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# WEIGHTED SOBOLEV SPACES, CAPACITIES AND EXCEPTIONAL SETS

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Abstract. We consider the weighted Sobolev space $W^{m,p}_\omega(\Omega)$, where $\Omega$ is an open subset of $R^n$, $n \geq 2$, and $\omega$ is a Muckenhoupt $A_p$-weight on $R^n$, $1 \leq p < \infty$, $m \in \mathbb{N}$. For the equalities $W^{m,p}_\omega(\Omega \setminus E) = W^{m,p}_\omega(\Omega)$, $\overset{\circ}{W}{}^{m,p}_\omega(\Omega \setminus E) = \overset{\circ}{W}{}^{m,p}_\omega(\Omega)$ to hold, conditions are obtained in terms of $E$ as a set of zero $(p, m, \omega)$-capacity, or an $NC_{p,\omega}$-set for the first equality. For the equality $W^{m,p}(\Omega) = \overset{\circ}{W}{}^{m,p}(\Omega)$, the conditions are established for $R^n \setminus \Omega$ as a set of zero $(p, m, \omega)$-capacity. Similar results are partially true for $W^m_{p,\omega}(\Omega)$, $L^m_{p,\omega}(\Omega)$.

**Keywords:** Sobolev space, capacity, Muckenhoupt weight, exceptional set.

## 1. Introduction

Suppose that $\Omega$ is an open set on the Euclidean space $R^n$, $n \geq 2$, and $E$ is a relatively closed subset on $\Omega$. Let $W(\Omega)$ be a Sobolev space with a norm (with a semi-norm) $\|\cdot\|_{W(\Omega)}$, whose elements are functions (classes of equivalent functions) defined on $\Omega$, and whose partial derivatives satisfy certain integrability conditions. We denote the closure of $C^\infty_0(G)$ on $W(\Omega)$ by $\overset{\circ}{W}(\Omega)$.

The following problems are well-known in the theory of Sobolev spaces: find the conditions for $E$ which need to be satisfied for the equalities $W(\Omega \setminus E) = W(\Omega)$ (problem (i)); $\overset{\circ}{W}(\Omega) = \overset{\circ}{W}(\Omega \setminus E)$ (problem (ii)); $W(\Omega) = \overset{\circ}{W}(R^n \setminus E)$, where $E = R^n \setminus \Omega$ (problem (iii)), to hold respectively. More information about the equality of spaces can be found in Remark 1 below.

In problems (i)–(iii), the set $E$, for which the equalities are realized, is called exceptional. In particular, with regard to the equation $W(\Omega) = L^1_p(\Omega)$, the criterion for the

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set $E$ to be exceptional in (i) was obtained by S. Vodop'yanov and V. Gol'dstein [15] in terms of $E$ as an $NC_p$-set, $1 < p < \infty$. L. Hedberg obtained a criterion for the exceptional set $E$ in problem (ii) as a set of zero $p$-capacity, where in our notation $W(\Omega) = L^1_p(\Omega)$, $1 < p < \infty$, and $\Omega$ is a bounded set in $R^n$ [7, Theorems 1,2].

For a weighted space $H^{1,p}(\Omega, \mu)$ with a $p$-admissible measure $\mu$, $1 < p < \infty$, all three problems (i)–(iii) were solved in [8, Theorems 2.43–2.45] in terms of $E$ as a set of zero Sobolev $(p, \mu)$-capacity. The necessary and sufficient conditions for an exceptional set $E$ in (ii)–(iii) for $W(\Omega) = W^{m,p}(\Omega)$ are provided in [1, Theorems 3.28, 3.33] in terms of $E$ as a $(m, p)$-polar set. The criteria for the exceptional set $E$ in (iii) for $W(\Omega) = W^m_p(\Omega)$ are obtained in [10, §9.2, Theorems 1,2] in terms of $E$ as a $(m, p)$-polar set and a set of zero $(p, m)$-capacity.

In this paper, the criteria for the exceptional set $E$ in (i) for $W(\Omega) = W^{m,p}_\omega(\Omega)$, $W^m_{p,\omega}(\Omega), L^m_{p,\omega}(\Omega)$, $1 < p < \infty$, and $m \in \mathbb{N}$, are established in terms of $E$ as an $NC_{p,\omega}$-set, see Theorems 6,7. The criteria for the exceptional set $E$ in (ii) for $W(\Omega) = W^{m,p}_\omega(\Omega), W^m_{p,\omega}(\Omega)$ and (iii) for $W^{m,p}_\omega(\Omega)$ are established in terms of $E$ as a set of zero $(p, m, \omega)$-capacity, where $1 \le p < \infty$, $m \in \mathbb{N}$, see Theorems 9,10.

In addition, sufficient condition for the exceptional set $E$ in (i) for

$$W(\Omega) = W^{m,p}_\omega(\Omega), W^m_{p,\omega}(\Omega), L^m_{p,\omega}(\Omega)$$

are given in terms of $E$ as a zero $(p, m, \omega)$-capacity set, where $1 \le p < \infty$, $m \in \mathbb{N}$, see Theorem 8, Corollary 6.

## 2. Preliminaries

2.1. **Some definitions and notations.** Throughout the text, $\Omega$ is used to denote an open set on $R^n = \{x = (x_1, \dots, x_n)\}$, while $E$ denotes a relatively closed subset on $\Omega$. The norm of a point $x = (x_1, \dots, x_n) \in R^n$ has the form $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. We put $\mathbb{N} = \{1, 2, \dots\}$, $R = (-\infty, +\infty)$. If $F \subset R^n$, then $\partial F, \bar{F}$ denote the boundary and the closure of $F$ on $R^n$, respectively. The distance between two sets $A, B \subset R^n$ is denoted by $\operatorname{dist}(A, B)$.

For an open set $U \subset R^n$, we use the notation $U \Subset \Omega$ in order to indicate that $U$ is bounded and $\bar{U} \subset \Omega$. The restriction of the function $f$ to the set $F$ is denoted by $f|_F$. Let $\chi_F$ be a characteristic function of the set $F$.

Given $x \in R^n$ and $r > 0$, suppose that $B(x, r)$ or $B_r(x) = \{y \in R^n : |y - x| < r\}$. If $a > 0$, then we have that $aB_r(x) = B_{ar}(x)$. We use the symbol $\mathcal{H}^s$ to denote an ordinary $s$-dimensional Hausdorff measure on $R^n$; $m_n$ is Lebesgue measure on $R^n$, and we put $m_n(F) = |F|$.

Let $C^\infty(\Omega)$ be a space of infinitely differentiable functions on $\Omega$; the space of functions in $C^\infty(R^n)$ with a compact support on $\Omega$ is denoted by $C^\infty_0(\Omega)$.

The support of a function $u$ will be denoted by $\operatorname{supp} u$.

For $1 \le p < \infty$, we define $L_p(\Omega)$ as a set of $m_n$-measurable functions $f$ on $\Omega$, such that

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f|^p \, dx\right)^{1/p} < \infty,$$

and suppose that $L_p(\Omega, loc)$ is a space of $m_n$-measurable functions $f$ on $\Omega$, such that $|f|^p$ is a locally integrable function on $\Omega$.

We will use the abbreviation "a.e." for the phrase "almost everywhere" with respect to $m_n$-measure. Similarly, when we use the words "measurable" and "locally integrable", we always mean "Lebesgue measurable" and "locally integrable with respect to $m_n$-measure".

For the case $\Omega = R^n$, we normally drop the reference to $\Omega$ in the notation of spaces and norms. Integration without specifying integration limits is extended to $R^n$ by agreement.

Within proofs of, say, theorems, the letter $C$ will be used to denote a generic positive constant which depends only on the parameters in the statement of the theorem. The quantities $A$ and $B$ are said to be "equivalent", if there exist two positive constants, $C_1$ and $C_2$, such that $C_1 A \le B \le C_2 A$.

If $\alpha = (\alpha_1, \dots, \alpha_n)$ is an $n$-tuple of non-negative integers $\alpha_i$, we call $\alpha$ a multi-index and denote by $x^\alpha$ the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$, which has a degree $|\alpha| = \sum\limits_{i=1}^{n} \alpha_i$.

Similarly, if $D_j = \frac{\partial}{\partial x_j}$, then $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ denotes a differential operator of order $|\alpha|$. Note that $D^{(0,0,\dots,0)} u = u$.

If $\alpha$ and $\beta$ are multi-indices, we say that $\beta \le \alpha$ provided that $\beta_i \le \alpha_i$ for $1 \le i \le n$. In this case, $\alpha - \beta$ is also a multi-index and $|\alpha - \beta| + |\beta| = |\alpha|$. Put $\alpha! = \alpha_1! \dots \alpha_n!$, then for $\beta \le \alpha$, we have that

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!} = \binom{\alpha_1}{\beta_1} \dots \binom{\alpha_n}{\beta_n}.$$

This allows us to write the Leibnitz formula in the form

$$D^\alpha (uv)(x) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u(x) D^{\alpha - \beta} v(x), \tag{1}$$

which holds for functions $u$ and $v$, that are $|\alpha|$ times continuously differentiable near $x$.

We use the notations $\nabla_m = \{D^\alpha : |\alpha| = m\}$, $\nabla = \nabla_1$.

By a weight we mean a locally integrable function $\omega$ on $R^n$, such that $\omega > 0$ for a.e. $x \in R^n$.

Then for $1 \le p < \infty$, we define $L_{p,\omega}(\Omega)$ as a set of measurable functions $f$ on $\Omega$, such that

$$\|f\|_{L_{p,\omega}(\Omega)} = \left( \int\limits_\Omega |f|^p \omega \, dx \right)^{1/p} < \infty.$$

As usual, any two functions $f$ and $g$ from $L_{p,\omega}(\Omega)$ that are equal a.e. on $\Omega$ will be identified. It is well-known (see [9, Theorem 2.7]) that $L_{p,\omega}(\Omega)$ is complete with respect to the norm $\| \cdot \|_{L_{p,\omega}(\Omega)}$.

Let $\mathcal{F}_1$ be a space of functions given on $\Omega$, and $\mathcal{F}_2$ be another space of functions given on $\Omega'$, where $\Omega' \subset \Omega$. Below, if $f \in \mathcal{F}_1$, then $f \in \mathcal{F}_2$ implies that $f|_{\Omega'} \in \mathcal{F}_2$.

We denote by $L_{p,\omega}(\Omega, loc)$ a set of all $m_n$-measurable functions $f$ on $\Omega$, such that $f \in L_{p,\omega}(\Omega')$ for all open sets $\Omega' \Subset \Omega$.

2.2. $A_p$**-weights.** Suppose that $1 \le p < \infty$. According to B. Muckenhoupt [11], a weight $\omega$ is called an $A_p$-weight, if there exists a positive constant $A$, such that for every ball $B \subset R^n$, the inequality

$$\left( \frac{1}{|B|} \int\limits_B \omega \, dx \right) \left( \frac{1}{|B|} \int\limits_B \omega^{-\frac{1}{p-1}} \, dx \right)^{p-1} \le A, \tag{2}$$

holds, if $p > 1$, and

$$\left( \frac{1}{|B|} \int\limits_B \omega \, dx \right) \operatorname*{ess\,sup}_{x \in B} \frac{1}{\omega(x)} \le A, \tag{3}$$

holds, if $p = 1$. The infimum of all such constants $A$ is called the $A_p$-constant of $\omega$. We denote by $A_p$, $1 \le p < \infty$, a set of $A_p$-weights. Throughout the text, suppose that $1 \le p < \infty$, $m \in \mathbb{N}$, $\omega \in A_p$, unless otherwise stated.

We should mention one result concerning $A_p$-weight [14, Remark 1.2.4].

**Proposition 1.** *If $\omega \in A_p$, then $L_{p,\omega}(\Omega)$ is a complete space with respect to the norm $\|\cdot\|_{L_{p,\omega}(\Omega)}$, and $L_{p,\omega}(\Omega) \subset L_1(\Omega, loc)$.*

2.3. **Weighted Sobolev spaces.** Suppose that $u : \Omega \to R$ is a function of class $L_1(\Omega, loc)$. The function $u$ on $\Omega$ has a weak derivative of order $|\alpha|$, if there is a locally integrable function (denoted by $D^\alpha u$), such that

$$\int_\Omega u \cdot D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u \cdot \varphi dx$$

for all $\varphi \in C_0^\infty(\Omega)$. For $1 \le p < \infty$, $m \in \mathbb{N}$ and every $\omega \in A_p$, $L^m_{p,\omega}(\Omega)$ is a space of functions which have weak derivatives $D^\alpha u$ of all orders $|\alpha|$, $|\alpha| \le m$, and that satisfy the condition

$$\|f\|_{L^m_{p,\omega}(\Omega)} = \left( \int_\Omega |\nabla_m u|^p \omega \, dx \right)^{1/p} < \infty,$$

where $|\nabla_m u| = \left( \sum_{|\alpha|=m} (D^\alpha u)^2 \right)^{1/2}$. For $m = 0$, set $L^m_{p,\omega}(\Omega) = L_{p,\omega}(\Omega)$, $\nabla_0 u = u$.

We introduce the spaces

$$W^m_{p,\omega}(\Omega) = L^m_{p,\omega}(\Omega) \cap L_{p,\omega}(\Omega), \qquad W^{m,p}_\omega(\Omega) = \bigcap_{k=0}^m L^k_{p,\omega}(\Omega),$$

equipped with the norms

$$\|u\|_{W^m_{p,\omega}(\Omega)} = \|u\|_{L^m_{p,\omega}(\Omega)} + \|u\|_{L_{p,\omega}(\Omega)}, \qquad \|u\|_{W^{m,p}_\omega(\Omega)} = \sum_{k=0}^m \|\nabla_k u\|_{L_{p,\omega}(\Omega)}.$$

We denote by $\overset{\circ}{L}{}^m_{p,\omega}(\Omega)$, $\overset{\circ}{W}{}^m_{p,\omega}(\omega)$, $\overset{\circ}{W}{}^{m,p}_\omega(\Omega)$ the closures of $C_0^\infty(\Omega)$ in $L^m_{p,\omega}(\Omega)$, $W^m_{p,\omega}(\omega)$, $W^{m,p}_\omega(\Omega)$, respectively. In addition, we set $W^{m,p}_\omega(\Omega, loc) = \bigcap_{\Omega'} W^{m,p}_\omega(\Omega')$, where the intersection is taken over all open sets $\Omega' \Subset \Omega$. Below, $W^{m,p}_\omega(\Omega, loc)$ will be considered a countably normed space with a system of semi-norms $\|u\|_{W^{m,p}_\omega(\Omega_k)}$. Here, $\{\Omega_k\}_{k\ge1}$ is a sequence of open sets $\Omega_k$, $\Omega_k \Subset \Omega_{k+1} \subset \Omega$, $\bigcup_k \Omega_k = \Omega$.

For the case when $\omega \equiv 1$, the weighted spaces considered above with the weight $\omega$ will be written below without the symbol $\omega$.

Next, let $\mathcal{P}_{m-1}$ be a collection of all polynomials of degree $\le m-1$. Consider the factor space $\check{L}^m_{p,\omega}(\Omega) = L^m_{p,\omega}(\Omega)/\mathcal{P}_{m-1}$ (with the norm $\|\cdot\|_{L^m_{p,\omega}(\Omega)}$). Elements of the space $\check{L}^m_{p,\omega}(\Omega)$ are classes $\check{u} = \{u + P\}$, where $u \in L^m_{p,\omega}(\Omega)$ and $P \in \mathcal{P}_{m-1}$.

Note that a number of important properties of spaces $W^{m,p}_\omega(\Omega)$, $L^m_{p,\omega}(\Omega)$ (in other notations and with equivalent norms) were obtained in [3, 14]. Below, we use the following properties.

**Proposition 2** ([3, Theorem 4.9])**.** *If $\Omega$ is an open connected set and $\omega \in A_p$, $1 \le p < \infty$, then $\check{L}^m_{p,\omega}(\Omega)$ is a Banach space. In particular, if $\{u_j\}$ is a Cauchy sequence in $L^m_{p,\omega}(\Omega)$, then there exists $u_0 \in L^m_{p,\omega}(\Omega)$, such that $\nabla_m u_j \to \nabla_m u_0$ in $L_{p,\omega}(\Omega)$ as $j \to \infty$.*

**Proposition 3** ([3, Corollary 4.10])**.** *Suppose that $\Omega$ is an open connected set, $\{u_j\}$ is a Cauchy sequence in $L^m_{p,\omega}(\Omega)$, and $u$ is a function in $L^m_{p,\omega}(\Omega)$, such that $\|\nabla_m(u_j - u)\|_{L_{p,\omega}(\Omega)} \to 0$. Then there exists a sequence of polynomials $\{P_j\} \subset \mathcal{P}_{m-1}$ with $u_i - P_j \to u$ in $L_{p,\omega}(K)$ for all compact sets $K \subset \Omega$.*

**Proposition 4** ([3, Theorem 4.2])**.** *Suppose that $1 \le p < \infty$, $\omega \in A_p$. If $u \in L^m_{p,\omega}(\Omega)$, then*

$$\int_K |D^\alpha u|^p \omega \, dx < \infty \tag{4}$$

*for all compact* $K \subset \Omega$, $0 \le |\alpha| \le m$.

**Proposition 5** ([14, Theorem 2.1.14]). *Suppose that* $\omega \in A_p$, $1 \le p < \infty$, *and* $k, m \in \mathbb{N}$, $1 \le k < m$. *Let* $B \subset R^n$ *be a ball. Then there is a positive constant* $C$ *depending only on* $k, m, p, n$ *and the* $A_p$*-constant of* $\omega$, *such that*

$$(5) \qquad \int_B |\nabla_k u|^p \omega\, dx \le C \left( |B|^{-\frac{kp}{n}} \int_B |u|^p \omega\, dx + |B|^{\frac{(m-k)p}{n}} \int_B |\nabla_m u|^p \omega\, dx \right)$$

*for all* $u \in W^{m,p}_{\omega}(B)$.

**Remark 1.** *If there is an isometric isomorphism between two normed or countably normed spaces* $X$ *and* $Y$, *then we have that* $X = Y$. *In particular,* $W^{m,p}_{\omega}(\Omega, loc) = W^{m,p}_{\omega}(\Omega \setminus E, loc)$ *implies that* $|E| = 0$, *and for every function* $u \in W^{m,p}_{\omega}(\Omega \setminus E, loc)$ *there is a function* $v \in W^{m,p}_{\omega}(\Omega, loc)$, *for which* $v|_{\Omega \setminus E} = u$. *Therefore, similar conditions can be written for* $W^{m,p}_{\omega}(\Omega)$, $W^m_{p,\omega}(\Omega)$, *and, by Proposition 4, for* $L^m_{p,\omega}(\Omega)$ *as subspaces of* $W^{m,p}_{\omega}(\Omega, loc)$. *For example,* $L^m_{p,\omega}(\Omega \setminus E) = L^m_{p,\omega}(\Omega)$ *implies that* $|E| = 0$, *and for every* $u \in L^m_{p,\omega}(\Omega \setminus E)$ *there is a function* $v \in L^m_{p,\omega}(\Omega)$, *for which* $v|_{\Omega \setminus E} = u$. *Similarly, for example, by* $\overset{\circ}{W}{}^m_{p,\omega}(\Omega \setminus E) = \overset{\circ}{W}{}^m_{p,\omega}(\Omega)$, *we mean that every function* $u \in \overset{\circ}{W}{}^m_{p,\omega}(\Omega)$ *can be approximated in* $\|\cdot\|_{W^m_{p,\omega}(\Omega)}$ *by functions from* $C_0^\infty(\Omega \setminus E)$. *Finally, for example, by* $W^m_{p,\omega}(\Omega) = \overset{\circ}{W}{}^m_{p,\omega}(\Omega)$, *we imply that every function* $u \in W^m_{p,\omega}(\Omega)$ *can be approximated in* $\|\cdot\|_{W^m_{p,\omega}(\Omega)}$ *by functions from* $C_0^\infty(\Omega)$.

**Remark 2.** *Suppose that* $u \in L^m_{p,\omega}(\Omega)$. *Then by virtue of Propositions 1,4, partial derivatives* $D^\alpha u$ *belong to the space* $W^{1,1}(\Omega, loc)$ *for all* $0 \le |\alpha| \le m-1$. *In addition,* $D^\alpha u$ *belongs to the space* $L^1_{p,\omega}(\Omega)$ *for every multi-index* $\alpha$ *of order* $m-1$. *Hence (see* [10, Sec. 1.1.3, Theorem 1], [13, Theorem 2.5]*), every partial derivative* $D^\alpha u$ *(perhaps, modified on a set of zero* $m_n$*-measure) is absolutely continuous in* $\Omega$ *on almost all straight lines (see* [13, p.19] *for a detailed discussion on "almost all straight lines") parallel to any coordinate axis,* $0 \le |\alpha| \le m-1$. *The weak gradient of* $D^\alpha u$ *coincides a.e. with the ordinary gradient. Conversely (see* [10, Sec. 1.1.3, Theorem 2]*), if every partial derivative* $D^\alpha u$ *is absolutely continuous on* $\Omega$ *on almost all lines which are parallel to the coordinate axes, and its first-order derivatives belong to* $L_{p,\omega}(\Omega, loc)$ *for* $0 \le |\alpha| < m-1$ *and to* $L_{p,\omega}(\Omega)$ *for* $|\alpha| = m-1$, *then* $u \in L^m_{p,\omega}(\Omega)$.

2.4. **Mollifications.** Let $\psi \in C_0^\infty(R^n)$ be a non-negative function, such that $\operatorname{supp}\psi \subset B_1(0)$ and $\int \psi(x)dx = 1$. For any function $u \in L_1(\Omega)$ extended by zero on $R^n \setminus \Omega$, we define the family of its mollifications by the equalities

$$(M_\varepsilon u)(x) = \varepsilon^{-n} \int u(y)\psi\left(\frac{y-x}{\varepsilon}\right) dy = \int_{|\xi|<1} u(x+\varepsilon\xi)\psi(\xi)\, d\xi, \quad 0 < \varepsilon \le 1.$$

The number $\varepsilon$ is called a radius of mollification.

The following result is well-known.

**Proposition 6** ([14, Theorem 2.1.4, Corollary 2.1.5]). *Suppose that* $u \in W^{m,p}_{\omega}(\Omega)$, *and let* $\Omega'$ *be an open set,* $\Omega' \Subset \Omega$. *Then* $(M_\varepsilon u)(x) \in C^\infty(\Omega) \cap L_{p,\omega}(\Omega)$, *and for* $0 < \varepsilon < \min(\operatorname{dist}(\Omega', \partial\Omega), 1)$ *the equality* $D^\alpha M_\varepsilon u = M_\varepsilon D^\alpha u$ *is true on* $\Omega'$, $1 \le |\alpha| \le m$; *and* $M_\varepsilon u \to u$ *holds on* $W^{m,p}_{\omega}(\Omega')$ *as* $\varepsilon \to 0$. *For the case when* $\Omega = R^n$, *we have a convergence* $M_\varepsilon u \to u$ *on* $W^{m,p}_{\omega}(R^n)$.

2.5. **Capacity and** $NC_{p,\omega}$**-sets.** A triple of sets $(F_0, F_1, \Omega)$, where $F_0$ and $F_1$ are disjoint compact subsets of $R^n$, is called a condenser. Suppose that $F_0 \cup F_1 \subset \bar{\Omega}$. Then we define (see [2, Proposition 5]) $(p,\omega)$-capacity of a condenser $(F_0, F_1, \Omega)$ by $C_{p,\omega}(F_0, F_1, \Omega) = 0$, if at least one of the following is true: $F_0 = \emptyset$ or $F_1 = \emptyset$. If $F_0$ and $F_1$ are nonempty sets, then the definition has the form

$$C_{p,\omega}(F_0, F_1, \Omega) = \inf_u \int_\Omega |\nabla u|^p \omega \, dx,$$

where the infimum is taken for all real-valued bounded functions $u$, such that $u|_\Omega \in C^\infty(\Omega) \cap L^1_{p,\omega}(\Omega)$ and $u = j$ in some neighborhood of $F_j$, $j = 0, 1$.

We denote the set of all admissible functions of this kind by $\mathrm{Adm}_{p,\omega}(F_0, F_1, \Omega)$.

In general, we define a $(p, \omega)$-capacity condenser $(F_0, F_1, \Omega)$ by

$$C_{p,\omega}(F_0, F_1, \Omega) = C_{p,\omega}(F_0 \cap \bar{\Omega}, F_1 \cap \bar{\Omega}, \Omega).$$

Consider a relatively closed subset $E \subset \Omega$, and let $\Pi$ be a coordinate rectangle

$$\{x = (x_1, \ldots, x_n) \in R^n : a_i < x_i < b_i, i = 1, \ldots, n\}.$$

We denote the facets of this rectangle parallel to the hyperplane $x_i = 0$ by $\sigma_{0i} \subset \{x : x_i = a_i\}$ and $\sigma_{1i} \subset \{x : x_i = b_i\}$. If

$$C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E) = C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi), \quad i = 1, 2, \ldots, n. \tag{6}$$

for every coordinate rectangle $\Pi$ with $\bar{\Pi} \subset \Omega$, then $E$ is called an $NC_{p,\omega}$-set in $\Omega$.

Similarly to the case $\omega = 1$ [15], the $NC_{p,\omega}$-set has zero $m_n$-measure (see [5, Lemma 5], [4, Theorem 1]) and $\tau \setminus E$ is an open connected set for every connected component $\tau$ of $\Omega$ [4, Theorem 8].

**Remark.** *We have provided a capacity definition of an $NC_{p,\omega}$-set. Since the capacity of condenser is equal to the modulus of this condenser* [4, Theorem 1]*, this definition is equivalent to the modulus definition of an $NC_{p,\omega}$-set in* [5, Sec. 3].

We now define another kind of capacity. For a compact set $e \subset \Omega$, we put $\mathfrak{M}(e, \Omega) = \{u \in C_0^\infty(\Omega) : u = 1 \text{ in some neighborhood of } e\}$ and suppose that $S^m_{p,\omega}(\Omega)$ is one of the spaces $W^{m,p}_\omega(\Omega)$, $W^m_{p,\omega}(\Omega)$, $L^m_{p,\omega}(\Omega)$. Following [10, §9.1], we define the capacity $\mathrm{Cap}(e, S^m_{p,\omega}(\Omega))$ of $e$ by $\inf \|u\|^p_{S^m_{p,\omega}(\Omega)}$, where the infimum is taken for all $u \in \mathfrak{M}(e, \Omega)$. The definition is extended to an arbitrary Borel set $F \subset \Omega$ by setting $\mathrm{Cap}(F, S^m_{p,\omega}(\Omega)) = \sup\{\mathrm{Cap}(e, S^m_{p,\omega}(\Omega)) : e \subset F, e \text{ compact}\}$. The number $\mathrm{Cap}(F, W^{m,p}_\omega(R^n))$ will be called a $(p, m, \omega)$-capacity of the Borel set $F \subset R^n$. As usual, for the case when $\Omega = R^n$, we will drop the reference to $\Omega$, as follows: $\mathrm{Cap}(F, S^m_{p,\omega})$.

**Remark 3.** *Using the truncation* [12, Theorem 4.14] $v = \min(\max(0, u), 1) \in W^{1,p}_\omega(\Omega)$ *for $u \in \mathfrak{M}(e, \Omega)$ and its subsequent mollification in $R^n$ (see Proposition 6), the $\mathfrak{M}(e, \Omega)$ class in the definition of* $\mathrm{Cap}(e, L^1_{p,\omega}(\Omega))$ *can be replaced by the class*

$$\tilde{\mathfrak{M}}(e, \Omega) = \{u \in C_0^\infty(\Omega) : 0 \le u \le 1 \text{ in } \Omega, u = 1 \text{ in some neighborhood of } e\}.$$

**Remark 4.** *From the definition of* $\mathrm{Cap}(F, W^{m,p}_\omega(\Omega))$*, it immediately follows that*

$$\mathrm{Cap}(F, L^1_{p,\omega}(\Omega)) \le \mathrm{Cap}(F, W^{1,p}_\omega(\Omega)) \le \mathrm{Cap}(F, W^{m,p}_\omega(\Omega)) \tag{7}$$

*for $1 \le p < \infty$ and $m \in \mathbb{N}$.*

2.6. **Coverings.** First, we will present the following version of Besicovitch theorem [6, Sec. 1, p. 5].

**Proposition 7** ([10, Sec. 1.2.1, Theorem 1])**.** *Let $S$ be a bounded set in $R^n$. For each $x \in S$, a ball $B_{r(x)}(x)$ is given, $r(x) > 0$. Then, one can choose among the given balls $\{B_{r(x)}(x)\}_{x \in S}$ a sequence $\{B_k\}$ (possibly finite), such that 1) $S \subset \bigcup_k B_k$; 2) no point of $R^n$ belongs to more than $\theta_n$ (a number which only depends on $n$) balls of the sequence $\{B_k\}$, i.e. for every $z \in R^n$, we have that $\sum_k \chi_{B_k}(z) \le \theta_n$.*

Now let $\Omega_j$ be a sequence of open sets, such that $\Omega_j \Subset \Omega_{j+1} \subset \Omega$ and $\Omega = \bigcup\limits_j \Omega_j$. Set $F_1 = \overline{\Omega_1}$, $F_2 = \overline{\Omega_2 \setminus \Omega_1}$, $F_3 = \overline{\Omega_3 \setminus \Omega_2}$, $\ldots$. Suppose that $D_j$ is another sequence of open sets, such that $D_j \Subset \Omega_j \Subset D_{j+1}$.

For each $x \in F_1$, we define a ball $B_{r_1(x)}(x)$, where $0 < r_1(x) < \min\left(1, \operatorname{dist}(F_1, \partial D_2)\right)$. For $j \geq 2$, we define an inequality $0 < r_j(x) < \min\left(\frac{1}{j}, \operatorname{dist}(F_j, \overline{D_{j-1}}), \operatorname{dist}(F_j, \partial D_{j+1})\right)$ and a ball $B_{r_j(x)}(x)$ if $x \in F_j$.

Assuming that in Proposition 7 we have that $S = F_j$, $\{B_{r_j(x)}(x)\}_{x \in F_j}$, we get a finite sequence of balls $B_{jk} \in \{B_{r_j(x)}(x)\}_{x \in F_j}$, $1 \leq k \leq k_j$, for which

1) $F_j \subset \bigcup\limits_k B_{jk}$;

2) $\sum\limits_k \chi_{B_{jk}}(x) \leq \theta_n$, $\forall x \in R^n$.

Next, choosing the value of $j$ from 1 to $\infty$, we come to another covering result.

**Corollary 1.** *Given $\Omega$, there exists a sequence of balls $B_j = B_{r_j}(x_j), r_j > 0$, such that*

*1) $\bigcup\limits_j B_j = \Omega$;*

*2) $\lim\limits_{j\to\infty} r_j = 0$;*

*3) for each $x \in R^n$, we have that $\sum\limits_j \chi_{B_j}(x) \leq 2\theta_n$, i.e. $\{B_j\}$ is a covering of the set $\Omega$ of bounded multiplicity;*

*4) $\{B_j\}$ is a locally finite covering of $\Omega$.*

## 3. Approximation by smooth functions on $\Omega$, completeness of $W^m_{p,\omega}(\Omega)$

3.1. **Approximation on $S^m_{p,\omega}(\Omega)$.** The following two theorems state that functions from $S^m_{p,\omega}(\Omega)$ can be approximated by smooth functions on $S^m_{p,\omega}(\Omega)$.

**Theorem 1.** *If $u \in L^m_{p,\omega}(\Omega)$, then there exists a sequence of functions $u_k \in L^m_{p,\omega}(\Omega) \cap C^\infty(\Omega)$, such that*

*1) $u_k \to u$ in $L^m_{p,\omega}(\Omega)$;*

*2) $u_k \to u$ in $W^{m,p}_\omega(\Omega')$ for every open set $\Omega' \Subset \Omega$;*

*3) from the estimate $|u| \leq C$ on $\Omega$ it follows that $|u_k| \leq 2C\theta_n$ on $\Omega$ for all $k \geq 1$, where $\theta_n$ is the constant from Proposition 7.*

*Proof.* We will prove assertions 1) and 2), using an approach by Maz'ya [10, Sec. 1.1.5, Theorem 1]. Suppose that $\{B_j\}$ is a covering of the bounded multiplicity on $\Omega$ from Corollary 1, and $\{\varphi_j\}$ is a partition of a unity subordinated to this covering. Consider a function $u \in L^m_{p,\omega}(\Omega)$. Then, $u \in W^{m,p}_\omega(B_j)$ by Proposition 5, and $u \in W^{m,1}(B_j)$ by Proposition 1 for every ball $B_j$.

Since $\varphi_j \in C_0^\infty(B_j)$, we have that $u\varphi_j \in W^{m,1}(B_j)$, and that every partial derivative $D^\alpha u$ is written, as usual, using the Leibnitz formula (1) [9, Sec. 6.12]. This implies that $u\varphi_j \in W^{m,p}_\omega(B_j)$. By virtue of $\operatorname{supp}\varphi_j \subset B_j$, we can assume that $u\varphi_j \in W^{m,p}_\omega(R^n)$, if we put $u\varphi_j = 0$ on $R^n \setminus B_j$, $j \geq 1$.

Here, note that $D^\alpha(u\varphi_j) = 0$ on $\Omega \setminus \operatorname{supp}\varphi_j$ for all $0 \leq |\alpha| \leq m$, $j \geq 1$. We take $0 < \varepsilon < \frac{1}{2}$ and denote by $z_j$ the mollification of the function $v_j = u\varphi_j$, where the radius of mollification is $\rho_j$, $0 < \rho_j < \min(1, \operatorname{dist}(\operatorname{supp}\varphi_j, \partial B_j))$. Moreover, according to Proposition 6, the choice of $\rho_j$ is made in a way that

$$\|v_j - z_j\|_{W^{m,p}_\omega(R^n)} < \varepsilon^j. \tag{8}$$

By the choice of $\{B_j\}$, on every open set $\Omega' \Subset \Omega$, the equality $u = \sum\limits_j v_j$ is valid. Here, the sum contains a finite number of non-zero terms $v_j$. The same property holds for the sum $g = \sum_j z_j$ on $\Omega'$, and therefore $g \in C^\infty(\Omega)$. Thus, by (8), we have that

$$\|u-g\|_{L^m_{p,\omega}(\Omega')} \le \|u-g\|_{W^{m,p}_\omega(\Omega')} \le \sum_j \|v_j - z_j\|_{W^{m,p}_\omega(R^n)} < \frac{\varepsilon}{1-\varepsilon} \le 2\varepsilon. \tag{9}$$

This inequality along with arbitrary choice of $\Omega'$ imply that $g \in L^m_{p,\omega}(\Omega)$ and $\|u - g\|_{L^m_{p,\omega}(\Omega)} \le 2\varepsilon$.

In (9), we set $\varepsilon = \frac{1}{4k}, k \in \mathbb{N}$ and denote the corresponding function $g$ by $u_k$. Then we have that

$$\|u-u_k\|_{L^m_{p,\omega}(\Omega)} \to 0, \qquad \|u-u_k\|_{W^{m,p}_\omega(\Omega')} \to 0$$

for every open set $\Omega' \Subset \Omega$ as $k \to \infty$. This implies assertions 1) and 2) of the Theorem.

Now, suppose that $|u(x)| \le C$ on $\Omega$, then obviously the estimates $|v_j(x)| \le C$, $|z_j(x)| \le C$ are valid at every point $x \in \Omega$ for all $j \ge 1$. In addition, in the equality $z(x) = \sum_j z_j(x)$ mentioned above, depending on the choice of $\{B_j\}$, the sum contains no more than $2\theta_n$ non-zero terms $z_j(x)$, $x \in \Omega$. Hence, we have that $|z(x)| \le 2C\theta_n$, and therefore $|u_k(x)| \le 2C\theta_n$ for all $x \in \Omega$ and $k \ge 1$. This completes the proof of the theorem. □

Using the same reasoning as in Theorem 1, we get another result.

**Theorem 2.** *If $u \in W^m_{p,\omega}(\Omega)$, then there exists a sequence of functions $u_k \in W^m_{p,\omega}(\Omega) \cap C^\infty(\Omega)$, such that 1) $u_k \to u$ in $W^m_{p,\omega}(\Omega)$; 2) $u_k \to u$ in $W^{m,p}_\omega(\Omega')$ for every open set, $\Omega' \Subset \Omega$; 3) from the estimate $|u| \le C$ on $\Omega$ it follows that $|u_k| \le 2C\theta_n$ on $\Omega$ for all $k \ge 1$, where $\theta_n$ is the constant from Proposition 7.*

*Similarly, if $u \in W^{m,p}_\omega(\Omega)$, then there exists a sequence of functions $u_k \in W^m_{p,\omega}(\Omega) \cap C^\infty(\Omega)$, such that 1) $u_k \to u$ in $W^{m,p}_\omega(\Omega)$; 2) from the estimate $|u| \le C$ on $\Omega$ it follows that $|u_k| \le 2C\theta_n$ on $\Omega$ for all $k \ge 1$.*

**Theorem 3.** *$W^{m,p}_\omega(R^n) = \overset{\circ}{W}{}^{m,p}_\omega(R^n)$.*

*Proof.* Here we use the same approach as in [1, Theorem 3.22]. According to Theorem 2, it is sufficient to show that every function $u \in C^\infty(R^n) \cap W^{m,p}_\omega(R^n)$ can be approximated in $W^{m,p}_\omega(R^n)$ by functions from $C^\infty_0(R^n)$. Let $f$ be a fixed function in $C^\infty_0(R^n)$, satisfying the following conditions: 1) $f(x) = 1$, if $|x| < 1$; 2) $f(x) = 0$, if $|x| \ge 2$; 3) $|D^\alpha f(x)| \le C_1$ (constant) for all $x \in R^n$, and $0 \le |\alpha| \le m$.

For $\varepsilon > 0$, suppose that $f_\varepsilon(x) = f(\varepsilon x)$. Then $f_\varepsilon(x) = 1$ for $|x| \le \frac{1}{\varepsilon}$, and also $|D^\alpha f_\varepsilon(x)| \le C_1 \varepsilon^{|\alpha|} \le C_1$, if $\varepsilon \le 1$.

If $u \in W^{m,p}_\omega(R^n) \cap C^\infty(R^n)$, then $u_\varepsilon = f_\varepsilon u$ belongs to $W^{m,p}_\omega(R^n)$ and has a bounded support. For $0 < \varepsilon \le 1$ and $|\alpha| \le m$ by (1), we have that

$$|D^\alpha u_\varepsilon(x)| = \left| \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u(x) \cdot D^{\alpha-\beta} f_\varepsilon(x) \right| \le C_1 \sum_{\beta \le \alpha} \binom{\alpha}{\beta} |D^\beta u(x)|.$$

Therefore, setting $\Omega_\varepsilon = \left\{x \in \Omega : |x| > \frac{1}{\varepsilon}\right\}$, we obtain

$$\|u-u_\varepsilon\|_{W^{m,p}_\omega(\Omega)} = \|u-u_\varepsilon\|_{W^{m,p}_\omega(\Omega_\varepsilon)} \le \|u\|_{W^{m,p}_\omega(\Omega_\varepsilon)} + \|u_\varepsilon\|_{W^{m,p}_\omega(\Omega_\varepsilon)} \le C_2 \|u\|_{W^{m,p}_\omega(\Omega_\varepsilon)},$$

where the constant $C_2$ does not depend on the choice of $u$. The right side approaches to zero as $\varepsilon \to 0$. The proof is complete. □

### 3.2. **Completeness of $W^m_{p,\omega}(\Omega)$.**

**Theorem 4.** *If $1 \le p < \infty, m \in \mathbb{N}$, then $W^m_{p,\omega}(\Omega)$ is a Banach space.*

*Proof.* Let $\{u_j\}$ be a Cauchy sequence in $W^m_{p,\omega}(\Omega)$, i.e. $\|u_i - u_j\|_{W^m_{p,\omega}(\Omega)} \to 0$ as $i, j \to \infty$. Because of completeness of $L_{p,\omega}(\Omega)$ (see Proposition 1), there exist functions $u$, $T^\alpha$, $|\alpha| = m$, such that $u_j \to u$, $D^\alpha u_j \to T^\alpha$ in $L_{p,\omega}(\Omega)$ for all $\alpha$, $|\alpha| = m$, as $j \to \infty$. Then by Proposition 4, we have that $u_j \in W^{m,p}_\omega(\Omega, loc)$ for all $j \ge 1$. This, with Proposition

5, allows us to conclude that $\{u_j\}$ is a Cauchy sequence in $W^{m,p}_\omega(B)$ for every $B \Subset \Omega$. Hence, $D^\beta u_j \to D^\beta u$ in $L_{p,\omega}(B)$ for these $B$, and $D^\beta u \in L_{p,\omega}(B) \cap L_1(\Omega, loc)$ for all multi-indices $\beta$, $1 \leq |\beta| \leq m-1$. In addition, we have that

$$\int\limits_\Omega u D^\alpha \varphi \, dx = \lim_{j\to\infty} \int\limits_\Omega u_j D^\alpha \varphi dx = (-1)^m \lim_{j\to\infty} \int\limits_\Omega \varphi D^\alpha u_j \, dx = (-1)^m \int\limits_\Omega \varphi T^\alpha \, dx$$

for any function $\varphi \in C_0^\infty(\Omega)$ and for every multi-index $\alpha$ of order $m$. Consequently, $D^\alpha u = T^\alpha$, and the sequence $\{u_j\}$ converges to the function $u$ in $W^m_{p,\omega}(\Omega)$. The theorem is proved. □

**Remark 5.** *The assertion about completeness of the space* $W^{m,p}_\omega(\Omega)$ *can be found in* [14, Proposition 2.1.2].

## 4. Some properties of capacity $\mathrm{Cap}(F, S^m_{p,\omega}(\Omega))$

Here we consider the question of equivalence of equalities $\mathrm{Cap}(F, W^{m,p}_\omega(\Omega)) = 0$, $\mathrm{Cap}(F, W^m_{p,\omega}(\Omega)) = 0$, $\mathrm{Cap}(F, L^m_{p,\omega}(\Omega)) = 0$ on a Borel set $F \subset \Omega$.

**Lemma 1.** *If $e$ is a compact subset of $\Omega$, then the equalities* $\mathrm{Cap}(e, W^m_{p,\omega}(\Omega)) = 0$ *and* $\mathrm{Cap}(e, W^{m,p}_\omega(\Omega)) = 0$ *are equivalent.*

*Proof.* If $\mathrm{Cap}(e, W^m_{p,\omega}(\Omega)) = 0$, then the inclusion $\mathfrak{M}(e,\Omega) \subset \mathfrak{M}(e, R^n)$ immediately implies that $\mathrm{Cap}(e, W^m_{p,\omega}) = 0$. Conversely, suppose that $\mathrm{Cap}(e, W^m_{p,\omega}) = 0$, and that $\Omega_1$ and $\Omega_2$ are open sets, such that $e \subset \Omega_1 \Subset \Omega_2 \Subset \Omega$. Consider the function $\varphi \in C_0^\infty(\Omega_2)$, satisfying the following conditions: 1) $0 \leq \varphi(x) \leq 1$, if $x \in R^n$; 2) $\varphi(x) = 1$, if $x \in \Omega_1$. By construction, $\max\limits_{|\alpha|\leq m} |D^\alpha u| \leq C_1$, if $x \in R^n$. Moreover, suppose that $u_j \in \mathfrak{M}(e, R^n)$, $j \geq 1$, and that $\|u_j\|^p_{W^m_{p,\omega}} \to \mathrm{Cap}(e, W^m_{p,\omega}) = 0$ as $j \to \infty$. Then from (5), we have that $\|u_j\|_{W^{m,p}_\omega(B)} \to 0$ for every ball $B \subset R^n$ as $j \to \infty$. Hence, $\lim\limits_{j\to\infty} \|u_j\|_{W^{m,p}_\omega(\Omega_2)} \to 0$.

Since $\varphi u_j \in \mathfrak{M}(e,\Omega)$, from (1) it follows that

$$\lim_{j\to\infty} \|\varphi u_j\|_{W^{m,p}_\omega(\Omega_2)} = \lim_{j\to\infty} \|\varphi u_j\|_{W^{m,p}_\omega(\Omega)} = 0 = \mathrm{Cap}(e, W^{m,p}_\omega(\Omega)).$$

It is obvious that $\mathrm{Cap}(e, W^m_{p,\omega}(\Omega)) \leq \mathrm{Cap}(e, W^{m,p}_\omega(\Omega))$. Therefore,

$$0 = \mathrm{Cap}(e, W^{m,p}_\omega(\Omega)) = \mathrm{Cap}(e, W^m_{p,\omega}(\Omega)).$$

Thus, the required assertion of the Lemma is proved along with the equivalence of the following equalities:

$$\mathrm{Cap}(e, W^m_{p,\omega}(\Omega)) = 0, \quad \mathrm{Cap}(e, W^{m,p}_\omega(\Omega)) = 0, \quad \mathrm{Cap}(e, W^{m,p}_\omega) = 0. \tag{10}$$

□

Taking into account (10), Lemma 1, and the definition of $\mathrm{Cap}(F, S^m_{p,\omega}(\Omega))$, we also obtain

**Corollary 2.** *Let $F$ be a Borel subset of $\Omega$. Then the equalities*

$$\mathrm{Cap}(F, W^m_{p,\omega}(\Omega)) = 0, \quad \mathrm{Cap}(F, W^{m,p}_\omega(\Omega)) = 0, \quad \mathrm{Cap}(F, W^{m,p}_\omega) = 0$$

*are equivalent.*

Consider some compact $e \subset \Omega$. For all $u \in \mathfrak{M}(e,\Omega)$, it is easily seen that

$$\|u\|_{L^m_{p,\omega}(\Omega)} \leq \|u\|_{W^{m,p}_\omega(\Omega)}.$$

In addition, for the case when $p = 1$, since we know the estimate $\omega(x) \geq \frac{C}{(1+|x|)^n}$ for a.e. $x \in R^n$ [14, Remark 1.2.4, Property 8], we obtain

$$m_n(e) = \int\limits_e |u| \, dx \leq C_1 \int\limits_{\Omega_1} |u|\omega \, dx \leq C_1 \|u\|_{W^{m,1}_\omega(\Omega)}. \tag{11}$$

Similarly, for the case when $p > 1$, due to Hölder's inequality, we get that

$$m_n(e) = \int_e |u|\,dx \le \left(\int_e |u|^p \omega\,dx\right)^{1/p} \left(\int_e \omega^{-\frac{1}{p-1}}\,dx\right)^{\frac{p-1}{p}} \le C_2 \|u\|_{W^{m,p}_\omega(\Omega)}. \tag{12}$$

Using these estimates for $u = \varphi u_j$ and taking into account the proof of Lemma 1, we easily deduce the following

**Lemma 2.** *If $e$ is a compact set of zero $(p, m, \omega)$-capacity on $\Omega$, then $m_n(e) = 0$, and $\mathrm{Cap}(e, L^m_{p,\omega}(\Omega)) = 0$.*

Replacing $e$ in Lemma 2 by a Borel set $F \subset \Omega$, we get a more general result.

**Corollary 3.** *If $F$ is a Borel set of zero $(p, m, \omega)$-capacity on $\Omega$, then $m_n(F) = 0$, and $\mathrm{Cap}(F, L^m_{p,\omega}(\Omega)) = 0$.*

**Lemma 3.** *Suppose that $e$ is a compact set of zero $(p, m, \omega)$-capacity and $e \subset B_1 = B(a, r_1)$, then $B_1 \setminus e$ is a connected open set.*

*Proof.* First, assume that $m = 1$. Suppose, on the contrary, that the set $B_1 \setminus e$ has a nonempty connected component $G$, where $G \Subset B_1$ and $\partial G \subset e$. By Lemma 2, $\mathrm{Cap}(e, L^1_{p,\omega}(B_1)) = 0$ and $m_n(e) = 0$.

According to Remark 3, there exists a sequence of functions $u_j \in \tilde{\mathfrak{M}}(e, B_1)$, $j \ge 1$, such that

$$\lim_{j\to\infty} \|u_j\|_{L^1_{p,\omega}(B_1)} = 0. \tag{13}$$

Since $u_j \in C_0^\infty(B_1)$, we have that $u_j = 0$ on $R^n \setminus B_1$. Therefore, in (13), the ball $B_1$ can be replaced with the ball $B_2 = B(a, r_2)$ for every $r_2 > r_1$. In other words,

$$\lim_{j\to\infty} \|u_j\|_{L^1_{p,\omega}(B_2)} = 0. \tag{14}$$

Set $v_j = 1$, if $x \in G$, and $v_j = u_j$, if $x \in R^n \setminus G$. It is obvious that $0 \le v_j \le 1$, $\mathrm{supp}\, v_j \subset B_1$, $v_j = 1$ in some neighborhood $O_j$ of a compact set $e \cup G$, where $O_j \Subset B_1$.

Moreover, $v_j$ satisfies the Lipschitz condition on $R^n$, and so $v_j \in W^{1,p}_\omega(B_2)$. By construction, $\int_{B_2} |\nabla v_j|^p \omega\,dx \le \int_{B_2} |\nabla u_j|^p \omega\,dx$ for all $j \ge 1$. From this relation and equality (14), we derive that

$$\lim_{j\to\infty} \|\nabla v_j\|_{L_{p,\omega}(B_2)} = 0. \tag{15}$$

Equality (15), Propositions 3,4, and the arbitrariness of $r_2 > r_1$ imply the existence of a sequence $\{c_j\}$ of constants, such that

$$\lim_{j\to\infty} \|v_j - c_j\|_{W^{1,p}_\omega(B_2)} = 0. \tag{16}$$

On the other hand, we know that $v_j = 0$ on $B_2 \setminus B_1$ for all $j \ge 1$. It follows that $\lim_{j\to\infty} c_j = 0$. Now, from (16) we deduce that $\lim_{j\to\infty} \|v_j\|_{W^{1,p}_\omega(B_2)} = 0$.

Using estimates (11), (12), we see that

$$m_n(G) \le C\|v_j\|_{W^{1,p}_\omega(B_2)}. \tag{17}$$

Since $m_n(G) > 0$, setting $j \to \infty$ in (17), we arrive at a contradiction. Thus, $B_1 \setminus e$ is a connected open set.

Now suppose that $m > 1$ and $\mathrm{Cap}(e, W^{m,p}_\omega) = 0$. Remark 4 implies that

$$\mathrm{Cap}(e, L^1_{p,\omega}(B_1)) = 0.$$

According to the above arguments, $B_1 \setminus e$ is a connected open set. The lemma is proved. □

**Lemma 4.** *Let $e$ be a compact subset of an open bounded set $\Omega$. Then the equalities $\mathrm{Cap}(e, L^m_{p,\omega}(\Omega)) = 0$ and $\mathrm{Cap}(e, W^{m,p}_\omega(\Omega)) = 0$ are equivalent.*

*Proof.* Taking into account Corollary 2 and Lemma 2, it is enough to show that the equality $Cap(e, L^m_{p,\omega}(\Omega)) = 0$ implies $Cap(e, W^{m,p}_\omega(\Omega)) = 0$. In fact, suppose that

$$\mathrm{Cap}(e, L^m_{p,\omega}(\Omega)) = 0$$

and $\{u_j\}$ is a sequence of functions $u_j \in \mathfrak{M}(e, \Omega)$, such that

$$\|\nabla_m u_j\|^p_{L_{p,\omega}(\Omega)} \to \mathrm{Cap}(e, L^m_{p,\omega}(\Omega)) = 0 \text{ as } j \to \infty. \tag{18}$$

By construction, $u_j \in C^\infty_0(\Omega)$, and therefore, $u_j \in C^\infty_0(B)$ for all $j \geq 1$, and for every ball, $B = B(0, r)$, where $\bar{\Omega} \subset B$. Obviously, $D^\alpha u_j \in C^\infty_0(\Omega)$, and $D^\alpha u_j = 0$ on $B \setminus \Omega$ for all multi-indices $\alpha$ of order at most $m$. Moreover, from (18), it follows that $\lim\limits_{j\to\infty} \|D^\alpha u_j\|_{L^1_{p,\omega}(B)} = 0$ for all multi-indices $\alpha$ of order $m - 1$. By Propositions 3, 4 there exists a sequence $\{c_j\}$ of constants, such that

$$\lim_{j\to\infty} \|D^\alpha u_j - c_j\|_{W^{1,p}_\omega(B)} = 0 \tag{19}$$

for all multi-indices $\alpha$ of order $m - 1$ and every ball $B(0, r)$, $\bar{\Omega} \subset B(0, r)$.

Here, the equality $\lim\limits_{j\to\infty} c_j = 0$ is obtained in a similar way as in the proof of Lemma 3. Therefore, (19) implies the relation

$$\lim_{j\to\infty} \|D^\alpha u_j\|_{W^{1,p}_\omega(B)} = 0 \tag{20}$$

with the same $\alpha$ and $B$.

Replacing the multi-index $\alpha$, $|\alpha| = m-1$, in the above argument sequentially by a multi-index of order less than $m - 1$, if necessary, we conclude that $\lim\limits_{j\to\infty} \|D^\alpha u_j\|_{W^{1,p}_\omega(B)} = 0$ with $|\alpha| \leq m - 1$, and $B \supset \bar{\Omega}$. Thus, $\lim\limits_{j\to\infty} \|u_j\|_{W^{m,p}_\omega(B)} = 0$.

Since $u_j \in \mathfrak{M}(e, B)$, we have that $\mathrm{Cap}(e, W^{m,p}_\omega(B)) = 0$, and therefore, by Corollary 2, we obtain $\mathrm{Cap}(e, W^{m,p}_\omega) = 0 = \mathrm{Cap}(e, W^{m,p}_\omega(\Omega))$. The Lemma is proved. □

For the case when $m = 1$, by Remark 3, the condition $u_j \in \mathfrak{M}(e, \Omega)$ in the proof of Lemma 4 can be replaced by the condition $u_j \in \tilde{\mathfrak{M}}(e, \Omega)$, which gives rise to the following result.

**Corollary 4.** *Suppose that $e$ is a compact subset of open bounded set $\Omega$, and* $\mathrm{Cap}(e, W^{1,p}_\omega) = 0$. *Then the class $\mathfrak{M}(e, \Omega)$ in the definition of* $\mathrm{Cap}(e, W^{1,p}_\omega)$ *can be replaced by the class $\tilde{\mathfrak{M}}(e, \Omega)$. Moreover, if $F$ is a Borel subset of open bounded set $\Omega$, then the equalities* $\mathrm{Cap}(F, L^m_{p,\omega}(\Omega)) = 0$ *and* $\mathrm{Cap}(F, W^{m,p}_\omega(\Omega)) = 0$ *are equivalent.*

**Remark 6.** *For the case when $\omega = 1$, the Lemma 4 was proved by Maz'ya* [10, Sec. 9.1.4].

## 5. Exceptional sets in problem (i) for $S^m_{p,\omega}(\Omega)$

By Remark 1, the equality $S^m_{p,\omega}(\Omega) = S^m_{p,\omega}(\Omega \setminus E)$ implies that $m_n(E) = 0$, and for each $u \in S^m_{p,\omega}(\Omega \setminus E)$ there exists $v \in S^m_{p,\omega}(\Omega)$, for which $v|_{\Omega\setminus E} = u$.

Recall that by definition, $E$ is a relatively closed subset on $\Omega$. In this case, the function $v$ will be called the extension of $u$ in $S^m_{p,\omega}(\Omega)$.

First, we refine the statement of Theorem 1 for $L^1_{p,\omega}(\Omega)$.

**Theorem 5.** *Let $u \in L^1_{p,\omega}(\Omega)$ and $\{\Omega_j\}$ be some sequence of open sets $\Omega_j$, such that $\Omega_j \Subset \Omega_{j+1} \subset \Omega$, and $\bigcup\limits_j \Omega_j = \Omega$. Then there exists a sequence of bounded functions $u_j \in L^1_{p,\omega}(\Omega) \cap C^\infty(\Omega)$, $j \geq 1$, such that*

$$\left(\int\limits_{\Omega_j} |u - u_j|^p \omega\, dx\right)^{1/p} < \frac{1}{j}, \quad \lim_{j\to\infty} \|u - u_j\|_{L^1_{p,\omega}(\Omega)} = 0. \tag{21}$$

*Proof.* By Theorem 1, there exists a function $v_j \in C^\infty(\Omega) \cap L^1_{p,\omega}(\Omega)$, for which the estimates

$$\int\limits_{\Omega_j} |u - v_j|^p \omega \, dx < \frac{1}{(3j)^p}, \tag{22}$$

$$\int\limits_{\Omega} |\nabla u - \nabla v_j|^p \omega \, dx < \frac{1}{(3j)^p} \tag{23}$$

are valid for all $j \geq 1$.

Now we choose an open set $G_j \Subset \Omega$, such that $\Omega_j \Subset G_j$ and

$$\int\limits_{\Omega \setminus G_j} |\nabla v_j|^p \omega \, dx < \frac{1}{2^{p+1}(3j)^p}. \tag{24}$$

For $l \in \mathbb{N}$, we set $\Omega_{j,l} = \{x \in \Omega : -l < v_j < l\}$ and choose $l = l_j$, such that

$$\int\limits_{\Omega_j \setminus \tilde{\Omega}_j} |v_j|^p \omega \, dx < \frac{1}{2^{p+1}(3j)^p}, \tag{25}$$

$$\int\limits_{G_j \setminus \tilde{\Omega}_j} |\nabla v_j|^p \omega \, dx < \frac{1}{2^{p+1}(3j)^p}, \tag{26}$$

where $\tilde{\Omega}_j = \Omega_{j,l_j}$.

Now suppose that $h_j = \max(-l_j, v_j)$, $g_j = \min(l_j, h_j)$. From the well-known properties of truncation (see [8, Theorem 1.20],[12, Theorem 4.14] for detailed information), it follows that $h_j = g_j = v_j$ and $\nabla h_j = \nabla g_j = \nabla v_j$ on $\tilde{\Omega}_j$, $|\nabla h_j| \leq |\nabla v_j|$ and $|\nabla g_j| \leq |\nabla h_j|$ a.e. on $\Omega$, $|g_j| \leq l_j$ on $\Omega$. Using these relations and the inequalities (22)–(26), we have that

$$\int\limits_{\Omega_j} |v_j - g_j|^p \omega \, dx \leq 2^p \int\limits_{\Omega_j \setminus \tilde{\Omega}_j} |v_j|^p \omega \, dx < \frac{1}{(3j)^p}, \tag{27}$$

$$\int\limits_{\Omega} |\nabla v_j - \nabla g_j|^p \omega \, dx \leq 2^p \int\limits_{\Omega \setminus G_j} |\nabla v_j|^p \omega \, dx + 2^p \int\limits_{G_j \setminus \tilde{\Omega}_j} |\nabla v_j|^p \omega \, dx < \frac{1}{(3j)^p}. \tag{28}$$

For the bounded function $g_j$, by Theorem 1, there exists a bounded function $u_j \in L^1_{p,\omega}(\Omega) \cap C^\infty(\Omega)$, $j \geq 1$, such that

$$\int\limits_{\Omega_j} |g_j - u_j|^p \omega \, dx < \frac{1}{(3j)^p}, \quad \int\limits_{\Omega} |\nabla g_j - \nabla u_j|^p \omega \, dx < \frac{1}{(3j)^p}. \tag{29}$$

Applying the properties of the norm $\| \cdot \|_{L_{p,\omega}(\Omega)}$ and combining (22)–(23) with (27)–(29), we obtain the relations in (21), which completes the proof of the Theorem. □

It is known that an $NC_{p,\omega}$-set is exceptional for $L^1_{p,\omega}(\Omega)$, $1 < p < \infty$ [5, Corollary 2]. Below, we will provide an addition to this result and extend it to $L^m_{p,\omega}(\Omega)$.

**Theorem 6.** *Suppose that $1 < p < \infty$ and $m \in \mathbb{N}$. Then $E$ is an exceptional set in problem (i) for $L^1_{p,\omega}(\Omega)$, if and only if $E$ is an $NC_{p,\omega}$-set in $\Omega$. If $E$ is an $NC_{p,\omega}$-set in $\Omega$, then $L^m_{p,\omega}(\Omega \setminus E) = L^m_{p,\omega}(\Omega)$ for all $m \in \mathbb{N}$.*

*Proof.* Step 1. Suppose that $E$ is an $NC_{p,\omega}$-set in $\Omega$, and $u \in L^1_{p,\omega}(\Omega \setminus E) \cap C^\infty(\Omega \setminus E)$ is a bounded function in $\Omega \setminus E$, where $E$ as an $NC_{p,\omega}$-set has zero $m_n$-measure (see Remark in Sec. 2.4). First, we prove that $u$ can be extended to a function in $L^1_{p,\omega}(\Omega)$. Indeed, consider a sequence $\{\Omega_j\}$ of open sets $\Omega_j$, where $\Omega_j \Subset \Omega_{j+1} \subset \Omega$ and $\bigcup\limits_j \Omega_j = \Omega$.

For a fixed coordinate $x_i$-axis, the function $u$ is absolutely continuous on every segment $e \subset \Omega_j \setminus E$ parallel to the $x_i$-axis, $i = 1, 2, \ldots, n$ and $j \geq 1$. Then (see the proof for sufficiency condition in Theorem 1 from [5]) the function $u$ can be further defined on $\Omega_j \cap E$ by $u_{ji}$, so that $u_{ji}$ is absolutely continuous in $\Omega_j$ on almost all straight lines parallel to the $x_i$-axis (see Remark 2).

Hence, the partial derivative $\dfrac{\partial u_{ji}}{\partial x_i}$ in the classical sense in $\Omega_j$ is equal to $\dfrac{\partial u}{\partial x_i}$ on $\Omega_j \setminus E$. In addition, $u_{ji}$, $\dfrac{\partial u_{ji}}{\partial x_i} \in L_1(\Omega_j, loc)$ (by virtue of $u$, $\dfrac{\partial u}{\partial x_i} \in L_1(\Omega_j, loc)$). Using integration by parts and Fubini's theorem, we obtain

$$\int\limits_{\Omega_j} \varphi \frac{\partial u_{ji}}{\partial x_i}\, dx = -\int\limits_{\Omega_j} u_{ji} \frac{\partial \varphi}{\partial x_i}\, dx \tag{30}$$

for all $\varphi \in C_0^\infty(\Omega_j)$.

Note now that for (30) it is possible to redefine the values of $u_{ji}$, $\dfrac{\partial u_{ji}}{\partial x_i}$ on a set of zero $m_n$-measure in $\Omega_j$. Then we change the values of $u_{ji}$ on $E \cap \Omega_j$, so that $u_{ji} = u_{j1}$ on $\Omega_j$ for all $i = 2, \ldots, n$.

We set $v_j = u_{j1}$ on $\Omega_j$ and suppose that

$$v = \begin{cases} v_1, & x \in \Omega_1; \\ v_j, & x \in \Omega_j \setminus \Omega_{j-1}, \text{ if } j \geq 2. \end{cases}$$

Obviously, $v \in L^1_{p,\omega}(\Omega)$ and $v|_{\Omega \setminus E} = u$. In other words, the function $u$ is extended to a function $v \in L^1_{p,\omega}(\Omega)$.

Step 2. Now, let $u$ be an arbitrary function in $L^1_{p,\omega}(\Omega \setminus E)$, and $\{\tau_k\}$ be a sequence (possibly, finite) of pairwise disjoint connected components of $\Omega$. Then, $\tau_k \setminus E$ is the connected component of $\Omega \setminus E$ (see Sec. 2.4), and $\Omega \setminus E = \bigcup\limits_k (\tau_k \setminus E)$.

By Theorem 5, there exists a sequence of bounded functions $u_j \in L^1_{p,\omega}(\Omega \setminus E) \cap C^\infty(\Omega \setminus E)$, $j \geq 1$, such that

$$\lim_{j \to \infty} \|u_j - u\|_{L^1_{p,\omega}(\Omega \setminus E)} = 0, \tag{31}$$

$$\lim_{j \to \infty} \|u_j - u\|_{L_{p,\omega}(\Omega')} = 0 \text{ for all } \Omega' \Subset \Omega \setminus E. \tag{32}$$

According to Step 1, we assume that $u_j \in L^1_{p,\omega}(\Omega)$ for all $j \geq 1$. Taking into account (31) and $m_n(E) = 0$, we get that $\{u_j\}$ is a Cauchy sequence in $L^1_{p,\omega}(\Omega)$. Then, by Proposition 2, $\{u_j\}$ converges in $L^1_{p,\omega}(\tau_k)$ to some function $v_k$, $k \geq 1$, as $j \to \infty$. Moreover, from (31), $|\nabla(u - v_k)| = 0$ a.e. on $\tau_k \setminus E$, and therefore, $u = v_k + c_k$ (see [3, Sec. 1.1.5]) on $\tau_k \setminus E$. Using (32), it is easy to show that $c_k = 0$, $k \geq 1$.

For all $x \in \Omega$, set $v(x) = v_k(x)$, if $x \in \tau_k$. By construction,

$$\|\nabla v\|_{L_{p,\omega}(\Omega)} = \|\nabla u\|_{L_{p,\omega}(\Omega \setminus E)}, \quad v(x)|_{\Omega \setminus E} = u(x).$$

Hence, $E$ is an exceptional set in problem (i) for $L^1_{p,\omega}(\Omega)$, $1 < p < \infty$.

Step 3. Let $E$ be an exceptional set in problem (i) for $L^1_{p,\omega}(\Omega)$. This implies that $E$ is an $NC_{p,\omega}$-set on $\Omega$. To establish this fact, we first prove that $\tau_k \setminus E$ is a domain for every $\tau_k$, $k \geq 1$. Here, $\{\tau_k\}$ is a sequence from Step 2.

Suppose, on the contrary, that for some $k$, the set $\tau_k \setminus E$ has a nonempty connected component $\eta_0$, for which $\eta_1 = (\tau_k \setminus E) \setminus \eta_0$ is a nonempty open set. Suppose that $u_0(x) = 0$ on $\eta_0 \cup (\Omega \setminus \tau_k)$ and $u_0(x) = 1$ on $\eta_1$. Obviously, $u_0 \in C^\infty(\Omega \setminus E) \cap L^1_{p,\omega}(\Omega \setminus E)$.

By the choice of $E$, $u_0$ can be extended to the function $v_0 \in L^1_{p,\omega}(\Omega)$. On the other hand, we will show that such extension is impossible. In fact, since $\tau_k$ is a domain, $m_n(E) = 0$, there exists a simple broken line $\gamma \subset \tau_k$ with a finite number of links, joining two given points, $a \in \eta_0$ and $b \in \eta_1$, for which $\mathcal{H}^1(\gamma \cap E) = 0$. By construction, $\gamma \cap E$

is a compact set in $\tau_k$. Hence, we can find a ball $B_0 = B(a^0, r_0)$ satisfying the following conditions: $a^0 \in \gamma \cap \overline{\eta_0}$ and $\overline{B_0} \subset \tau_k$, $B_0 \cap \eta_0 \neq \emptyset$ and $B_0 \cap \eta_1 \neq \emptyset$.

We consider $a^1 \in B_0 \cap \eta_0$, $b^1 \in B_0 \cap \eta_1$ and an arbitrary orthogonal transformation $\mathcal{P} : R^n \to R^n$ [5, Sec. 3.1]. Set $T = a^0 + \mathcal{P}$.

By the choice of $T$, $T(B_0) = B_0$, $T(B(x,r))$ is a ball $B(a^0+P(x), r)$ for all $B(x,r) \subset R^n$, and the determinant of the Jacobian matrix is equal to 1: $\det(T'(x)) = 1$. Hence, applying the change of the variable $x = T^{-1}(y)$, in (2), (3), we deduce that $\omega \circ T^{-1}$ is also an $A_p$-weight for $1 \le p < \infty$.

Then the linear operator $T_{p,\omega} : L^1_{p,\omega}(B_0) \to L^1_{p,\omega\circ T^{-1}}(B_0)$, defined by $T_{p,\omega}(u) = u \circ T^{-1}$, transforms $L^1_{p,\omega}(B_0)$ boundedly into $L^1_{p,\omega\circ T^{-1}}(B_0)$ and has a bounded inverse operator [5, Theorem 3], [12, Corollary 6.1.6]. From Remark 2, it follows that every function $u \in L^1_{p,\omega}(B_0)$ is absolutely continuous on almost all straight lines parallel to an arbitrary pre-given straight line in $R^n$, and, in particular, to the line $a^1b^1$.

Suppose that $P$ is a closed rectangle in $B_0$, and that $\sigma_0$, $\sigma_1$ are its opposite facets, where $a^1 \in \sigma_0 \subset \eta_0$, $b^1 \in \sigma_1 \subset \eta_1$, and the straight line $a^1b^1 \perp \sigma_0, \sigma_1$. We denote by $\Gamma$ the family of all straight segments $e$ joining the facets $\sigma_0, \sigma_1$ in $P$ and parallel to the straight line $a^1b^1$.

According to the mentioned above, the function $v_0$ is absolutely continuous on almost every segment $e \in \Gamma$ satisfying the additional condition $\mathcal{H}^1(e \cap E) = 0$. This implies the existence of a limit point $x_e \in E$ on each of such segments $e$ simultaneously for $e \cap \eta_0$ and $e \cap \eta_1$. Consequently, $v_0(x_e) = 0$ and $v_0(x_e) = 1$, which contradicts the definition of the function $v_0$. Thus, $\tau_k \setminus E$ is a domain for all $k \ge 1$.

Finally, we will prove that $E$ is an $NC_{p,\omega}$-set in $\Omega$ for $1 < p < \infty$. Suppose that $\Pi = \{x = (x_1, \dots, x_n) : a_i < x_i < b_i, i = 1, 2, \dots, n\}$ is a coordinate rectangle with the facets $\sigma_{0i}$, $\sigma_{1i}$, from the definition of an $NC_{p,\omega}$-set (see Sec. 2.4), $\bar{\Pi} \subset \Omega$. According to $m_n(E) = 0$, we get that $\sigma_{0i} \cup \sigma_{1i} \subset \partial(\Pi \setminus E)$ for all $i = 1, \dots, n$. Now we choose the connected component $\tau_k$ of the set $\Omega$, for which $\bar{\Pi} \subset \tau_k$.

In order to prove equality (6) for $\Pi$, given $\varepsilon > 0$ and $i = 1, \dots, n$, we find an admissible function $u \in \mathrm{Adm}(\sigma_{0i}, \sigma_{li}, \Pi)$, such that

$$C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E) \le \int\limits_{\Pi\setminus E} |\nabla u|^p \omega \, dx \le C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E) + \varepsilon.$$

Suppose that $G_l$ is an open neighborhood of the facet $\sigma_{li}$, in which $u = l$, and that $G'_l$ is another neighborhood of the facet $\sigma_{li}$, where $G'_l \Subset G_l \Subset \tau_k$, $l = 0, 1$. Set $G = G_0 \cup G_1 \cup \Pi$ and let the sequence $\{B_j\}$ be a locally finite covering of the set $G$ by the balls $B_j = B(a_j, r_j) \subset G$.

By Corollary 1, we assume that the covering $\{B_j\}$ has a bounded multiplicity, and that the balls from $\{B_j\}$, which have common points with $\overline{G'_l}$, belong to $G_l$, $l = 0, 1$.

Let $\{\varphi_j\}$ be a $C^\infty$-partition of the unity for $G$, subordinating to the covering $\{B_j\}$. Here, by definition, $\varphi_j \in C_0^\infty(B_j)$, and therefore there is a ball $B'_j = B(a_j, \rho_j)$, such that $0 < \rho_j < r_j$ and $\operatorname{supp}\varphi_j \subset B'_j$. Suppose that $e_j = E \cap \overline{B'_j}$ and $u_j(x) = u(x)\varphi_j(x)$, if $x \in \overline{B'_j} \setminus e_j$; $u_j(x) = 0$ if $x \in \Omega \setminus \overline{B'_j}$.

According to the above arguments, $B_j \setminus e_j$ is a domain. Moreover, it follows from the inclusions $u \in C^\infty(G \setminus E)$, $\varphi_j \in C_0^\infty(B_j)$, that $u_j|_{B_j \setminus E} = u\varphi_j$, and $u_j$ satisfies locally the Lipschitz condition on $\Omega \setminus e_j$. This implies $u_j \in L^1_{p,\omega}(\Omega \setminus e_j)$.

With an appropriate choice of $E$, the function $u_j$ extends to the function $v_j \in L^1_{p,\omega}(\Omega)$. Moreover, $v_j \in L^1_{p,\omega}(G)$ and, by construction,

$$v_j|_{B_j \setminus E} = u\varphi_j, \quad \|v_j\|_{L^1_{p,\omega}(\Omega)} = \|u\varphi_j\|_{L^1_{p,\omega}(B_j \setminus E)}.$$

Since $u = \sum_j u\varphi_j$ on $G \setminus E$, then, setting $v = \sum_j v_j$ on $G$, we conclude, similarly to the proof of Theorem 1, that $v \in L^1_{p,\omega}(G)$ and that

$$v|_{G\setminus E} = u, \quad \|v\|_{L^1_{p,\omega}(G)} = \|u\|_{L^1_{p,\omega}(G\setminus E)}.$$

Below, for convenience of calculations, we denote the function $v$ by $u$. In the proof of Theorem 1, we replace the set $\Omega$, the covering $\{B_j\}$, the partition $\{\varphi_j\}$ of the unity for $\Omega$, $u$, with the set $G$, the covering $\{B_j\}$, the partition $\{\varphi_j\}$ of the unity for $G$, and $u$, considered here, respectively. In addition, note in this case, that $z_j = u\varphi_j = 0$ on every $B_j$, $B_j \cap \overline{G'_0} \neq \emptyset$, and $z_j = u\varphi_j = \varphi_j$ on every $B_j$, $B_j \cap \overline{G'_1} \neq \emptyset$.

Using the same reasoning as in Theorem 1, we get the proper function $z = \sum_j z_j \in \mathrm{Adm}_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi)$, such that $z = 0$ on $G'_0$, $z = 1$ on $G'_1$, and $\int_\Pi |\nabla(u - z)|^p \omega\, dx \to 0$ as $\varepsilon \to 0$. Hence, we have that

$$C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi) \leq \int_\Pi |\nabla z|^p \omega\, dx = \int_\Pi |\nabla u|^p \omega\, dx + o(1) =$$

$$= \int_{\Pi\setminus E} |\nabla u|^p \omega\, dx + o(1) < C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E) + \varepsilon + o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Here, we suppose that $\varepsilon \to 0$ and conclude that

$$C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi) = C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E)$$

for all $i = 1, \ldots, n$ and $\Pi, \bar{\Pi} \subset \Omega$. Thus, $E$ is an $NC_{p,\omega}$-set in $\Omega$.

Step 4. Here we show that an $NC_{p,\omega}$-set in $\Omega$ is exceptional in problem (i) for $L^m_{p,\omega}(\Omega)$, $m = 2, 3, \ldots$.

Let $u$ be a function in $L^m_{p,\omega}(\Omega \setminus E)$, where $E$ is an $NC_{p,\omega}$-set in $\Omega$, and $m = 2, 3, \ldots$. Below we will keep the previous notation for the extended functions. Obviously, if $D^\alpha u$ is a weak partial derivative of order $|\alpha| = m - 1$ in $\Omega \setminus E$, then $D^\alpha u \in L^1_{p,\omega}(\Omega \setminus E)$. According to Step 1, it follows that $D^\alpha u \in L^1_{p,\omega}(\Omega)$. Hence, by Proposition 4, we have that $D^\alpha u \in L_{p,\omega}(\Omega, loc)$ for all $\alpha$ of order $|\alpha| = m-1$. Replacing the set $\Omega$ in above arguments with an arbitrary open set $\Omega' \Subset \Omega$, the derivative $D^\alpha u$ – by $D^\nu u$, $|\nu| = m - 2$, we deduce that $D^\nu u \in L^1_{p,\omega}(\Omega')$. By Proposition 1, it follows that $D^\nu u \in W^{1,p}_\omega(\Omega, loc) \cap W^{1,1}(\Omega, loc)$. Taking into account the continuity of this process, we obtain that $D^\alpha u \in W^{1,p}_\omega(\Omega, loc) \cap W^{1,1}(\Omega, loc)$ for all $\alpha, |\alpha| \leq m - 2$. In other words, $u \in L^m_{p,\omega}(\Omega)$, which completes the proof of the Theorem. □

Since $W^{m,p}_\omega(\Omega)$ and $W^m_{p,\omega}(\Omega) \subset L^m_{p,\omega}(\Omega)$, a simple modification of the arguments in the proof of Theorem 6 gives rise to another statement.

**Theorem 7.** *If $E$ is an $NC_{p,\omega}$-set in $\Omega$, $1 < p < \infty$, then $W^m_{p,\omega}(\Omega \setminus E) = W^m_{p,\omega}(\Omega)$, $W^{m,p}_\omega(\Omega \setminus E) = W^{m,p}_\omega(\Omega)$.*

The following theorem states that the set of zero $(p, m, \omega)$-capacity is also exceptional in problem (i) for $L^m_{p,\omega}(\Omega)$ for $1 \leq p < \infty$.

**Theorem 8.** *If $E$ is a set of zero $(p, m, \omega)$-capacity in $\Omega$, $1 \leq p < \infty$ and $m \in \mathbb{N}$, then $L^m_{p,\omega}(\Omega \setminus E) = L^m_{p,\omega}(\Omega)$, and $E$ is an $NC_{p,\omega}$-set in $\Omega$ for $1 < p < \infty$.*

*Proof.* First, suppose that $m = 1$ and $E \subset \Omega$ is a set of zero $(p, 1, \omega)$-capacity or, in other words, by Corollary 2, we obtain $\mathrm{Cap}(E, W^{1,p}_\omega(\Omega)) = 0$. In addition, by Corollary 4, we have that $m_n(E) = 0$. We first prove that every bounded function $u \in C^\infty(\Omega \setminus E) \cap L^1_{p,\omega}(\Omega \setminus E)$ can be extended to a function in $L^1_{p,\omega}(\Omega)$.

Here we use the construction from Step 3 of the proof of Theorem 6. Let the sequence $\{B_j\}$ be a locally finite covering of $\Omega$, similar to the one in Corollary 1, where, in particular, $B_j = B(a_j, r_j) \subset \Omega$. Suppose that $\{\varphi_j\}$ is a $C^\infty$-partition of the unity for $\Omega$, subordinated to the covering $\{B_j\}$. We fix $j$ and take a ball $B'_j = B(a_j, \rho_j)$, such that $0 < \rho_j < r_j$ and $\operatorname{supp} \varphi_j \subset B'_j$.

Suppose that $e_j = E \cap \overline{B'_j}$ and set $u_j(x) = u(x)\varphi_j(x)$, if $x \in \overline{B'_j} \setminus e_j$, and $u_j(x) = 0$, if $x \in \Omega \setminus \overline{B'_j}$. By the choice of $u_j$, $u$, and by Lemma 3, we have that $u_j$ satisfies locally the Lipschitz condition, and that $u_j \in W^{1,p}_\omega(\Omega \setminus e_j)$. By construction, we see that $u_j = u\varphi_j$ on $\Omega \setminus E$.

We need to show that $u_j$ can be extended to a function in $W^{1,p}_\omega(\Omega)$. By virtue of $u_j = 0$ on $\Omega \setminus \overline{B'_j}$, it is sufficient to show that $u_j$ can be extended to a function in $W^{1,p}_\omega(B_j)$. Using Corollary 4, we find a sequence $\{\psi_k\}$, $k \geq 1$, such that $\psi_k \in \tilde{\mathfrak{M}}(e_j, B_j)$ and

$$\|\psi_k\|^p_{W^{1,p}_\omega(B_j)} \to \operatorname{Cap}(e_j, W^{1,p}_\omega(B_j)) = 0. \tag{33}$$

Here, by definition, $\psi_k \in C^\infty_0(B_j)$, $0 \leq \psi_k \leq 1$ on $B_j$ and $\psi_k = 1$ on some open neighborhood $O_k$ of a compact set $e_j$. Let $O'_k$ be another open neighborhood of the set $e_j$, where $O'_k \Subset O_k$ and $O'_{k+1} \Subset O'_k$, $\bigcap_k O'_k = e_j$.

We set $v_{jk} = u_j(1 - \psi_k)$ on $B_j \setminus e_j$, and for a given $\varepsilon > 0$, we choose $k_0 \in \mathbb{N}$, such that

$$\int_{O'_{k_0} \setminus e_j} |\nabla u_j|^p \omega \, dx < \varepsilon. \tag{34}$$

In addition, note that $\nabla(u_j\psi_k) = u_j \nabla\psi_k + \psi_k \nabla u_j$ a.e. on $B_j \setminus e_j$; $u_j$, $\psi_k$ are bounded functions on $B_j \setminus e_j$, $|\nabla u_j|$ is a bounded function on $B_j \setminus O'_{k_0}$. Hence, (33) and (34) imply the existence of $k_1 \in \mathbb{N}$

$$\int_{O'_{k_0} \setminus e_j} |\nabla(u_j\psi_k)|^p \omega \, dx = o(1), \tag{35}$$

$$\int_{B_j \setminus O'_{k_0}} |\nabla(u_j\psi_k)|^p \omega \, dx = o(1) \tag{36}$$

for all $k \geq k_1$. Here, $o(1) \to 0$, if $\varepsilon \to 0$.

The equalities (35) and (36) imply that

$$v_{jk} \to u_j \text{ in } W^{1,p}_\omega(B_j \setminus e_j) \text{ as } k \to \infty. \tag{37}$$

On the other hand, set $g_{jk} = u_j(1 - \psi_k)$ on $B_j \setminus O'_k$, $g_{jk} = 0$ on $O'_k$. Obviously, $g_{jk} \in W^{1,p}_\omega(B_j)$ and $v_{jk} = g_{jk}$ on $B_j \setminus e_j$.

Hence, from (33) and $m_n(e_j) = 0$, we conclude that $\{g_{jk}\}$ is a Cauchy sequence in $W^{1,p}_\omega(B_j)$. Due to the completeness of the space $W^{1,p}_\omega(B_j)$, there exists a function $v_j$, for which $v_j = \lim_{k\to\infty} g_{jk}$ in $W^{1,p}_\omega(B_j)$, and, along with that, in $L^1_{p,\omega}(B_j)$. According to (37), $v_j = u_j$ on $B_j \setminus e_j$, $j \geq 1$. Therefore, $u_j$ is extended to $v_j$ in $L^1_{p,\omega}(B_j)$, $j \geq 1$.

Setting $v = \sum_j v_j$ in $\Omega$, where $v_j = 0$ on $\Omega \setminus B_j$, $j \geq 1$, as in the proof of Theorem 6 (see Step 3 there), we get that $v \in L^1_{p,\omega}(\Omega)$ and $v|_{\Omega\setminus E} = u$.

Now let $u$ be an arbitrary function in $L^1_{p,\omega}(\Omega \setminus E)$. Repeating verbatim the reasoning in Step 2 of the proof of Theorem 6, we obtain that $u$ extends to a function $v \in L^1_{p,\omega}(\Omega)$, and $v|_{\Omega\setminus E} = u$.

Finally, suppose that $u$ is a function in $L^m_{p,\omega}(\Omega \setminus E)$, where $m \geq 2$, and that $E$ is a set of zero $(p, m, \omega)$-capacity in $\Omega$. By Remark 4, it follows that $E$ is the set of zero $(p, 1, \omega)$-capacity, and $m_n(E) = 0$. As was proved above, any function $h \in L^1_{p,\omega}(\Omega \setminus E)$ extends to the function $z$ from $L^1_{p,\omega}(\Omega)$, and $z|_{\Omega\setminus E} = h$. Using this fact and the arguments from Step 4 of the proof of Theorem 6, we deduce that the function $u \in L^m_{p,\omega}(\Omega \setminus E)$ extends to the function $v \in L^m_{p,\omega}(\Omega)$, $v|_{\Omega\setminus E} = u$. Consequently, $L^m_{p,\omega}(\Omega \setminus E) = L^m_{p,\omega}(\Omega)$ for $1 \leq p < \infty$, and $m \in \mathbb{N}$.

Now, suppose that $1 < p < \infty$, and that $E$ is a set of zero $(p, m, \omega)$-capacity. Then $E$ is a set of zero $(p, 1, \omega)$-capacity, and $L^1_{p,\omega}(\Omega \setminus E) = L^1_{p,\omega}(\Omega)$. By Theorem 6, $E$ is an $NC_{p,\omega}$-set. Thus, the second part and, hence, the entire theorem is proved. $\square$

We will mention two other insertions that can be proved by a simple modification of the arguments in the proof of Theorems 6, 8.

**Corollary 5.** *If* $\mathrm{Cap}(E, W^{m,p}_\omega(\Omega)) = 0$, $1 \le p < \infty$, *and* $m \in \mathbb{N}$, *then* $\tau \setminus E$ *is a domain for every connected component* $\tau$ *of* $\Omega$.

**Corollary 6.** *If* $\mathrm{Cap}(E, W^{m,p}_\omega(\Omega)) = 0$, $1 \le p < \infty$, *and* $m \in \mathbb{N}$, *then* $W^{m,p}_\omega(\Omega \setminus E) = W^{m,p}_\omega(\Omega)$, $W^m_{p,\omega}(\Omega \setminus E) = W^m_{p,\omega}(\Omega)$.

## 6. Exceptional sets in problems (ii)-(iii) for $S^m_{p,\omega}(\Omega)$

According to Remark 3, the equality $\overset{\circ}{S}{}^m_{p,\omega}(\Omega) = \overset{\circ}{S}{}^m_{p,\omega}(\Omega \setminus E)$ implies that for every function $u \in \overset{\circ}{S}{}^m_{p,\omega}(\Omega)$, there exists a sequence of functions $u_j \in C^\infty_0(\Omega \setminus E)$, $j \ge 1$, for which $\lim\limits_{j\to\infty} \|u - u_j\|_{S^m_{p,\omega}(\Omega)} = 0$. Similarly, from the equality $S^m_{p,\omega}(\Omega) = \overset{\circ}{S}{}^m_{p,\omega}(\Omega)$ it follows that for every function $u \in S^m_{p,\omega}(\Omega)$ there exists a sequence of functions $u_j \in C^\infty_0(\Omega)$, $j \ge 1$, for which $\lim\limits_{j\to\infty} \|u - u_j\|_{S^m_{p,\omega}(\Omega)} = 0$.

Here and below, as always, $E$ is a relatively closed subset of the open set $\Omega$.

First, we will give the conditions under which the set $E$ will be exceptional in problem (ii) for $S^m_{p,\omega}(\Omega)$.

**Theorem 9.** *The equalities* $\overset{\circ}{W}{}^m_{p,\omega}(\Omega \setminus E) = \overset{\circ}{W}{}^m_{p,\omega}(\Omega)$, $\overset{\circ}{W}{}^{m,p}_\omega(\Omega \setminus E) = \overset{\circ}{W}{}^{m,p}_\omega(\Omega)$ *hold if and only if* $\mathrm{Cap}(E, W^{m,p}_\omega) = 0$. *In order for the equality* $\overset{\circ}{L}{}^m_{p,\omega}(\Omega \setminus E) = \overset{\circ}{L}{}^m_{p,\omega}(\Omega)$ *to hold, it is necessary that* $\mathrm{Cap}(E, L^m_{p,\omega}(\Omega)) = 0$ *and it is sufficient that* $\mathrm{Cap}(E, W^{m,p}_\omega) = 0$. *In addition, for the case of a bounded set* $\Omega$, *the equality* $\overset{\circ}{L}{}^m_{p,\omega}(\Omega \setminus E) = \overset{\circ}{L}{}^m_{p,\omega}(\Omega)$ *holds if and only if* $\mathrm{Cap}(E, W^{m,p}_\omega) = 0$.

*Proof.* Necessity. Suppose that $\overset{\circ}{W}{}^m_{p,\omega}(\Omega \setminus E) = \overset{\circ}{W}{}^m_{p,\omega}(\Omega)$. Let $e$ be a compact set in $E$. We choose $u \in C^\infty_0(\Omega)$ with $u = 1$ in a neighborhood of $e$.

Since $\overset{\circ}{W}{}^m_{p,\omega}(\Omega \setminus E) = \overset{\circ}{W}{}^m_{p,\omega}(\Omega)$, we can choose a sequence of functions $u_j \in C^\infty_0(\Omega \setminus E)$, such that $u_j \to u$ in $W^m_{p,\omega}(\Omega)$. By construction, $u - u_j \in \mathfrak{M}(e, \Omega)$ for all $j \ge 1$. This implies

$$0 \le \mathrm{Cap}(e, W^m_{p,\omega}(\Omega)) \le \lim_{j\to\infty} \|u - u_j\|^p_{W^m_{p,\omega}(\Omega)} = 0.$$

Hence, by Lemma 1 and Corollary 2, we have that

$$\mathrm{Cap}(E, W^m_{p,\omega}(\Omega)) = \mathrm{Cap}(E, W^{m,p}_\omega(\Omega)) = 0.$$

Similarly, from the equalities $\overset{\circ}{W}{}^{m,p}_\omega(\Omega \setminus E) = \overset{\circ}{W}{}^{m,p}_\omega(\Omega)$, $\overset{\circ}{L}{}^m_{p,\omega}(\Omega \setminus E) = \overset{\circ}{L}{}^m_{p,\omega}(\Omega)$, we obtain

$$\mathrm{Cap}(E, W^{m,p}_\omega) = 0, \quad \mathrm{Cap}(E, L^m_{p,\omega}(\Omega)) = 0,$$

respectively. Moreover, if $\Omega$ is a bounded set, from Corollaries 3 and 4, we have the equivalence of the equalities $\mathrm{Cap}(E, L^m_{p,\omega}(\Omega)) = 0$ and $\mathrm{Cap}(E, W^{m,p}_\omega) = 0$. The necessity condition of the theorem is proved.

Sufficiency. Now suppose that $\mathrm{Cap}(E, W^{m,p}_\omega) = 0$. We need to prove that $\overset{\circ}{L}{}^m_{p,\omega}(\Omega \setminus E) = \overset{\circ}{L}{}^m_{p,\omega}(\Omega)$. To do this, it is sufficient to prove that every function $u \in C^\infty_0(\Omega)$ can be approximated in $L^m_{p,\omega}(\Omega)$ by functions from $C^\infty_0(\Omega \setminus E)$. Indeed, we take a function $u \in C^\infty_0(\Omega)$ and suppose that $\Omega'$ is an open set, such that $\operatorname{supp} u \subset \Omega' \Subset \Omega$. Put $e = \overline{\Omega'} \cap E$ and note that $e$ is a compact set of zero $(p, m, \omega)$-capacity. Then there exists a sequence of functions $\varphi_j \in \mathfrak{M}(e, R^n)$, $j \ge 1$, such that $\|\varphi_j\|_{W^{m,p}_\omega} \to 0 = \mathrm{Cap}(e, W^{m,p}_\omega)$ as $j \to \infty$.

Since all partial derivatives $D^\alpha u$, $|\alpha| \le m$, are uniformly bounded in $\Omega$, we have that $\|u\varphi_j\|_{W^{m,p}_\omega(\Omega)} \to 0$. This implies $u(1-\varphi_j) \to u$ in $W^{m,p}_\omega(\Omega)$, and therefore $u(1-\varphi_j) \to u$ in $L^m_{p,\omega}(\Omega)$ as $j \to \infty$. Obviously, $u(1-\varphi_j) \in C_0^\infty(\Omega \setminus E)$. Consequently, $\overset{\circ}{L}{}^m_{p,\omega}(\Omega \setminus E) = \overset{\circ}{L}{}^m_{p,\omega}(\Omega)$. Similarly, we deduce that $\overset{\circ}{W}{}^m_{p,\omega}(\Omega \setminus E) = \overset{\circ}{W}{}^m_{p,\omega}(\Omega)$, $\overset{\circ}{W}{}^{m,p}_\omega(\Omega \setminus E) = \overset{\circ}{W}{}^{m,p}_\omega(\Omega)$. Hence, the sufficiency condition of the theorem is proved. Thus, the theorem is also proved. □

Next, we give the conditions under which the equality $S^m_{p,\omega}(\Omega) = \overset{\circ}{S}{}^m_{p,\omega}(\Omega)$ holds.

**Theorem 10.** *The equality* $W^{m,p}_\omega(\Omega) = \overset{\circ}{W}{}^{m,p}_\omega(\Omega)$ *holds true if and only if* $\mathrm{Cap}(R^n \setminus \Omega, W^{m,p}_\omega) = 0$. *If* $W^m_{p,\omega}(\Omega) = \overset{\circ}{W}{}^m_{p,\omega}(\Omega)$ *or* $L^m_{p,\omega}(\Omega) = \overset{\circ}{L}{}^m_{p,\omega}(\Omega)$, *then* $\mathrm{Cap}(R^n \setminus \Omega, W^{m,p}_\omega) = 0$ *or* $\mathrm{Cap}(R^n \setminus \Omega, L^m_{p,\omega}) = 0$, *respectively.*

*Proof.* Necessity. Suppose that, for example, $L^m_{p,\omega}(\Omega) = \overset{\circ}{L}{}^m_{p,\omega}(\Omega)$. Then we have that

$$\overset{\circ}{L}{}^m_{p,\omega}(R^n) \subset L^m_{p,\omega}(R^n) \subset L^m_{p,\omega}(\Omega) = \overset{\circ}{L}{}^m_{p,\omega}(\Omega) \subset \overset{\circ}{L}{}^m_{p,\omega}(R^n).$$

This implies $\overset{\circ}{L}{}^m_{p,\omega}(R^n) = \overset{\circ}{L}{}^m_{p,\omega}(\Omega) = \overset{\circ}{L}{}^m_{p,\omega}(R^n \setminus (R^n \setminus \Omega))$, and, consequently, by Theorem 9, we get $\mathrm{Cap}(R^n \setminus \Omega, L^m_{p,\omega}) = 0$.

Similarly, from $W^{m,p}_\omega(\Omega) = \overset{\circ}{W}{}^{m,p}_\omega(\Omega)$ or $W^m_{p,\omega}(\Omega) = \overset{\circ}{W}{}^m_{p,\omega}(\Omega)$, by Corollary 2, we deduce that $\mathrm{Cap}(R^n \setminus \Omega, W^m_{p,\omega}) = \mathrm{Cap}(R^n \setminus \Omega, W^{m,p}_\omega) = 0$. The necessity condition of the theorem is proved.

Sufficiency. Suppose that $\mathrm{Cap}(R^n \setminus \Omega, W^{m,p}_\omega) = 0$. By Theorems 3, 9, and Corollary 6, we infer that

$$W^{m,p}_\omega(\Omega) = W^{m,p}_\omega(R^n \setminus (R^n \setminus \Omega)) = W^{m,p}_\omega(R^n) =$$
$$\overset{\circ}{W}{}^{m,p}_\omega(R^n) = \overset{\circ}{W}{}^{m,p}_\omega(R^n \setminus (R^n \setminus \Omega)) = \overset{\circ}{W}{}^{m,p}_\omega(\Omega).$$

Theorem 10 is proved. □

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