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SOME REPRESENTATIONS OF SOLUTIONS TO BLOKHINTSEV EQUATION

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Abstract. In the paper, we obtain some representations for solutions and coefficients of Blokhintsev equation under condition the solutions satisfy to supplementary quasi-linear equation. These results may be used in the problems of identification of solutions and coefficients given supplementary initial-boundary information.

Keywords: Blokhintsev equation, method of differential constrains, overdetermined systems of partial differential equations.

1. Introduction

In this paper, we study Blokhintsev equation [1] on the base of method of differential constrains [2].

Blokhintsev equation is written by

$$D_t^2 w = \nu(x,t)\Delta w + (\nabla P, \nabla w) + (\overline{V}(x,t), \nabla \ln \nu(x,t)) D_t w, \tag{1}$$

for $x \in D$, $t \geq 0$, D is a domain in Euclidean space $\mathbb{R}^n$. Here, the function $w(x,t)$ is a potential of velocity of acoustic oscillations, P is a potential of pressure, $\overline{V}(x,t)$ is vector of stream velocity, $\nu(x,t) = c^2(x,t)$ is sound velocity squared, ∇u is gradient of function $u(x,t)$ with respect to variable x, that is

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right).$$

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If $a = (a_1, a_2, \dots, a_n)$, $b = (b_1, b_2, \dots, b_n)$ are two vectors then $\sum_{i=1}^{n} a_i b_i = (a, b)$ is a scalar product.

Derivatives $D_t w$, $D_t^2 w$ are equal to

$$D_t w = \frac{\partial w}{\partial t} + (\overline{V}(x,t), \nabla w) \equiv \psi(x,t), \quad D_t^2 w = \frac{\partial \psi}{\partial t} + (\overline{V}(x,t), \nabla \psi) \tag{2}$$

simultaneously. Substituting derivatives (2) in the equation (1), we obtain an equation

$$\frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t} + (\overline{V}(x,t), \nabla w)\right] + \left(\overline{V}(x,t), \nabla\left(\frac{\partial w}{\partial t} + (\overline{V}(x,t), \nabla w)\right)\right) = \tag{3}$$
$$= \nu(x,t)\Delta w + (\nabla P, \nabla w) + \left[\frac{\partial w}{\partial t} + (\overline{V}(x,t), \nabla w)\right]\big(\overline{V}(x,t), \nabla \ln \nu(x,t)\big).$$

In [3], the Blokhintsev equation was represented as non-homogeneous hyperbolic equation with a source function $R(x,t)$ which depends on solution and coefficients

$$\frac{\partial^2 w}{\partial t^2} = \nu(x,t)\Delta w + (\nabla P, \nabla w) + R(x,t)$$

where

$$R(x,t) = \left[\frac{\partial w}{\partial t} + (\overline{V}(x,t), \nabla w)\right]\big(\overline{V}(x,t), \nabla \ln \nu(x,t)\big) - \frac{\partial}{\partial t}(\overline{V}(x,t), \nabla w)$$
$$- \left(\overline{V}(x,t), \nabla\left(\frac{\partial w}{\partial t} + (\overline{V}(x,t), \nabla w)\right)\right).$$

General and explicit representations of solutions to equation were obtained for cases $w(x,t) = f(u(x,t))$ and $w(x,t) = \varphi(x,t)F\left(\frac{\psi(x,t)}{\varphi(x,t)}\right)$.

We obtain the main results of this paper to on the hypotheses that a solution $w = w(x,t)$ to the equation (1) satisfies to supplementary quasi-linear equation

$$D_t w = \psi, \tag{4}$$

where $\psi = \psi(x,t)$ is a function of variables (x,t). If the function $\psi(x,t)$ is arbitrary then system (1), (4) is equivalent to equation (1), that is condition (4) does not apply supplementary restriction to the function $w(x,t)$. In this way, for separation some class of solutions from all solutions to equation (1), it is necessary to apply some supplementary restrictions to the function $\psi(x,t)$. For example, if $\psi = 0$, we obtain the solution $w(x,t)$ is a Lagrange variable of operator D_t. It is reasonable to require $\psi(x,t)$ is a function of w, that is $\psi = \Psi(w)$, or does not depend on part of variables (x,t) and so on. Supposing w satisfies to condition (4) with some restrictions right-hand part ψ, we obtain overdetermined system of equations for function w. Reducing of this system to completely integrable type is a classical problem of the theory of overdetermined systems of partial differential equations [4, 5], and obviously the final result should depend on collection of functional parameters-coefficients ν, V, P, ψ. Hence, problem (1), (4) consists in determination of conditions for coefficients ν, V, P, ψ which lead to nontrivial sub-classes of solutions to equation (1).

In the second section of this paper, we consider so called one-parametric solutions [6] to equation (1), such that all considered functions depend on Lagrange variable of operator D_t and (4) is fulfilled in the case $\psi = 0$. Explicit formulas for solutions and coefficients are obtained, in general in complex-valued functions. Note, in some applications, the solutions to classic wave equation is given by complex-valued function [7] – [15]. Note, the solutions constructed in theorem 3 are determined by function of distance to some surface in $\mathbb{R}^n$.

In sections 3, 4, we consider one-dimensional equation (1), $n = 1$, under conditions $D_t w = 0$ and $D_t w = w$ and by methods of system of partial differential equations these systems are reduced to completely integrable type [4, 5, 16]. Also, in section 5, we state a general remark on possibility of reducing system (1), (4) in the case $\psi = 0$ to completely integrable type. More exactly, we obtain a system of two second order partial equations which have only derivatives with respect to variables x of the function w. It appears that reducing of this new system to completely integrable type is the main problem.

In final part of the paper, we discuss a problem of finding of functional-invariant solutions to Blokhintsev equations.

We suppose all considered functions are sufficiently differentiable.

2. One-parametric solutions to Blokhintsev equation

Let functions w, P and a vector-function $\overline{V}$ depend on a function $u = u(x,t)$

$$w = f(u), \quad P = g(u), \quad V_k = V_k(u), \quad k = 1, \dots, n,$$

where f, g, V_k, $k = 1, \dots, n$ are some functions of one variable. Then

$$\frac{\partial w}{\partial t} = f'\frac{\partial u}{\partial t}, \quad \frac{\partial w}{\partial x_k} = f'\frac{\partial u}{\partial x_k}, \quad \nabla w = f'\nabla u, \quad \nabla P = g'\nabla u,$$

$$\frac{\partial^2 w}{\partial t^2} = f'\frac{\partial^2 u}{\partial t^2} + f''\left(\frac{\partial u}{\partial t}\right)^2, \quad \Delta w = f'\Delta u + f''|\nabla u|^2.$$

The equation (3) takes the form

(5)
$$\begin{aligned} f'u_{tt} + f''u_t^2 = \nu\left(f'\Delta u + f''|\nabla u|^2\right) + f'g'|\nabla u|^2 + f'\left(u_t + \langle\overline{V}, \nabla u\rangle\right)\langle\overline{V}, \nabla \ln\nu\rangle \\ - \frac{\partial}{\partial t}\left(f'\langle\overline{V}, \nabla u\rangle\right) - \left\langle\overline{V}, \nabla\left(f'\left(u_t + \langle\overline{V}, \nabla u\rangle\right)\right)\right\rangle. \end{aligned}$$

In addition suppose, the function $u(x,t)$ is a solution to a quasi-linear equation

$$u_t + \left(\overline{V}, \nabla u\right) = 0, \tag{6}$$

that is $D_t w = 0$. Then we rewrite the equation (5) in the form

$$\nu\left(f'\Delta u + f''|\nabla u|^2\right) + f'g'|\nabla u|^2 = 0. \tag{7}$$

It is possible to write a general solution to the equation (6) implicitly

$$u = F(y_1, \dots, y_n), \tag{8}$$

where $y_k = x_k - tV_k(u)$, $k = 1, \dots, n$ and $F(y)$ is an arbitrary function of variables $y = (y_1, \dots, y_n)$.

We deduce from (8) that

$$\frac{\partial u}{\partial x_i} = \frac{\partial F}{\partial y_i} - t\frac{\partial u}{\partial x_i}\sum_{k=1}^{n}\frac{\partial F}{\partial y_k}V_k',$$

and

(9)
$$\frac{\partial u}{\partial x_i} = \frac{1}{m}\frac{\partial F}{\partial y_i},$$

where $m = 1 + t\sum_{k=1}^{n} V_k' \frac{\partial F}{\partial y_k}$. Therefore,

(10)
$$|\nabla u|^2 = \frac{1}{m^2}|\nabla F|^2.$$

Next, it follows from (9)

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i^2} &= \frac{1}{m}\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_i}\right) - \frac{1}{m^2}\frac{\partial F}{\partial y_i}\frac{\partial m}{\partial x_i} \\ &= \frac{1}{m}\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_i}\right) - \frac{t}{m^2}\frac{\partial F}{\partial y_i}\sum_{k=1}^{n}\left(V_k''\frac{\partial u}{\partial x_i}\frac{\partial F}{\partial y_k} + V_k'\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_k}\right)\right) \\ &= \frac{1}{m}\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_i}\right) - \frac{t}{m^2}\frac{\partial F}{\partial y_i}\sum_{k=1}^{n}V_k'\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_k}\right) - \frac{t}{m^3}\left(\frac{\partial F}{\partial y_i}\right)^2\sum_{k=1}^{n}V_k''\frac{\partial F}{\partial y_k}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_k}\right) &= \sum_{j=1}^{n}\frac{\partial^2 F}{\partial y_k \partial y_j}\frac{\partial y_j}{\partial x_i} = \sum_{j=1}^{n}\frac{\partial^2 F}{\partial y_k \partial y_j}\left(-tV_j'\frac{\partial u}{\partial x_i}\right) + \frac{\partial^2 F}{\partial y_k \partial y_i} \\ &= -\frac{t}{m}\sum_{j=1}^{n}\frac{\partial^2 F}{\partial y_k \partial y_j}V_j'\frac{\partial F}{\partial y_i} + \frac{\partial^2 F}{\partial y_k \partial y_i}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i^2} &= -\frac{t}{m^2}\frac{\partial F}{\partial y_i}\sum_{j=1}^{n}\frac{\partial^2 F}{\partial y_i \partial y_j}V_j' + \frac{1}{m}\frac{\partial^2 F}{\partial y_i^2} \\ &- \frac{t}{m^2}\frac{\partial F}{\partial y_i}\sum_{k=1}^{n}V_k'\left(-\frac{t}{m}\sum_{j=1}^{n}\frac{\partial^2 F}{\partial y_k \partial y_j}V_j'\frac{\partial F}{\partial y_i} + \frac{\partial^2 F}{\partial y_k \partial y_i}\right) - \frac{t}{m^3}\left(\frac{\partial F}{\partial y_i}\right)^2\sum_{k=1}^{n}V_k''\frac{\partial F}{\partial y_k} \\ &= -\frac{2t}{m^2}\frac{\partial F}{\partial y_i}\sum_{k=1}^{n}\frac{\partial^2 F}{\partial y_i \partial y_k}V_k' + \frac{1}{m}\frac{\partial^2 F}{\partial y_i^2} + \frac{t^2}{m^3}\left(\frac{\partial F}{\partial y_i}\right)^2\sum_{k,j=1}^{n}V_k'V_j'\frac{\partial^2 F}{\partial y_k \partial y_j} \\ &- \frac{t}{m^3}\left(\frac{\partial F}{\partial y_i}\right)^2\sum_{k=1}^{n}V_k''\frac{\partial F}{\partial y_k}.\end{aligned}$$

In this way,

(11)
$$\begin{aligned}\Delta u = &-\frac{t}{m^2}\sum_{k=1}^{n}V_k'\frac{\partial}{\partial y_k}\left(|\nabla F|^2\right) + \frac{1}{m}\Delta F \\ &+ \frac{t^2}{m^3}|\nabla F|^2\sum_{k,j=1}^{n}V_k'V_j'\frac{\partial^2 F}{\partial y_k \partial y_j} - \frac{t}{m^3}|\nabla F|^2\sum_{k=1}^{n}V_k''\frac{\partial F}{\partial y_k}.\end{aligned}$$

If $\Delta F = |\nabla F|^2 = 0$ then the relation (7) is fulfilled identically by (10) and (11). Thus, a following theorem takes place

Theorem 1. *Suppose*

1) $F(y_1, \dots, y_n)$ be a function of complex variables $y_1, \dots, y_n$ such that

$$\Delta F = |\nabla F|^2 = 0;$$

2) $f(z)$, $g(z)$, $V_k(z)$, $k = 1, \dots, n$ be arbitrary functions of one variable z;

3) function $u = u(x,t)$ be an implicit solution to equation

$$u = F(x_1 - tV_1(u), \dots, x_n - tV_n(u)).$$

Then functions $w = f(u)$, $P = g(u)$, $\overline{V} = \overline{V}(u)$ for arbitrary $\nu = \nu(x,t)$ reduce (3) *to identity.*

Example. Let $F = \varphi(y_1 + \mathrm{i}y_2)$. Then $\Delta F = |\nabla F|^2 = 0$. We can find $u(x_1, x_2, t)$ from an equation

$$u = \varphi(x_1 + \mathrm{i}x_2 - t(V_1(u) + \mathrm{i}V_2(u))).$$

By taking $\varphi(z) = z$, $V_1(u) = a_1 u$, $V_2(u) = a_2 u$, $a_1, a_2 \in \mathbb{C}$, we find

$$u = \frac{x_1 + \mathrm{i}x_2}{1 + t(a_1 + \mathrm{i}a_2)}.$$

By taking $\varphi(z) = z$, $V_1(u) = a_1 u^2$, $V_2(u) = a_2 u^2$, $a_1, a_2 \in \mathbb{C}$, we find multivalued solution

$$u = \frac{1 + \sqrt{1 + 4t(a_1 + \mathrm{i}a_2)(x_1 + \mathrm{i}x_2)}}{2t(a_1 + \mathrm{i}a_2)}.$$

By taking $V_1(u) = a_1\varphi^{-1}(u)$, $V_2(u) = a_2\varphi^{-1}(u)$, $a_1, a_2 \in \mathbb{C}$, we find

$$u = \varphi\left(\frac{x_1 + \mathrm{i}x_2}{1 + t(a_1 + \mathrm{i}a_2)}\right),$$

where $\varphi(z)$ is arbitrary function and $\varphi^{-1}(z)$ is inverse function.

We can now return to (7) and rewrite it in the form

$$\nu = -\frac{f'g'|\nabla u|^2}{\Delta f(u)},$$

allowing denominator is not zero. Like this the equality assumes formula for function ν.

Theorem 2. *Let*

1) $F(y_1, \dots, y_n)$ be an arbitrary function of variables $y_1, \dots, y_n$;

2) $f(z)$, $g(z)$, $V_k(z)$, $k = 1, \dots, n$ be an arbitrary function of one variable z;

3) a function $u = u(x,t)$ be an implicit solution to equation

$$u = F(x_1 - tV_1(u), \dots, x_n - tV_n(u)).$$

Then functions $w = f(u)$, $P = g(u)$, $\overline{V} = \overline{V}(u)$ and a function $\nu = \nu(x,t)$ which are defined by formula

$$\nu = -\frac{f'g'|\nabla u|^2}{\Delta f(u)},$$

reduce (3) *to an identity. Here $|\nabla u|^2$ and Δu are defined by formulas* (10) *and* (11)*.*

Rewrite the relation (7) in the form

$$\nu f' \Delta u + (\nu f'' + f'g')\,|\nabla u|^2 = 0.$$

Substitute $|\nabla u|^2$ and Δu from (10) and (11):

$$\nu f' \left(\frac{1}{m}\Delta F - \frac{t}{m^2}\sum_{k=1}^{n} V_k' \frac{\partial}{\partial y_k}\left(|\nabla F|^2\right) + \frac{t^2}{m^3}|\nabla F|^2 \sum_{k,j=1}^{n} V_k' V_j' \frac{\partial^2 F}{\partial y_k \partial y_j} - \frac{t}{m^3}|\nabla F|^2 \sum_{k=1}^{n} V_k'' \frac{\partial F}{\partial y_k} \right) + (\nu f'' + f'g')\frac{|\nabla F|^2}{m^2} = 0.$$

The following statement immediate from this equality:

Theorem 3. *Let*

1) $F(y_1, \dots, y_n)$ be a function of variables $y_1, \dots, y_n$ which satisfies to an equation

$$|\nabla F|^2 = \kappa^2 = const;$$

2) $f(z)$, $g(z)$, $V_k(z)$, $k = 1, \dots, n$ be arbitrary functions of one variable z;

3) a function $u = u(x,t)$ be implicit solution to an equation

$$u = F(x_1 - tV_1(u), \dots, x_n - tV_n(u)).$$

Then functions $w = f(u)$, $P = g(u)$, $\overline{V} = \overline{V}(u)$ and function $\nu = \nu(x,t)$ which is defined by formula

$$\nu = \frac{\kappa^2}{m} \cdot \frac{t\sum\limits_{k=1}^{n} V_k'' \dfrac{\partial F}{\partial y_k} - t^2 \sum\limits_{k,j=1}^{n} V_k' V_j' \dfrac{\partial^2 F}{\partial y_k \partial y_j} - mf'g'}{mf' + \kappa^2 f''},$$

where $m = 1 + t\sum\limits_{k=1}^{n} V_k' \dfrac{\partial F}{\partial y_k}$, reduce (3) to identity.

Remark. If $\Omega = \{\, y \in \mathbb{R}^n \,|\, \omega(y) = 0 \,\}$ is some surface in $\mathbb{R}^n(y)$, then function $F(y) = \rho(y, \Omega)$ of distance from point y to the surface Ω satisfies to the equation $|\nabla F|^2 = 1$.

3. Solution to one-dimensional Blokhintsev equation under condition $D_t w = 0$

Consider one-dimensional Blokhintsev equation

$$D_t^2 w = \nu w_{xx} + P_x w_x + V D_t w (\ln \nu)_x, \tag{12}$$

where $D_t = \partial_t + V\partial_x$, under supplementary condition

$$D_t w \equiv \partial_t w + V\partial_x w = 0. \tag{13}$$

By (13), the equation (12) takes form

$$\nu w_{xx} + P_x w_x = 0.$$

Now then, consider a system

$$w_{xx} = -\frac{P_x}{\nu} w_x, \tag{14}$$

$$w_t = -V w_x \tag{15}$$

and test it for consistency.

From(15) by virtue (14), we get

$$w_{tx} = -V w_{xx} - V_x w_x = V\frac{P_x}{\nu} w_x - V_x w_x = w_x \left(V\frac{P_x}{\nu} - V_x \right), \tag{16}$$

$$w_{txx} = w_{xx}\left(V\frac{P_x}{\nu} - V_x\right) + w_x\left(V\frac{P_x}{\nu} - V_x\right)_x = w_x\left(\left(V\frac{P_x}{\nu} - V_x\right)_x - \frac{P_x}{\nu}\left(V\frac{P_x}{\nu} - V_x\right)\right). \tag{17}$$

From (14) by virtue (16), we obtain

$$w_{xxt} = -\frac{P_x}{\nu}w_{xt} - \left(\frac{P_x}{\nu}\right)_t w_x = w_x\left(-\left(\frac{P_x}{\nu}\right)_t - \frac{P_x}{\nu}\left(V\frac{P_x}{\nu} - V_x\right)\right). \tag{18}$$

Generate consistency condition $w_{xxt} = w_{txx}$ by virtue (17) and (18)

$$w_x\left(\left(V\frac{P_x}{\nu} - V_x\right)_x + \left(\frac{P_x}{\nu}\right)_t\right) = 0.$$

As far as $w_x = 0$ and $w_t = 0$, that is w is a constant function, we can assume $w_x \neq 0$ and, it follows that,

$$\left(V\frac{P_x}{\nu} - V_x\right)_x + \left(\frac{P_x}{\nu}\right)_t = 0. \tag{19}$$

Thus, we proved the following theorem.

Theorem 4. *Blokhintsev equation* (12) *under supplementary condition* (13) *has nontrivial solution if and only if, coefficients* $V(x,t)$, $P(x,t)$, $\nu(x,t)$ *satisfy to the equation* (19). *If* (19) *is fulfilled, then solution* $w(x,t)$ *to the system* (12), (13) *is determined from totally-integrable system* (14), (15) *by initial data*

$$w|_{t=x=0} = w_0, \quad w_x|_{t=x=0} = w_1,$$

where w_0, w_1 *are constants.*

In the case of arbitrary dimension $n \geq 1$, suggestion the solution $w = w(x,t)$ to equation (1) satisfies to supplementary condition $D_t w = 0$ leads to system of equations

$$\nu\Delta w + \langle\nabla P, \nabla w\rangle = 0, \tag{20}$$

$$D_t w = 0. \tag{21}$$

Denote

$$L = \sum_{k=1}^{n} V_k \frac{\partial}{\partial x_k}, \quad L_i = \sum_{k=1}^{n} \frac{\partial V_k}{\partial x_i}\frac{\partial}{\partial x_k}, \quad L_{ii} = \sum_{k=1}^{n} \frac{\partial^2 V_k}{\partial x_i^2}\frac{\partial}{\partial x_k},$$

$i = 1, \ldots, n$. Then the operator D_t is written in the form

$$D_t = \frac{\partial}{\partial t} + L.$$

Applying operator D_t to the equation (20), we get

$$D_t(\nu)\Delta w + \nu D_t(\Delta w) + \langle D_t(\nabla P), \nabla w\rangle + \langle\nabla P, D_t(\nabla w)\rangle = 0. \tag{22}$$

As far as

$$D_t\left(\frac{\partial w}{\partial x_i}\right) = \frac{\partial}{\partial x_i}(D_t w) - L_i w,$$

then

$$D_t(\nabla w) = \nabla(D_t w) - \mathcal{L}w, \quad D_t(\Delta w) = \Delta(D_t w) - 2\mathrm{div}(\mathcal{L}w) + \Delta(L)w, \tag{23}$$

where $\mathcal{L}w = (L_1 w, \ldots, L_n w)$.

Owing to (23) and (21), the relation (22) takes form

$$D_t(\nu)\Delta w + \nu\left(\Delta(L)w - 2\mathrm{div}(\mathcal{L}w)\right) + \langle D_t(\nabla P), \nabla w\rangle - \langle \nabla P, \mathcal{L}w\rangle = 0. \quad (24)$$

Owing to (20), it is possible to exclude Δw from (24). Now then, finally we obtain system of two equations to function w:

$$\Delta w = -\frac{1}{\nu}\langle \nabla P, \nabla w\rangle = 0,$$

$$2\mathrm{div}(\mathcal{L}w) = \Delta(L)w - \frac{1}{\nu^2}D_t(\nu)\langle \nabla P, \nabla w\rangle + \frac{1}{\nu}\langle D_t(\nabla P), \nabla w\rangle - \frac{1}{\nu}\langle \nabla P, \mathcal{L}w\rangle.$$

There are only derivatives of function w with respect to variables x in this system.

4. Solution to one-dimensional Blokhintsev equation under condition $D_t w = w$

Consider one-dimensional Blokhintsev equation

$$D_t^2 w = \nu w_{xx} + P_x w_x + D_t w V(\ln \nu)_x, \quad (25)$$

where $D_t = \partial_t + V\partial_x$, under supplementary condition

$$D_t w \equiv \partial_t w + V\partial_x w = w. \quad (26)$$

Owing to (26), the equation (25) takes form

$$w = \nu w_{xx} + P_x w_x + wV(\ln \nu)_x. \quad (27)$$

Denote

$$\nu = e^A.$$

Then we can rewrite (27) as

$$w = e^A w_{xx} + P_x w_x + wVA_x$$

or

$$w_{xx} = -e^{-A}P_x w_x + e^{-A}(1 - VA_x)w. \quad (28)$$

Denote

$$B = -e^{-A}P_x, \quad C = e^{-A}(1 - VA_x). \quad (29)$$

Lemma 1. *If $B(x,t)$, $C(x,t)$, $V(x,t)$ are given functions then, by virtue (29), functions A and P are defined by formulas*

$$A = a - \ln\left(\int \frac{Ce^a}{V}dx\right),$$

$$P = -\int Be^a\left(\int \frac{Ce^a}{V}dx\right)^{-1}dx,$$

where $a = \int \frac{dx}{V}$.

Proof. Consider the second relation of (29)

$$VA_x = 1 - Ce^A. \quad (30)$$

Make the change of variables

$$A = A_0 + a, \ \text{where} \ a = \int \frac{dx}{V}.$$

Then (30) takes form

$$V\left(A_{0x}+\frac{1}{V}\right)=1-Ce^{A_0+a},$$

$$VA_{0x}=-Ce^{A_0}e^{a},$$

$$-e^{-A_0}A_{0x}=\frac{Ce^a}{V}.$$

Therefore

$$e^{-A_0}=\int\frac{Ce^a}{V}dx,$$

$$A_0=-\ln\left(\int\frac{Ce^a}{V}dx\right).$$

Now then,

$$A=a-\ln\left(\int\frac{Ce^a}{V}dx\right).$$

Now, from the first relation of (29)

$$P_x=-Be^{A},$$

we get

$$P=-\int Be^a\left(\int\frac{Ce^a}{V}dx\right)^{-1}dx.$$

The lemma is proved. □

Owing to notations (29), the equation (28) takes form

$$w_{xx}=Bw_x+Cw.$$

Now, consider a problem of reduction of system

$$w_{xx}=Bw_x+Cw, \tag{31}$$

$$w_t=-Vw_x+w, \tag{32}$$

in involution.

Owing to (32) and (31),

$$w_{tx}=-Vw_{xx}-V_xw_x+w_x=-V(Bw_x+Cw)+w_x(1-V_x)=(1-V_x-VB)w_x-CVw.$$

Now then,

$$w_{tx}=(1-V_x-VB)w_x-CVw. \tag{33}$$

Owing to (33) and (31),

$$\begin{aligned}w_{txx}&=(1-V_x-BV)w_{xx}-(V_{xx}+B_xV+BV_x)w_x-CVw_x-(C_xV+CV_x)w\\&=(1-V_x-BV)(Bw_x+Cw)-(V_{xx}+B_xV+BV_x+CV)w_x-(C_xV+CV_x)w\\&=(B-2BV_x-B^2V-V_{xx}-B_xV-CV)w_x+(C-2CV_x-BCV-C_xV)w.\end{aligned}$$

Now then,

$$w_{txx}=(B-2BV_x-B^2V-V_{xx}-B_xV-CV)w_x+(C-2CV_x-BCV-C_xV)w. \tag{34}$$

Owing to (31), (32) and (33),

$$\begin{aligned} w_{xxt} &= Bw_{xt} + B_t w_x + C w_t + C_t w \\ &= B\left((1 - V_x - VB)w_x - CVw\right) + B_t w_x + C(-Vw_x + w) + C_t w \\ &= (B - BV_x - B^2V + B_t - CV)w_x + (C + C_t - BCV)w. \end{aligned}$$

Now then,

$$w_{xxt} = (B - BV_x - B^2V + B_t - CV)w_x + (C + C_t - BCV)w. \tag{35}$$

By (34) and (35), we obtain

$$\begin{aligned} &(B - 2BV_x - B^2V - V_{xx} - B_xV - CV)w_x + (C - 2CV_x - BCV - C_xV)w \\ &\qquad = (B - BV_x - B^2V + B_t - CV)w_x + (C + C_t - BCV)w, \end{aligned}$$

$$-(BV_x + V_{xx} + B_xV)w_x - (2CV_x + C_xV)w = B_t w_x + C_t w.$$

It follows that,

$$(B_t + (BV)_x + V_{xx})w_x + (C_t + 2CV_x + C_xV)w = 0. \tag{36}$$

If

$$B_t + (BV)_x + V_{xx} = 0 \text{ and } C_t + 2CV_x + C_xV = 0$$

then (36) is fulfilled and the system (31), (32) is in involution. Taking account of lemma 1, we obtain the following statement

Theorem 5. *Let*

1) $V(x,t) \neq 0$ be a given function;

2) functions $B(x,t)$, $C(x,t)$ be solutions of Cauchy problem

$$B_t = -(BV)_x - V_{xx},$$

$$C_t = -2CV_x - C_xV,$$

$$B|_{t=0} = B_0(x), \quad C|_{t=0} = C_0(x)$$

for $B_0(x)$, $C_0(x)$;

3) functions $P(x,t)$, $\nu(x,t)$ be defined by formulas

$$P = -\int Be^a \left(\int \frac{Ce^a}{V} dx\right)^{-1} dx,$$

$$\nu = e^a \left(\int \frac{Ce^a}{V} dx\right)^{-1},$$

where $a = \int \frac{dx}{V}$.

Then the function $w(x,t)$ is a solution to involutive system

$$w_{xx} = Bw_x + Cw, \quad w_t = -Vw_x + w,$$

$$w|_{t=x=0} = w_0, \quad w_x|_{t=x=0} = w_1,$$

where w_0, w_1 are constants, satisfy to the Blokhintsev equation (25), where coefficients $P(x,t)$, $\nu(x,t)$ are defined by the formulas above.

Now, if

$$B_t + (BV)_x + V_{xx} \neq 0,$$

then from (36), we get

$$w_x = Dw,$$

where

$$D = -\frac{C_t + (CV)_x + CV_x}{B_t + (BV)_x + V_{xx}}. \tag{37}$$

In this case, the system (31), (32) is equivalent to the system

$$w_t = (1 - DV)w, \quad w_x = Dw. \tag{38}$$

Give consistency condition $w_{xt} = w_{tx}$:

$$(1 - DV)w_x - (DV)_x w = Dw_t + D_t w.$$

Owing to (38),

$$(1 - DV)Dw - (DV)_x w = D(1 - DV)w + D_t w.$$

It follows that,

$$D_t + (DV)_x = 0. \tag{39}$$

The relation (39) is a unique equation for functions B, C. Owing to (37), we can give system of Cauchy type:

$$C_t = -(CV)_x - CV_x - D(B_t + (BV)_x + V_{xx}), \quad D_t = -(DV)_x$$

for functions C, D. In the process the function B leaves over arbitrary.

Taking account of lemma 1, we get the following result.

Theorem 6. *Let*

1) $V(x,t)$, $B(x,t)$ be arbitrary functions such that

$$B_t + (BV)_x + V_{xx} \neq 0;$$

2) functions $C(x,t)$, $D(x,t)$ be a solution of Cauchy problem

$$C_t = -(CV)_x - CV_x - D(B_t + (BV)_x + V_{xx}), \quad D_t = -(DV)_x,$$

$$C|_{t=0} = C_0(x), \quad D|_{t=0} = D_0(x)$$

for some analytical functions $C_0(x)$, $D_0(x)$;

3) functions $P(x,t)$, $\nu(x,t)$ be defined by formulas

$$P = -\int Be^a \left(\int \frac{Ce^a}{V} d\,x\right)^{-1} d\,x,$$

$$\nu = e^a \left(\int \frac{Ce^a}{V} d\,x\right)^{-1},$$

where $a = \int \frac{d\,x}{V}$.

Then the function $w(x,t)$ which is a solution to an involutive system

$$w_t = (1 - DV)w, \quad w_x = Dw,$$

$$w|_{t=x=0} = w_0,$$

where w_0 is a constant, satisfies to Blokhintsev equation (25), coefficients $P(x,t)$, $\nu(x,t)$ of which are defined by formulas above.

5. Final remarks

Methods of construction of explicit solutions to mathematical physics equations based on entering of supplementary differential relation are the basis of differential constrains method [2] and has wide application area. So, for example, construction of invariant, partly-invariant solutions and so on by methods of group analysis of differential equations is direct application of differential constrains method with the use of algebraic structure of symmetry group of considered equation [6]. Differential constrains which have natural origin and are connected with assumed equation directly, are of the main interest. In this context, we formulate a problem for Blokhintsev equation which in our opinion be of interest, be connected with differential constrains methods, and directly enlarge analogous problem for classic wave equation

$$w_{tt} = \Delta w. \tag{40}$$

Problem. To describe all functional-invariant solutions to equation (1).

For equation (40), requirement of functional invariant property of a solution w leads to supplementary nonlinear differential equation of the first order

$$w_t^2 = |\nabla w|^2. \tag{41}$$

The system (40), (41) was studied intensively by many authors [7] – [10]. The final description for real-valued solutions was obtained in paper [17]. There are not a final description for complex-valued solutions, there are only special results [9].

Note, the equation (1) has variable coefficients ν, V, P contrary to wave equation (40). In this context, we note the problem setting by R. Courant: to find all linear hyperbolic equations of the second order which have collection of generalized functional-invariant solutions (GFIS). In two-dimensional case, a necessary and sufficient condition for coefficients of linear differential equation of the second order with two independent variables for existing GFIS was obtained [9]. In high dimensional cases, it is known only special classes of these equations, and in general case, this problem leaves unsolved. In papers [18, 19], special classes of equations and second-order systems with variable coefficients admitting GFIS are constructed and obtained some applications of these results to inverse problems.

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