equiv \mathfrak{M}_1$ for all models $\mathfrak{M}_0, \mathfrak{M}_1 \in \mathrm{Mod}(T_1)$.

Study main properties of isostone interpretations.

Lemma 1.1. [16, Lem. 5.2.1] *Let I be an isostone interpretation of a theory T_0 of signature σ_0 in a theory T_1. Then, mapping μ from $\mathcal{L}(T_0)$ into $\mathcal{L}(T_1)$ defined by the rule*

$$\mu([\varphi]_{T_0}) = [I(\varphi)]_{T_1}, \quad \varphi \in SL(\sigma_0), \tag{1.1}$$

is an isomorphism between these Tarski-Lindenbaum algebras. In the case when interpretation I is effective, the rule (1.1) *determines a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(S)$ between the Tarski-Lindenbaum algebras of theories T and S.*

Proof. Immediately. □

An interpretation I of theory T_0 in the domain $U(x)$ of theory T_1 is said to be *auto-free*, if the following condition is satisfied:

$$\big(\forall\, \mathfrak{M} \in \mathrm{Mod}\, T_1\,\big)\big(\forall \mu \in \mathrm{Aut}\, \mathbb{K}(\mathfrak{M})\,\big)\big(\exists \mu^* \in \mathrm{Aut}\, \mathfrak{M}\big)\;\big[\, \mu = \mu^* \restriction U(\mathfrak{M})\,\big]. \tag{1.2}$$

We give an important technical fact.

Lemma 1.2. [19, Lem. 1.4] *Let I be an isostone interpretation of theory T_0 of signature σ_0 in the domain $U(x)$ of theory T_1 of signature σ_1, such that, I is an auto-free interpretation. If $\varphi(x_1, ..., x_n)$ is a formula of signature σ_1 satisfying*

$$T_1 \vdash \varphi(x_1, ..., x_n) \to U(x_1) \,\&\, ... \,\&\, U(x_n),$$

then, there is a formula $\psi(x_1, ..., x_n)$ of signature σ_0 such that

$$T_1 \vdash \varphi(x_1, ..., x_n) \leftrightarrow I\psi(x_1, ..., x_n).$$

Proof. Given a formula $\varphi(\bar{x})$, $\bar{x} = (x_1, ..., x_n)$, of signature σ_1 satisfying in theory T_1 the following condition

$$\varphi(\bar{x}) \to \bar{x} \subseteq U. \tag{1.3}$$

We prove that for any complete type $p(\bar{x})$ in theory T_0 and any formula $\varphi(\bar{x})$ of signature σ_1 satisfying (1.3), one of the following cases must take place:

$$\texttt{(a)}\; T_1 \cup Ip(\bar{x}) \vdash \varphi(\bar{x}), \;\; \textit{or} \;\; \texttt{(b)}\; T_1 \cup Ip(\bar{x}) \vdash \neg\varphi(\bar{x}). \tag{1.4}$$

Suppose (1.4) were false for $\varphi(\bar{x})$; i.e., there is a type $p(\bar{x})$ in a complete extension T' of T_0, such that $T_1 \cup Ip(\bar{x}) \nvdash \varphi(\bar{x})$ and $T_1 \cup Ip(\bar{x}) \nvdash \neg\varphi(\bar{x})$. Since $p(\bar{x})$ is a complete type in T_0, each sentence Φ of signature σ or its negation $\neg\Phi$ must belong to $p(\bar{x})$. Interpretation I is isostone; thus, I-image of $p(\bar{x})$ must generate a complete extension T'' of T_1. By assumption each of the sets $Ip(\bar{x}) \cup \{\varphi(\bar{x})\}$ and $Ip(\bar{x}) \cup \{\neg\varphi(\bar{x})\}$ is compatible with T_1; therefore, they are compatible with T''. Hence, we can find in T'' complete types $q_1(\bar{x})$ and $q_2(\bar{x})$, such that $q_1(\bar{x})$ is compatible with $Ip(\bar{x}) \cup \{\varphi(\bar{x})\}$, and $q_2(\bar{x})$ is compatible with $Ip(\bar{x}) \cup \{\neg\varphi(\bar{x})\}$. Consider a countable homogeneous model $\mathfrak{M}$ of theory T'' that realizes both types $q_1(\bar{x})$ and $q_2(\bar{x})$ on tuples, respectively, $\bar{c}_1$ and $\bar{c}_2$. By (1.3), the tuples $\bar{c}_1$ and $\bar{c}_2$ are located in the kernel domain $U(\bar{x})$ and realize the same type $p(\bar{x})$ in theory $\mathrm{Th}\, \mathbb{K}(\mathfrak{M})$. Since the model $\mathfrak{M}$ is homogeneous, its kernel $\mathbb{K}(\mathfrak{M})$ is also homogeneous.

Since $\bar{c}_1$ and $\bar{c}_2$ realize the same type in $\mathbb{K}(\mathfrak{M})$, there is an automorphism $\mu : \mathbb{K}(\mathfrak{M}) \to \mathbb{K}(\mathfrak{M})$ that maps $\bar{c}_1$ into $\bar{c}_2$. However, no automorphism $\mu^* : \mathfrak{M} \to \mathfrak{M}$ extending μ can exist since $\bar{c}_1$ and $\bar{c}_2$ realize different types $q_1(\bar{x})$ and $q_2(\bar{x})$ in theory $\mathrm{Th}(\mathfrak{M})$. This contradiction establishes that (1.4) is indeed true.

In the first case (1.4)(a), by using standard methods of model theory, we can find a formula $\theta(\bar{x})$ in type $p(\bar{x})$ such that $T_1 \cup \{I\theta(\bar{x})\} \vdash \varphi(\bar{x})$, while in another case (1.4)(b), we can find a formula $\lambda(\bar{x})$ in the type $p(\bar{x})$, such that $T_1 \cup \{I\lambda(\bar{x})\} \vdash \neg\varphi(\bar{x})$. Consider the set θ_p, $p \in P$, of all formulas obtained by this rule for different types p satisfying (1.4)(a), and the set λ_q, $q \in Q$, of all formulas found from types q satisfying (1.4)(b). By construction, the following disjunction (possible, infinitary)

$$\bigvee\nolimits_{p\in P}\theta_p(\bar{x}) \ \vee\ \bigvee\nolimits_{q\in Q}\lambda_q(\bar{x})$$

is true in any tuple of variables $\bar{x}$ in any model $\mathfrak{N}$ of theory T. By Maltsev's Compactness Theorem, there are finite subsets $P_0 \subseteq P$ and $Q_0 \subseteq Q$ such that

$$T \vdash (\forall\bar{x})[\bigvee\nolimits_{p\in P_0}\theta_p(\bar{x}) \vee \bigvee\nolimits_{q\in Q_0}\lambda_q(\bar{x})].$$

We have obtained finite sets of formulas $\{\theta_0(\bar{x}), ..., \theta_k(\bar{x})\}$ and $\{\lambda_0(\bar{x}), ..., \lambda_t(\bar{x})\}$ of signature σ_0 such that

$$\begin{aligned}
&T_1 \vdash I\theta_0(\bar{x}) \vee ... \vee I\theta_k(\bar{x}) \to \varphi(\bar{x}),\\
&T_1 \vdash I\lambda_0(\bar{x}) \vee ... \vee I\lambda_t(\bar{x}) \to \neg\varphi(\bar{x}),\\
&T_0 \vdash (\forall\bar{x})\left[\theta_0(\bar{x}) \vee ... \vee \theta_k(\bar{x}) \ \vee\ \lambda_0(\bar{x}) \vee ... \vee \lambda_t(\bar{x})\right],\\
&T_0 \vdash (\forall\bar{x})\left[\left(\theta_0(\bar{x}) \vee ... \vee \theta_k(\bar{x})\right) \leftrightarrow \neg\left(\lambda_0(\bar{x}) \vee ... \vee \lambda_t(\bar{x})\right)\right].
\end{aligned}$$

Thereby, by putting $\theta(\bar{x}) = \theta_0(\bar{x}) \vee ... \vee \theta_k(\bar{x})$, we obtain the required relation $T_1 \vdash I\theta(\bar{x}) \leftrightarrow \varphi(\bar{x})$. □

Interpretation I of a theory T_0 in a theory T_1 is called *model bijective* if the following requirements are held:

$$\text{(1.5)}\qquad \begin{aligned}
&\text{(a)}\quad \mathrm{Mod}(T_0) \ =\ \{\mathbb{K}(\mathfrak{M}) | \mathfrak{M} \in \mathrm{Mod}(T_1)\},\\
&\text{(b)}\quad \mathbb{K}(\mathfrak{M}) \cong \mathbb{K}(\mathfrak{M}') \ \Leftrightarrow\ \mathfrak{M} \cong \mathfrak{M}',\ \textit{for all}\ \mathfrak{M}, \mathfrak{M}' \in \mathrm{Mod}(T_1).
\end{aligned}$$

Lemma 1.3. [19, Lem. 1.5] *Let I be a model bijective interpretation of a theory T_0 in a theory T_1. Then, I is faithful, model free, and isostone. Besides, the following relations take place*:

(a) $||\mathbb{K}(\mathfrak{M})|| < \omega \ \Leftrightarrow\ ||\mathfrak{M}|| < \omega$, for all $\mathfrak{M} \in \mathrm{Mod}(T_1)$,

(b) $||\mathbb{K}(\mathfrak{M})|| = ||\mathfrak{M}||$, *for all infinite models* $\mathfrak{M} \in \mathrm{Mod}(T_1)$.

Proof. Immediately. □

2. Cartesian-type interpretations

In this section, we introduce the operation of a Cartesian-quotient extension of a theory and study some technical properties of the operation. The idea behind the operation was considered by Leslaw Szczerba in the work [20, p. 130, lines 17-24], where significance of this construction is also discussed. The operation in detail was described in [21, Sec. 1.5]. A weak version of the operation is presented in [7, pp. 89-90]. Essence of the operation of a Cartesian-quotient extension is close to that of the operation $T \mapsto T^{eq}$, cf. [22], [23], [24], [25], and others. The operation 'eq' attaches imaginary elements to the universe for classes of first-order definable equivalence relations. In this paper, we generally use simpler operation of a Cartesian extension of a theory doing without quotients. As for the general version of the operation of a

Cartesian-quotient extension of a theory that is indeed close to the operation 'eq', we concern this version just for the comparison purposes.

We start to describe the operation of a Cartesian-type extension of a theory.

Given a signature σ and a finite sequence of formulas of this signature of either of the following forms:

$$\text{(2.1)}\qquad \begin{aligned}&\text{(a)}\ \varkappa = \langle \varphi_1^{m_1}/\varepsilon_1, \varphi_2^{m_2}/\varepsilon_2, \ldots, \varphi_s^{m_s}/\varepsilon_s\rangle,\\ &\text{(b)}\ \varkappa = \langle \varphi_1^{m_1}, \varphi_2^{m_2}, \ldots, \varphi_s^{m_s}\rangle,\end{aligned}$$

where $\varphi_k(\bar{x}_k)$ is a formula with m_k free variables, $\varepsilon_k(\bar{y}_k, \bar{z}_k)$ is a formula with $2m_k$ free variables such that $\mathrm{Len}\,(\bar{y}_k) = \mathrm{Len}\,(\bar{z}_k) = m_k$; moreover, (2.1)(b) is a simplified notation instead of the common entry (2.1)(a) in the case when $\varepsilon_k(\bar{y}_k, \bar{z}_k)$ coincides with $\bar{y}_k = \bar{z}_k$ for all $k \leqslant s$.

Starting from a model $\mathfrak{M}$ of signature σ together with a tuple $\varkappa$ of any of the forms (2.1)(a,b), we are going to construct a new model $\mathfrak{M}_1$ of signature

$$\text{(2.2)}\qquad \sigma_1 \;=\; \sigma \,\cup\, \{U^1, U_1^1, U_2^1, \ldots, U_s^1\} \,\cup \{K_1^{m_1+1}, K_2^{m_2+1}, \ldots, K_s^{m_s+1}\}$$

as follows. As the universe, we take $|\mathfrak{M}_1| = |\mathfrak{M}| \cup A_1 \cup A_2 \cup \ldots \cup A_s$, where all specified parts are pairwise disjoint sets. On the set $|\mathfrak{M}|$, all symbols of signature σ are defined exactly as they were defined in $\mathfrak{M}$; in the remainder, they are defined trivially; predicate $U(x)$ distinguishes $|\mathfrak{M}|$; predicate $U_k(x)$ distinguishes A_k; the other predicates are defined by specific rules depending on the case. In the case (2.1)(b), each predicate $K_k(\bar{x}_k, u)$ in (2.2) should be defined so that it would represent a one-to-one correspondence between the set of tuples $\{\bar{a} \mid \mathfrak{M} \models \varphi_k(\bar{a})\}$ and the set $A_k = U_k(\mathfrak{M}_1)$. Turn to the most common case (2.1)(a). Denote by $\mathrm{Equiv}(\varepsilon_k, \varphi_k)$ a sentence stating that ε_k is an equivalence relation on the set of tuples distinguished by the formula $\varphi_k(\bar{x}_k)$ in $\mathfrak{M}$. In this case, $(m_k{+}1)$-ary predicate $K_k(\bar{x}_k, u)$ should be defined so that it would represent a one-to-one correspondence between the quotient set $\{\bar{a} \mid \mathfrak{M} \models \varphi_k(\bar{a})\}/\varepsilon_k'$ and the set $U_k(\mathfrak{M}_1)$, where

$$\text{(2.3)}\qquad \varepsilon_k'(\bar{y}_k, \bar{z}_k) = \varepsilon_k(\bar{y}_k, \bar{z}_k) \vee \neg\, \mathrm{Equiv}(\varepsilon_k, \varphi_k).$$

The model $\mathfrak{M}_1$ obtained from $\mathfrak{M}$ and $\varkappa$ as explained above is denoted by $\mathfrak{M}\langle\varkappa\rangle$.

The aim of replacement of ε_k by ε_k' using $\mathrm{Equiv}(\varepsilon_k, \varphi_k)$ is to provide the total definiteness of the operation $\mathfrak{M} \mapsto \mathfrak{M}\langle\varkappa\rangle$ independently of whether the formulas ε_k, $k = 1, 2, ..., s$, represent equivalence relations in corresponding domains or not. In the case (2.1)(a), $\mathfrak{M}\langle\varkappa\rangle$ is said to be a *Cartesian-quotient extension* of $\mathfrak{M}$, while in the case (2.1)(b), the model $\mathfrak{M}\langle\varkappa\rangle$ is said to be a *Cartesian extension* of $\mathfrak{M}$ *by a sequence of formulas* $\varkappa$.

Mention some kind of determinism for the operation under consideration.

Lemma 2.1. [26, Lem. 2.1+Sect. 3] *Given a model $\mathfrak{N}$ of signature σ and a tuple $\varkappa$ of the form* (2.1)(*a*). *For a fixed choice of signature* (2.2), *Cartesian-quotient extension $\mathfrak{M} = \mathfrak{N}\langle\varkappa\rangle$ of the model $\mathfrak{N}$ is defined uniquely, up to an isomorphism over $\mathfrak{N}$. Moreover, we have $|\mathfrak{M}| = \mathrm{acl}(U(\mathfrak{M}))$. Thus, any automorphism $\lambda : \mathfrak{N} \to \mathfrak{N}$ can be extended, by a unique way, up to an automorphism $\lambda^* : \mathfrak{N}\langle\varkappa\rangle \to \mathfrak{N}\langle\varkappa\rangle$.*

Proof. This statement is a simple consequence of the construction. □

Expand the operation of an extension (initially defined for models) on theories. Given a theory T and a tuple $\varkappa$ of the form (2.1). Using a fixed signature (2.2) for extensions of models, we define a new theory $T' = T\langle\varkappa\rangle$ as follows: $T' = \mathrm{Th}(K)$, $K = \{\mathfrak{M}\langle\varkappa\rangle \mid \mathfrak{M} \in \mathrm{Mod}(T)\}$. In the case (2.1)(a) it is called a *Cartesian-quotient* *extension*, while in the case (2.1)(b) it is called a *Cartesian extension* of T *by a sequence* $\varkappa$.

Lemma 2.2. [26, Lem. 2.2] *For any model* $\mathfrak{M}$ *of theory* $T\langle\varkappa\rangle$*, there is a model* $\mathfrak{N}$ *of theory* T *such that* $\mathfrak{M} \cong \mathfrak{N}\langle\varkappa\rangle$.

Proof. Immediately, from the description of the operation $T \mapsto T\langle\varkappa\rangle$. □

In theory $T\langle\varkappa\rangle$, the domain $U(x)$ represents a model of theory T. Particularly, the transformation $T \mapsto T\langle\varkappa\rangle$ defines a natural interpretation $I_{T,\varkappa}$ of T in $T\langle\varkappa\rangle$. It is called a *plain Cartesian-quotient* interpretation. Similar definition applies to the other case of the tuple $\varkappa$; thereby, the concept of a *plain Cartesian* interpretation is also defined. Considering theories up to an algebraic isomorphism, we may use a simpler term *Cartesian-quotient* or, respectively, *Cartesian* interpretation, cf. [26, Def. 2.A].

We study main properties of plain Cartesian-type interpretations.

Lemma 2.3. [26, Lem. 2.3] *Given a theory* T *of signature* σ *and a tuple* $\varkappa$ *of the form* (2.1)(*a*)*. For a fixed choice of signature* (2.2)*, Cartesian-quotient interpretation* $I_{T,\varkappa} : T \rightarrowtail T\langle\varkappa\rangle$ *has the following properties*:

(a) *the model-kernel passage is defined by rule* $\mathbb{K}(\mathfrak{N}\langle\varkappa\rangle) = \mathfrak{N}$*, for all* $\mathfrak{N} \in \mathrm{Mod}(T)$,

(b) $I_{T,\varkappa}(\varphi) = (\varphi)_U$*, for all* $\varphi \in SL(\sigma)$,

(c) $I_{T,\varkappa}$ *is* $\exists\cap\forall$*-presentable*,

(d) $I_{T,\varkappa}$ *is effective, faithful, auto-free, model-bijective, and isostone*,

(e) *interpretation* $I_{T,\varkappa}$ *determines in accordance with rule* (1.1) *a computable isomorphism* $\mu_{T,\varkappa} : \mathcal{L}(T) \to \mathcal{L}(T\langle\varkappa\rangle)$ *between the Tarski-Lindenbaum algebras.*

Proof. (a), (b), (c) Immediately, from construction.

(d) Effectiveness of the interpretation is checked immediately. By Lemma 2.1 and Lemma 2.2, the mapping of passage to the model-kernel is a one-to-one correspondence between isomorphism types of models of the classes $\mathrm{Mod}(T\langle\varkappa\rangle)$ and $\mathrm{Mod}(T)$; thereby, interpretation $I_{T,\varkappa}$ is model bijective. By Lemma 1.3, the interpretation $I_{T,\varkappa}$ is faithful, model-free, and isostone.

(e) By applying Lemma 1.3. □

Normally, we consider passages $T \mapsto T\langle\varkappa\rangle$ for which sequence (2.1) satisfies the following technical condition:

$$\varphi_k(\bar{x}_k) \textit{ and } \varepsilon_k(\bar{y}_k, \bar{z}_k) \textit{ are } \exists\cap\forall\textit{-presentable}, \textit{ for all } k \leqslant s. \tag{2.4}$$

Denote by $\mathcal{K}D(\sigma)$ and $\mathcal{K}C(\sigma)$ the sets of tuples of formulas of signature σ of the forms, respectively, (2.1)(a) and (2.1)(b), while $\mathcal{K}D$ and $\mathcal{K}C$ are unions of these sets for all possible (enumerable) signatures σ. We denote by $\mathcal{K}C_{\exists\cap\forall}$ the set of all tuples (2.1)(b) satisfying (2.4), while $\mathcal{K}D^{\varepsilon}_{\exists\cap\forall}$ denotes the set of all tuples (2.1)(a) satisfying (2.4). By applying an entry $T\langle\varkappa\rangle$, we always suppose that theory T is applicable to the tuple $\varkappa$, while if we use an entry $T\langle\varkappa\rangle$ with $\varkappa$ in either $\mathcal{K}C_{\exists\cap\forall}$ or $\mathcal{K}D^{\varepsilon}_{\exists\cap\forall}$, we count that $T \vdash \mathit{DomEA}(\varkappa)$ ensuring that each of the formulas $\varphi_k(\bar{x}_k)$ and $\varepsilon_k(\bar{y}_k\bar{z}_k)$, $i = 1, ..., m$, in the tuple $\varkappa$ is $\exists\cap\forall$-presentable in T.

When using an extra specifier *algebraic*, we explicitly indicate that the *algebraic approach* is accepted, i.e., demands (2.4) for the passage $T \mapsto T\langle\varkappa\rangle$ take place. For instance, passage $T \mapsto T\langle\varkappa\rangle$ is called an *algebraic Cartesian-quotient extension* whenever $\varkappa \in \mathcal{K}D^{\varepsilon}_{\exists\cap\forall}$, interpretation $I_{T,\varkappa}$ is called a *plain algebraic Cartesian interpretation* if $\varkappa \in \mathcal{K}C_{\exists\cap\forall}$, etc.

In this paper, we systematically follow the algebraic approach. Moreover, we focus our attention on the case of Cartesian extensions (2.1)(b). As for the common case (2.1)(a) of a Cartesian-quotient extension, we concern this case of the operation just for comparison purposes.

Let us study formal properties of Cartesian-quotient extensions of theories.

Consider a theory T of signature σ together with a sequence $\varkappa \in \mathit{KD}(\sigma)$. New domains $U_i(x)$, $i = 1, 2, ..., s$, are obtained by applying the standard quotient construction of first-order definable sets modulo definable equivalences in theory T. These relations are presented in theory $T\langle\varkappa\rangle$ by the following formulas:

$$(2.5) \qquad \begin{aligned} &\text{(a)} \quad \check{\varphi}_k(\bar{x}) = (\bar{x} \subseteq U) \,\&\, (\varphi_k(\bar{x}))_U\,, \\ &\text{(b)} \quad \check{\varepsilon}'_k(\bar{x}, \bar{y}) = (\bar{x} \subseteq U) \,\&\, (\bar{y} \subseteq U) \,\&\, (\varepsilon'_k(\bar{x}, \bar{y}))_U\,. \end{aligned}$$

Now, we formalize the operation of a Cartesian-quotient extension $T \mapsto T\langle\varkappa\rangle$, $\varkappa \in \mathit{KD}$, in accordance with the informal description given earlier in this section.

System of axioms of theory $T\langle\varkappa\rangle$ includes the following sentences:

1°. $(\exists x)\, U(x)$,

2°. $(\exists x)\, U_i(x)$, $i = 1, 2, ..., s$,

3°. $(\forall x)\, [\, U(x) \to \neg U_i(x)\,]$, $i = 1, 2, ..., s$,

4°. $(\forall x)\, [\, U_i(x) \to \neg U_j(x)\,]$, $1 \leqslant i < j \leqslant s$,

5°. All σ-predicates are defined trivially outside the domain $U(x)$,

6°. All σ-functions are defined trivially outside the domain $U(x)$,

7°. $(\Phi)_U$, for all $\Phi \in \mathit{SL}(\sigma)$, such that $\Phi \in \Sigma$ (Σ is a set of axioms of T),

8°. $(\forall x_1 ... x_{m_k} z)\, \big[\, K_k(x_1, ..., x_{m_k}, z) \to U(x_1) \,\&\, ... \,\&\, U(x_{m_k}) \,\&\, U_k(z)\, \big]$, $k = 1, ..., s$,

9°. $(\forall \bar{x} z)\, \big[\, K_k(\bar{x}, z) \to \bar{x} \subseteq U \,\&\, \check{\varphi}_k(\bar{x}) \,\&\, U_k(z)\, \big]$, $k = 1, ..., s$,

10°. $(\forall \bar{x})\, \big[\, \bar{x} \subseteq U \,\&\, \check{\varphi}_k(\bar{x}) \to (\exists z) K_k(\bar{x}, z)\, \big]$, $k = 1, ..., s$,

11°. $(\forall z)\, \big[U_k(z) \to (\exists \bar{x})\, \big(\, \bar{x} \subseteq U \,\&\, \check{\varphi}_k(\bar{x}) \,\&\, K_k(\bar{x}, z)\, \big)\, \big]$, $k = 1, ..., s$,

12°. $(\forall \bar{x}\, \bar{y} z u)\, \big[\, \check{\varphi}_k(\bar{x}) \,\&\, \check{\varphi}_k(\bar{y}) \,\&\, \check{\varepsilon}'_k(\bar{x}, \bar{y}) \,\&\, K_k(\bar{x}, z) \,\&\, K_k(\bar{y}, u) \;\to\; z = u \big]$, $k = 1, ..., s$,

13°. $(\forall \bar{x}\, \bar{y}\, z)\, \big[\check{\varphi}_k(\bar{x}) \,\&\, \check{\varphi}_k(\bar{y}) \,\&\, K_k(\bar{x}, z) \,\&\, K_k(\bar{y}, z) \to \check{\varepsilon}'_k(\bar{x}, \bar{y})\, \big]$, $k = 1, ..., s$,

14°. $(\forall \bar{x}\, \bar{y}\, z)\, \big[\, \check{\varphi}_k(\bar{x}) \,\&\, \check{\varphi}_k(\bar{y}) \,\&\, K_k(\bar{x}, z) \,\&\, \check{\varepsilon}'(\bar{x}, \bar{y}) \to K_k(\bar{y}, z)\, \big]$, $k = 1, ..., s$.

By $\mathit{FRM}(\varkappa)$, we denote the set of sentences included in Axioms 1°-6° and 8°-14°. The set $\mathit{FRM}(\varkappa)$ is called the *framework* of the operation $T \mapsto T\langle\varkappa\rangle$. This part of axioms participates in the operation with the same tuple $\varkappa$ for all input theories T. By construction, the set of sentences $\mathit{FRM}(\varkappa)$ is finite, it does not depend on theory T, and we have the following presentation for all theories T of signature σ:

$$(2.6) \qquad T\langle\varkappa\rangle = \big[\; \mathit{FRM}(\varkappa) + \{\, I(\varphi) \mid \varphi \in \mathit{SL}(\sigma),\; T \vdash \varphi\, \}\, \big]^{\sigma_1}.$$

Actually, $\mathit{FRM}(\varkappa)$ depends not only on $\varkappa$, but also on signature σ of theory T, and on a signature (2.2) fixed for the construction $T \mapsto T\langle\varkappa\rangle$.

Lemma 2.4. *Given a theory T of signature σ together with a tuple of formulas $\varkappa \in \mathit{KC}_{\exists\cap\forall}(\sigma)$. Consider computable isomorphism $\mu_{T,\varkappa}$: $\mathcal{L}(T) \to \mathcal{L}(T\langle\varkappa\rangle)$ defined in Lemma* 2.3. *For an arbitrary theory $T' \supseteq T$ and corresponding theory $S' \supseteq T\langle\varkappa\rangle$ linked by $S' = \mu_{T,\varkappa}(T')$, as pointed out in* (0.2), *we have $S' = T'\langle\varkappa\rangle$.*

Proof. By Lemma 2.3(d), interpretation $I_{T,\varkappa} : T \rightarrowtail T\langle\varkappa\rangle$ is model-bijective, while Lemma 2.3(a) establishes details of the model-kernel operation for the interpretation. On the one hand, we obtain from requirement (1.5)(a) that the class of models of theory $S' = \mu_{T,\varkappa}(T')$ equals to $M = \{\mathbb{K}(\mathfrak{N}) \mid \mathfrak{N} \in \mathrm{Mod}(T')\}$. On the other hand, from relation (2.6) considered with respect to theories T' and $T'\langle\varkappa\rangle$ we obtain that $\mathrm{Mod}(T'\langle\varkappa\rangle)$ is equal to the same class M. From this we obtain finally that $\mu_{T,\varkappa}(T') = S' = T'\langle\varkappa\rangle$. $\square$

3. Preservation of the property of being a model complete theory

In this section, we present main statements of the paper.

Theorem 3.1. *Given a theory T of signature σ together with a tuple of formulas $\varkappa \in \mathcal{K}D(\sigma)$. The following assertions are held*:

(a) *if theory T is model complete, the theory $T\langle\varkappa\rangle$ is also model complete,*

(b) *if formulas $\varphi_k(\bar{x}_k)$ and $\varepsilon'_k(\bar{y}_k, \bar{z}_k)$, $k = 1, 2, ..., s$, are $\exists \cap \forall$-presentable in T and theory $T\langle\varkappa\rangle$ is model complete, the theory T is also model complete,*

(c) *if formulas $\varphi_k(\bar{x}_k)$ and $\varepsilon_k(\bar{y}_k, \bar{z}_k)$, $k = 1, 2, ..., s$, are $\exists \cap \forall$-presentable in theory T and T is complete, $T\langle\varkappa\rangle$ is model complete if and only if T is model complete.*

Proof. (a) Suppose that T is model complete. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models of theory $T\langle\varkappa\rangle$ such that $\mathfrak{M} \subseteq \mathfrak{M}'$. By construction, signature symbols of T are defined in the domain $U(x)$ of theory $T\langle\varkappa\rangle$. From this, we obtain $U(\mathfrak{M}) = U(\mathfrak{M}') \cap |\mathfrak{M}|$; moreover, we have $\mathbb{K}(\mathfrak{M}) \subseteq \mathbb{K}(\mathfrak{M}')$. By virtue of model completeness of T, we have $\mathbb{K}(\mathfrak{M}) \preccurlyeq \mathbb{K}(\mathfrak{M}')$ ensuring that the identical mapping $f : \mathbb{K}(\mathfrak{M}) \to \mathbb{K}(\mathfrak{M}')$ is an elementary embedding of these models of theory T. By applying Lemma 1.2, we obtain that $f : \mathfrak{M} \restriction U(\mathfrak{M}) \to \mathfrak{M}' \restriction U(\mathfrak{M}')$ is an elementary embedding of subsets in the models of theory $T\langle\varkappa\rangle$. Therefore, it is possible to extend f to an elementary embedding f^* of the whole model $\mathfrak{M}$ into a suitable elementary extension $\mathfrak{M}''$ of $\mathfrak{M}'$. By virtue of Lemma 2.1, each element in the image $f^*(\mathfrak{M})$ is first-order definable over its domain $U(f^*(\mathfrak{M}))$ coinciding with $U(\mathfrak{M})$, thus, the set $f^*(\mathfrak{M})$ must be a subset of $|\mathfrak{M}'|$. As a result, we obtain $\mathfrak{M} \preccurlyeq \mathfrak{M}'$ ensuring that the target theory $T\langle\varkappa\rangle$ is model complete.

(b) Now, we suppose that conditions of Part (b) are held and theory $T\langle\varkappa\rangle$ is model complete. Consider models $\mathfrak{N}$ and $\mathfrak{N}'$ of theory T such, that $\mathfrak{N} \subseteq \mathfrak{N}'$. By construction, we can find a model $\mathfrak{M}'$ of theory $T\langle\varkappa\rangle$ such that $\mathfrak{N}' = \mathbb{K}(\mathfrak{M}')$. Since formulas $\varphi_i(\bar{x}_i)$ and $\varepsilon_i(\bar{y}_i, \bar{z}_i)$, $i = 1, 2, ..., s$, are $\exists \cap \forall$-presentable in theory T, their domains of true in $\mathfrak{N}$ are restrictions on $|\mathfrak{N}|$ of their domains of true computed in $\mathfrak{N}'$. This allows us to define a model $\mathfrak{M}$ of theory $T\langle\varkappa\rangle$ with the kernel $\mathfrak{N}$ as a submodel in the available model $\mathfrak{M}'$, i.e., we have $U(\mathfrak{M}) = |\mathfrak{N}|$ and $\mathfrak{N} = \mathbb{K}(\mathfrak{M})$ for the model $\mathfrak{M}$. Since theory $T\langle\varkappa\rangle$ is model complete, embedding $\mathfrak{M} \subseteq \mathfrak{M}'$ implies elementary embedding $\mathfrak{M} \preccurlyeq \mathfrak{M}'$. From this, we immediately obtain $\mathfrak{N} \preccurlyeq \mathfrak{N}'$ ensuring that the theory T is model complete.

(c) In the case when theory T is complete, each sentence $\mathrm{Equiv}(\varepsilon_i, \varphi_i)$, $i = 1, 2, ..., s$, is either identically true or identically false in T. From (2.3) we obtain that, for each i, formula $\varepsilon'_i(\bar{y}_i, \bar{z}_i)$ is $\exists \cap \forall$-presentable in T if and only if the formula $\varepsilon_i(\bar{y}_i, \bar{z}_i)$ is $\exists \cap \forall$-presentable in T. By applying Part (b), we obtain exactly what is required. $\square$

Corollary 3.2. *Given a theory T and a tuple of formulas $\varkappa \in \mathcal{K}C_{\exists\cap\forall}$. Theory $T\langle\varkappa\rangle$ is model complete if and only if the theory T is model complete.*

Proof. In this case, formulas $\varepsilon_i(\bar{y}_i, \bar{z}_i)$ are simple equalities $\bar{y}_i = \bar{z}_i$. In particular, each formula $\varepsilon_i(\bar{y}_i, \bar{z}_i)$ coincides with $\varepsilon'_i(\bar{y}_i, \bar{z}_i)$, $i = 1, 2, ..., s$. Thus, these formulas are $\exists\cap\forall$-presentable in T. By applying Theorem 3.1(a,b), we obtain exactly what is required. $\square$

Corollary 3.3. *Given a complete theory T and a tuple of formulas $\varkappa \in \mathcal{K}D^{\varepsilon}_{\exists\cap\forall}$. Theory $T\langle\varkappa\rangle$ is model complete if and only if the theory T is model complete.*

Proof. By applying Theorem 3.1(a,c). $\square$

Theorem 3.4. *Given a theory T of signature σ together with a tuple of formulas $\varkappa \in \mathcal{K}C_{\exists\cap\forall}(\sigma)$. Consider computable isomorphism $\mu_{T,\varkappa}\colon \mathcal{L}(T) \to \mathcal{L}(T\langle\varkappa\rangle)$ defined in Lemma 2.3. For an arbitrary theory $T' \supseteq T$ and corresponding theory $S' \supseteq T\langle\varkappa\rangle$ linked by $S' = \mu_{T,\varkappa}(T')$, as pointed out in (0.2), we have: theory S' is model complete if and only if theory T' is model complete.*

Proof. Relations (0.3) characterize transition rules (0.2) between the extension $T' \supseteq T$ and corresponding extension $S' \supseteq T\langle\varkappa\rangle$, $S' = \mu_{T,\varkappa}(T')$. By virtue of Lemma 2.4, the theories T' and S' are linked by the equality $S' = T'\langle\varkappa\rangle$. Thus, Corollary 3.2 ensures that theory S' is model complete if and only if the theory T' is model complete. $\square$

Proposition 3.5. *There is a theory T of a finite signature together with a tuple of formulas $\varkappa \in \mathcal{K}D^{\varepsilon}_{\exists\cap\forall}$ such that theory T is not model complete; however, theory $T\langle\varkappa\rangle$ is model complete.*

Proof. For theory T, we use a pure predicate signature $\sigma = \{A^1, B^1, \varGamma^2\}$.

Axioms of T are the following statements:

1°. $A(x) \leftrightarrow \neg B(x)$,
2°. $(\exists^{\geqslant k} x)A(x)$, $k < \omega$,
3°. $(\exists^{\geqslant k} x)B(x)$, $k < \omega$,
4°. predicate $\varGamma(x,y)$ is symmetric and antireflexive,
5°. $\varGamma(x,y) \to \big(A(x) \,\&\, A(y)\big) \vee \big(B(x) \,\&\, B(y)\big)$,
6°. $(\exists x\, y {\in} A)\,\big[x \neq y \;\&\; (\forall u\, v {\in} A)\big(u \neq v \,\&\, \{u,v\} \neq \{x,y\} \to \varGamma(u,v)\big)\big]$,
7°. $(\exists x\, y {\in} B)\,\big[x \neq y \;\&\; (\forall u\, v {\in} B)\big(u \neq v \,\&\, \{u,v\} \neq \{x,y\} \to \varGamma(u,v)\big)\big]$,
8°. $(\exists x\, y {\in} A)\,\big[x \neq y \;\&\; \neg\varGamma(x,y)\big] \leftrightarrow (\exists u\, v {\in} B)\,\big[u \neq v \;\&\; \neg\varGamma(u,v)\big]$.

We put $\varkappa$ to be equal to $\langle\varphi(x)/\varepsilon(y,z)\rangle$, where $\varphi(x) = (x = x)$, and $\varepsilon(y,z) = (y = z) \vee \varGamma(y,z)$. Obviously, both $\varphi(x)$ and $\varepsilon(y,z)$ are $\exists\cap\forall$-presentable in T. Any model $\mathfrak{N}$ of theory T consists of two disjoin domains A and B, each having an infinite cardinality; moreover, $\varGamma$ represents two separate graphs within A and within B, such that these graphs are either total, or almost total, linking all elements excepting just one pair. Isomorphism types of models of theory T can be characterized by the expressions $(\alpha^{\ominus}, \beta^{\ominus})$ and (α, β), where α and β are infinite cardinals indicating powers of the domains A and, respectively, B, while an upper index $\ominus$ points out that $\varGamma$ is not a full graph in the corresponding domain. There is an isomorphic embedding of a model of type (α, β) into a model of type $(\alpha^{\ominus}, \beta^{\ominus})$; moreover, this embedding is not elementary. Thus, theory T is not model complete. As for theory $T\langle\varkappa\rangle$, embeddings between its models become limited by virtue of the Cartesian superstructure. The sentence $\mathrm{Equiv}(\varepsilon, \varphi)$ is true in models of type (α, β). In these models, $\varepsilon'(y,z)$ is an equivalence relation consisting of two classes. In models of the other type $(\alpha^{\ominus}, \beta^{\ominus})$, $\mathrm{Equiv}(\varepsilon, \varphi)$ is failed, thus, ensuring $\varepsilon'(y,z)$ to be an equivalence relation with the only class. Based on this, it is possible to establish that theory $T\langle\varkappa\rangle$ is model complete. $\square$

4. Final Remarks

Section 3 presents main results of this paper establishing that algebraic Cartesian interpretations preserve the property of being a model complete theory. In particular, Theorem 3.4 establishes statement of Item 5 from Theorem 1 in the abstract [10]. An extra example in Proposition 3.5 shows that a more common class of algebraic Cartesian-quotient interpretations does not preserve the property of being a model complete theory.

Investigations on the expressive possibilities of first-order logic operate with many concepts closely interacting with each other. Therefore, proofs of the statements lead to large texts. In this paper, one specific result is presented in a compact close presentation as part of a more general result in this direction, [11], [10].

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