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ESTIMATES OF A SINGLE PROBLEM OF ELECTRODYNAMICS ARISING IN MAGNETIC HYDRODYNAMICS IN SPACE $W_p^{2,1}(Q_T)$, $p > 1$

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Abstract. In this paper, unique solvability is obtained and estimates of solutions to the problem of magnetic hydrodynamics are obtained.

Keywords: boundary value problem, magnetic hydrodynamics, Hölder conditions, Volterra–Fredholm type, heat equation.

Let us introduce the subspace of solenoidal vectors.

As is known [1], that $L_2(\Omega) = G(\Omega) \oplus \overset{0}{J}(\Omega) = \overset{0}{G}(\Omega) \oplus J(\Omega)$, where $G(\Omega)$, $\overset{0}{G}(\Omega)$ are vector spaces in the form of $\vec{u} = \text{grad}\varphi$ and $\vec{u} = \text{grad}\psi$, and φ, $\psi \in W_2^1(\Omega)$, $\psi\big|_S = 0$, $J(\Omega)$ – the closure in the L_2 set of smooth solenoid vectors (i.e. such that), and $\overset{0}{J}(\Omega)$ is the closure of the set of smooth solenoid vectors satisfying the condition $u_n\big|_S = 0$.

Denote by a $\overset{0}{J}(Q_T)$, $J(Q_T)$ subspace of vectors from $L_2(Q_T)$, which almost all belong to $\overset{0}{J}(\Omega)$ and $J(\Omega)$ and define the following spaces $\overset{0}{J}{}^\alpha(\Omega) = \overset{0}{J}(\Omega) \cap C^\alpha(\Omega)$,

$$\overset{0}{J}{}^\alpha(Q_T) = \overset{0}{J}(Q_T) \cap C^{\alpha,\frac{\alpha}{2}}, \quad J^\alpha(\Omega) = J(\Omega) \cap C^\alpha(\Omega), \tag{1}$$

$$J(Q_T) = J(Q_T) \cap C^{\alpha,\frac{\alpha}{2}}(Q_T).$$

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Consider the following redefined inhomogeneous initial boundary value problem (electrodynamics) that arises from the system of equations of magnetic hydrodynamics during linearization:

$$\begin{aligned} &\sigma\mu\frac{\partial\overrightarrow{H}}{\partial t} - \Delta\overrightarrow{H} = \mathrm{rot}\vec{j},\ \ \mathrm{div}\overrightarrow{H} = 0, \\ &H_n\big|_S = 0,\ \ \mathrm{rot}_\tau\overrightarrow{H}\big|_S = \overrightarrow{\underset{0}{J}}\big|_S,\ \ \overrightarrow{H}\big|_{t=0} = \overrightarrow{H}_0(x). \end{aligned} \tag{2}$$

Also consider the following "auxiliary problem" to determine the vector $\overrightarrow{E}$:

$$\begin{aligned} &\sigma\mu\frac{\partial\overrightarrow{E}}{\partial t} - \Delta\overrightarrow{E} = 0,\ \ \mathrm{div}\overrightarrow{E} = 0, \\ &\overrightarrow{E}_\tau\big|_S = \overrightarrow{\psi}\ \ \overrightarrow{E}\big|_{t=0} = 0, \end{aligned} \tag{3}$$

studied in [2, 3]. In [2] it is shown that for any vector $\vec{\psi}(x,t)$ with $\psi_n\big|_S = 0$ that satisfies certain Hölder conditions for both arguments, problem (3) has a unique classical solution that can be represented as a potential

$$\overrightarrow{E}(x,t) = \mathrm{rot}\,\overrightarrow{A}(x,t), \tag{4}$$

$$\overrightarrow{A}(x,t) = \int\limits_0^t d\tau \int\limits_S \Gamma(x-y,t-\tau)\vec{\lambda}(\eta,\tau)dS_\eta, \tag{5}$$

where

$$\Gamma(x,t) = \begin{cases} -\dfrac{\sqrt{\sigma\mu}}{(4\pi t)^{\frac{3}{2}}}\cdot e^{-\frac{\sigma\mu x^2}{4t}} & \text{at} \quad t>0, \\ \quad 0 & \text{at} \quad t<0, \end{cases} \tag{6}$$

the fundamental solution of the heat equation, $\vec{\lambda}$ is vector defined from the system of integral equations of the Volterra–Fredholm type (the same as in the solution of initial boundary value problems for one heat equation) with the right side $\vec{\omega} = \left[\vec{\psi}\times\vec{n}\right] = 0$ (1).

In addition, the following theorem is proved in [3].

Theorem 1. If $S \in C^2$, then for all $\vec{\psi} \in W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(S_T)$ with $\psi_n\big|_S = 0$ satisfying the matching conditions :

$$\vec{\psi}\big|_{t=0} = 0 \quad \text{by} \quad p>3,$$

$$\int\limits_{S_T}\left|\vec{\psi}(x,t)\right|^3\cdot t^{-1}dSdt < +\infty \quad \text{by} \quad p=3,$$

vector $\overrightarrow{A}(x,t) \in W_p^{2,1}(Q_T)$ and

$$\begin{aligned} &\left\|\overrightarrow{A}\right\|_{W_p^{2,1}(Q_T)} \le C\left\|\vec{\psi}\right\|_{W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(S_T)} \qquad p\neq 3 \\ &\left\|\overrightarrow{A}\right\|_{W_3^{2,1}(Q_T)} \le C\left[\left\|\vec{\psi}\right\|_{W_3^{\frac{2}{3},\frac{1}{3}}(S_T)} + \left(\int\limits_{S_T}\left|\vec{\psi}\right|^3 t^{-1}dSdt\right)^{\frac{1}{3}}\right]. \end{aligned} \tag{7}$$

To use this result to evaluate solutions to problem (2), we must first reduce the system and the initial condition to homogeneous ones.

Lemma 1. Every vector $\vec{u}(x) \in W_p^r(\Omega) \cap J_p(\Omega)$ $(r \geq 0)$ at $r = 0$ $W_p^r\big|_{\tau=0} = L_p$ can be extended to all R^3 so that $\vec{u} \in W_p^\tau(R^3) \cap J_p(R^3)$ and

$$\|\vec{u}\|_{W_p^\tau(R^3)} \leq C\,\|\vec{u}\|_{W_p^\tau(\Omega)} \tag{8}$$

with a constant that does not depend on $\vec{u}(x)$. It is assumed that $S \in C^{1+\tau'}$ $r' > r$.

Evidence. Let be $\vec{v}(x)$ is a solution of the problem

$$\operatorname{rot}\vec{v} = \vec{u} \;\; \operatorname{div}\vec{v} = 0 \;\; v_n\big|_S = 0. \tag{9}$$

It is solvable for any $\overrightarrow{u}(x) \in J_p(\Omega)$ as shown in [4], the solution has a fair estimate

$$\|\vec{v}\|_{W_p^{\tau+1}(\Omega)} \leq C\,\|\vec{u}\|_{W_p^\tau(\Omega)}\,. \tag{10}$$

Let's now continue the vector $\vec{v}$ to the entire space R^3 with class preserving, i.e. so that

$$\|\vec{v}\|_{W_p^{\tau+1}}(R^3) \leq C\,\|\vec{v}\|_{W_p^{\tau+1}(\Omega)} \tag{11}$$

at $p = 2$ when this is done in [5], and when $p \neq 2$ we will consider here. We show that the vector $\operatorname{rot}\vec{v}$ will be the desired continuation of the vector $\vec{u}(x)$.

We apply this Lemma to a vector $\overrightarrow{H}_0(x) \in W_p^{2-\frac{2}{p}}(\Omega) \cap \overset{0}{J}_p(\Omega$ from problem (2), whose continuation we construct without using problem (9).

Here is the continuation construction. First of all, we need to continue vector $\overrightarrow{H}_0(x)$ with the class-preserving $W_p^{p-\frac{2}{p}}(\Omega)$ over the entire space R^3 and solve the Cauchy problem:

$$H_0(x) \to B = -\int\limits_\Omega G(x-y)\vec{\omega}(y)dy \Rightarrow \mathrm{R}^3 \to \tag{12}$$

$$\frac{\partial\vec{\omega}}{\partial t} - \Delta\vec{\omega} = 0,\;\; \vec{\omega}(x,t)\big|_{t=0} = \overrightarrow{H}_0(x).$$

Let $G(x) = (4\pi|x|)^{-1}$ be the fundamental solution of the Laplace equation. Consider the vector

$$\begin{aligned}\vec{z}(x,t) &= -\operatorname{rot}\operatorname{rot}\int\limits_\Omega G(x-y)\vec{\omega}(y,t)dy = \\ &= \vec{\omega}(x,t) - \operatorname{grad}\operatorname{div}\int\limits_\Omega G(x-y)\vec{\omega}(y,t)dy.\end{aligned} \tag{13}$$

At $x \in \Omega$ vector (13) satisfies the condition $\vec{z}(x,0) = \vec{\omega}(x,0) = \overrightarrow{H}_0(x)$.

Let's continue the vector $\overrightarrow{B}(x,t) = -\operatorname{rot}\int\limits_\Omega G(x-y)\vec{\omega}(y,t)dy$ for all R^3; then $\operatorname{rot}\overrightarrow{B}(x,0)$ will Show it as a continuation $\overrightarrow{H}_0(x)$.

First of all, for the vector $\vec{\omega}(x,t)$ from problem (12) we have [6]:

$$\|\vec{\omega}(x,t)\|_{W_p^{2,1}(\Pi_T)} \leq C\left\|\overrightarrow{H}_0(x)\right\|_{W_p^{2-\frac{2}{p}}(R^3)} \leq C\left\|\overrightarrow{H}_0\right\|_{W_p^{2-\frac{2}{p}}(\Omega)}, \tag{13*}$$

where $\Pi_T = R^3 \times [0,T]$.

Consider the vector $\vec{z}(x,t)$ by formula (13) and differentiate it by t. Then we have $\vec{z}_t(x,t) = -\operatorname{rot}\operatorname{rot}\int\limits_\Omega G(x-y)\vec{\omega}_t(y,t)dy$, and (13*).

Hence, using the Calderon–Zygmund theorem [7] on the limitation of singular integrals in [8], we obtain

$$\|\vec{z}_t(x,t)\|_{L_p(Q_T)} \le C\,\|\vec{\omega}_t\|_{L_p(Q_T)}\,,$$

i.e.

$$\|\vec{z}_t(x,t)\|_{L_p(Q_T)} \le C\,\|\vec{\omega}(x,t)\|_{W_p^1(0,T)}\,. \tag{14}$$

Let's rewrite expression (13) as

$$\vec{z}(x,t) = \vec{\omega}(x,t) - \operatorname{grad}\operatorname{div}\int\limits_{\Omega} G(x-y)\vec{\omega}(y,t)dy = \vec{\omega}(x,t) - \vec{\omega'}(x,t), \tag{15}$$

where

$$\begin{aligned}
&\vec{\omega'}(x,t) = \operatorname{grad}\operatorname{div}\int\limits_{\Omega} G(x-y)\vec{\omega}(y,t)dy = \\
&= \operatorname{grad}\int\limits_{\Omega} G(x-y)\operatorname{div}\vec{\omega}(x,t)dy + \operatorname{grad}_x\int\limits_{S} G(x-y)\vec{\omega}(y,t)\cdot\vec{n}(y)dS_y = \\
&= \int\limits_{\Omega} G(x-y)\operatorname{grad}\operatorname{div}\vec{\omega}(y,t)dy + \int\limits_{S} G(x-y)\vec{n}\operatorname{div}\vec{\omega}(y,t)dS_y + \\
&+ \operatorname{grad}\int\limits_{S} G(x-y)\omega_n(y,t)dS_y
\end{aligned}$$

and from here, taking derivatives by $x_i,\ x_j,\ (i,j=1,2,3)$ we have

$$\begin{aligned}
\frac{\partial^2\vec{\omega'}(x,t)}{\partial x_i\partial x_j} &= \frac{\partial^2}{\partial x_i\partial x_j}\int\limits_{\Omega} G(x-y)\operatorname{grad}\operatorname{div}\vec{\omega}(y,t)dy + \\
&+ \frac{\partial^2}{\partial x_i\partial x_j}\int\limits_{\Omega} G(x-y)\vec{n}(y)\operatorname{div}\vec{\omega}(y,t)dS_y + \\
&+ \frac{\partial^2}{\partial x_i\partial x_j}\operatorname{grad}\int\limits_{S} G(x-y)\omega_n(y,t)dS_y = \sum_{k=1}^{3}\vec{J}_k(x,t).
\end{aligned} \tag{16}$$

Show that $\vec{J}_k(x,t) \in L_p(Q_T)$. $\vec{J}_1(x,t)$ evaluated by the Calderon–Zygmund theorem [7] and therefore, we have

$$\left\|\vec{J}_1(x,t)\right\|_{L_p(\Omega)} \le C\,\|\operatorname{grad}\operatorname{div}\vec{\omega}(y,t)\|_{L_p(\Omega)} \le C\,\|\vec{\omega}\|_{W_p^2(\Omega)}\,. \tag{17}$$

for $\vec{J}_2(x,t)$ applying the potential estimation in L_p we have a chain of inequalities

$$\left\|\vec{J}_2(x,t)\right\|_{L_p(\Omega)} \le C\,\|\operatorname{div}\vec{\omega}\|_{W_p^{1-\frac{1}{p}}(S)} \le C\,\|\vec{\omega}\|_{W_p^{2-\frac{1}{p}}(S)} \le C\,\|\vec{\omega}\|_{W_p^2(\Omega)}\,. \tag{18}$$

$\vec{J}_3(x,t)$ is evaluated as follows. First, we transfer the derivative to the density of a simple layer, using the formula for differentiating the potential of a simple layer with a differentiated density (see, Chapter II, §8). Next, the resulting expression is applied to the estimation of potentials $L_p(\Omega)$.

$$\left\|\vec{J}_3(x,t)\right\|_{L_p(\Omega)} \le C\,\|D_y\vec{\omega}\|_{W_p^{2-\frac{1}{p}}(S)} + \|\vec{\omega}\|_{W_p^{2-\frac{1}{p}}(\Omega)} \le C\,\|\vec{\omega}\|_{W_p^2(\Omega)}\,. \tag{19}$$

Combining inequalities (17)–(19) and integrating by t, we have

$$\left\|\vec{\omega}_{x_ix_j}\right\|_{L_P(Q_T)} \le C\,\|\vec{\omega}\|_{W_p^{2,1}(Q_T)}\,. \tag{20}$$

Collecting estimates $(13^*, 14, 14^*)$ and (20), we have

$$\|\vec{z}(x,t)\|_{W_p^{2,1}(Q_T)} \le C\,\|\vec{\omega}\|_{W_p^{2,1}(Q_T)} \tag{21}$$

Now let's continue the vector $\overrightarrow{B}(x,t) = \text{rot}\int\limits_{\Omega} G(x-y)\vec{\omega}(y,t)dy$ from Q_T in Π_T. Let M be a continuation operator with the properties: if $M\overrightarrow{B} = \overrightarrow{B}^*$ and $S \in C^3$, then

1) $\left\|D^l_{x_k}\overrightarrow{B}^*\right\|_{L_p(\Omega)} \leq C\left\|D^l_{x_k}\overrightarrow{B}\right\|_{L_p(\Omega)}$ with $k = 0,1,2,3$;

2) M – the linear operator;

3) the operator M does not depend on t.

Let $\vec{z}^* = \text{rot}\overrightarrow{B}^*$, then by virtue of (11) we have

$$\int\limits_0^T \|\vec{z}^*\|_{W^2_p(R^3)}\,dt \leq C\int\limits_0^T \|\vec{z}\|_{W^2_p(\Omega)}\,dt \leq C\int\limits_0^T \|\vec{\omega}\|_{W^2_p(\Omega)}\,dt,$$

$$\|D_{x_n}\vec{z}^*\|_{W^m_p(R^3)} \leq C\left\|\overrightarrow{B}^*\right\|_{W^{m+1}_p(R^3)} \quad \text{with} \quad m = 0,1,2. \tag{22}$$

Next, check the software derivative by t. To do this, we make a difference (keeping in mind that $B^* = MB$)

$$\vec{z}^*(x,t+\Delta t) - \vec{z}^*(x,t) =$$
$$= \text{rot}\left[\overrightarrow{B}^*(x,t+\Delta t) - \overrightarrow{B}^*(x,t)\right] = \text{rot}M\cdot\left[\overrightarrow{B}(x,t+\Delta t) - \overrightarrow{B}(x,t)\right].$$

Hence we have chains of inequalities

$$\|\vec{z}^*(x,t+\Delta t) - \vec{z}^*(x,t)\|_{L_p(\Omega)} \leq C\left\|\overrightarrow{B}(x,t+\Delta t) - \overrightarrow{B}(x,t)\right\|_{W^1_p(\Omega)} \leq$$
$$\leq C\left\|\overrightarrow{B}(x,t+\Delta t) - \overrightarrow{B}(x,t)\right\|_{L_p(\Omega)} + \left\|\overrightarrow{B}_{x_k}(x,t+\Delta t) - \overrightarrow{B}_{x_k}(x,t)\right\|_{L_p(\Omega)} \leq$$
$$\leq C\left\|\overrightarrow{B}_t\right\|_{L_p(\Omega)}\Delta t + C\left\|\overrightarrow{B}_{tx}\right\|_{L_p(\Omega)}\cdot\Delta t,$$

of which when dividing both parts by we get

$$\|\vec{z}^*_t(x,t)\|_{L_p(R^3)} \leq C\left(\left\|\overrightarrow{B}_t\right\|_{L_p(\Omega)} + \left\|\overrightarrow{B}_{tx}\right\|_{L_p(\Omega)}\right) \leq C\,\|\vec{\omega}_t\|_{L_p(\Omega)} \tag{23}$$

and integrating the software by t, we have

$$\int\limits_0^E \|\vec{z}^*_t(x,t)\|_{L_p(R^3)}\,dt \leq C\int\limits_0^T \|\vec{\omega}_t\|_{L_p(\Omega)}\,dt \tag{24}$$

Thus, as a result of the established estimates (14), (21) and (24), we assert that the desired continuation will be $\vec{z}^* = \text{rot}\overrightarrow{B}^*$, for which equality is true $\vec{z}^*(x,t)\big|_{t=0} = \overrightarrow{H}^*_0(x)$, where $\overrightarrow{H}^*(x) = \overrightarrow{H}_0(x)$ at $x \in \Omega$ and inequality

$$\left\|\overrightarrow{H}^*(x,t)\right\|_{W^{2-\frac{2}{p}}_p(\Pi_T)} \leq C\,\|\vec{z}^*\|_{W^{2,1}_p(\Pi_T)} \leq C\left\|\overrightarrow{H}_0(x)\right\|_{W^{2-\frac{2}{p}}_p(\Omega)}.$$

So, the result is $r = 2 - \dfrac{2}{p}$, $\vec{u} = \overrightarrow{H}_0$

In particular, the solvability of stationary problems for systems of equations of electrodynamics and magnetohydrodynamics was studied in [9] - [12] and in a number of others.

Let us now begin to study the main task.

Theorem 2. If $S \in C^{(3)}$, then problem (2) has a single solution in the class $W_p^{2,1}(Q_T)$ at any $\overrightarrow{H}_0(x) \in W_p^{2-\frac{2}{p}}(\Omega) \cap \overrightarrow{J}_p(\Omega)$, $\mathrm{rot}\vec{j}_\tau \in W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(S_T)$ satisfying the matching condition:

$$\mathrm{rot}_\tau \overrightarrow{H}_0(x)\big|_{x\in S} = \vec{j}_\tau\big|_{x\in S} \quad \text{when} \quad p > 3,$$

$$\int\limits_0^T \int\limits_S \int\limits_\Omega \frac{\left|\mathrm{rot}\overrightarrow{H}_0(y) - \vec{j}_\tau(x,t)\right|^3}{(|x-y|^2+t)^{\frac{5}{2}}} dS_x dy dt < \infty \quad \text{when} \quad p = 3$$

(here $\vec{j}_\tau = \vec{j} - \vec{n}(y)(\vec{j},\vec{n})$, $\mathrm{rot}_\tau \overrightarrow{H}_0 = \mathrm{rot}\overrightarrow{H}_0 - \vec{n}(y)\mathrm{rot}\overrightarrow{H}_0$, the vector $\vec{n}(y)$ is continued with S inwards Ω so that $\vec{n} \in C^2(\Omega)$).

The solution of problem (2) obeys the inequality

$$\left\|\overrightarrow{H}\right\|_{W_p^{2,1}(Q_T)} \leq \tag{25}$$

$$\leq C\left[\left\|\mathrm{rot}\vec{j}\right\|_{L_p(Q_T)} + \left\|\overrightarrow{H}_0\right\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \left\|\vec{j}\right\|_{W_p^{1-\frac{1}{p},2-\frac{1}{2p}}(S_T)}\right], \quad p \neq 3,$$

$$\left\|\overrightarrow{H}\right\|_{W_3^{2,1}(Q_T)} \leq C\left[\left\|\mathrm{rot}\vec{j}\right\|_{L_3(Q_T)} + \left\|\overrightarrow{H}_0\right\|_{W_3^{\frac{2}{3}}(\Omega)} + \left\|\vec{j}_\tau\right\|_{W_3^{\frac{2}{3},\frac{1}{3}}(S_T)} + \right. \tag{26}$$

$$\left. + \left(\int\limits_0^T \int\limits_S \int\limits_\Omega \frac{\left|\mathrm{rot}\overrightarrow{H}_0(y) - \vec{j}_\tau(x,t)\right|^3}{(|x-y|^2+t)^{\frac{5}{2}}} dS_x dy dt\right)^{\frac{1}{3}}\right].$$

Proof. We continue the vector $\overrightarrow{H}_0(x)$ and $\vec{g} = \mathrm{rot}\vec{j}$ with class and solenoidally preserving in R^3, according to Lemma 1 with $r = 2 - \frac{2}{p}$ and $r = 0$, and let the $\vec{h}_0(x,t)$ is solution of the Cauchy problem

$$\sigma\mu\frac{\partial \vec{h}_0}{\partial t} - \Delta\vec{h}_0 = \vec{g}(x,t), \quad \vec{h}_0(x,t)\big|_{t=0} = \overrightarrow{H}_0(x). \tag{27}$$

Since $\mathrm{div}\vec{h}_0$ satisfies the homogeneous equation of thermal conductivity and vanishes at $t = 0$, then

$$\mathrm{div}\vec{h}_0(x,t) = 0 \tag{28}$$

Let $\overrightarrow{E}(x,t)$ is the solution of the "memory problem"

$$\begin{aligned} &\sigma\mu\frac{\partial \overrightarrow{E}}{\partial t} - \Delta\overrightarrow{E} = 0, \quad \mathrm{div}\overrightarrow{E} = 0, \\ &\overrightarrow{E}\big|_{t=0} = 0, \quad E_\tau\big|_{x\in S} = \left(-\mathrm{rot}_\tau\vec{h}_0 - \vec{j}\right)\big|_{x\in S}, \end{aligned} \tag{29}$$

a $\vec{h}(x,t)$ is defined by $\overrightarrow{E}$ as the solution to the problem

$$\mathrm{rot}\vec{h} = \overrightarrow{E}, \quad \mathrm{div}\vec{h} = 0, \quad \vec{h},\vec{n}\big|_{x\in S} = -\vec{h}_0\cdot\vec{n}\big|_{x\in S}. \tag{30}$$

It is solvable since $\overrightarrow{E} \in J_p(Q_T)$; also, for $\overrightarrow{E}(x,t)$ the formula (5) is fair, so that

$$\vec{h}(x,t) = \overrightarrow{A}(x,t) + \mathrm{grad}\varphi(x,t), \tag{31}$$

where $\varphi(x,t)$ is the solution to the next Neumann problem

$$\Delta\varphi = -\mathrm{div}\overrightarrow{A}, \quad \frac{\partial\varphi}{\partial n}\big|_{x\in S} = -\left(\overrightarrow{A}\cdot\vec{n} + \vec{h}_0\cdot\vec{n}\right)\big|_{x\in S}. \tag{32}$$

We show that $\vec{h}(x,t)$ is satisfies the relations

$$\sigma\mu\frac{\partial\vec{h}}{\partial t}-\Delta\vec{h}=0,\quad \mathrm{div}\vec{h}=0, \tag{33}$$

$$\vec{h}(x,t)\big|_{t=0}=0,\quad \mathrm{rot}_\tau\vec{h}\big|_{x\in S}=\left(-\mathrm{rot}_\tau\vec{h}_0-\vec{j}_\tau\right)\big|_{x\in S}, \tag{34}$$
$$\vec{h}\cdot\vec{n}\big|_{x\in S}=-\vec{h}_0\cdot\vec{n}\big|_{x\in S}.$$

All of them, except the heat equation, are obvious.

We show that this equation also holds. For a vector

$$\vec{\omega}(x,t)=\sigma\mu\vec{h}(x,t)-\int\limits_0^T\Delta\vec{h}(x,\tau)d\tau=0 \tag{35}$$

fair equality

$$\begin{aligned}&\mathrm{rot}\vec{\omega}=\sigma\mu\overrightarrow{E}(x,t)-\int\limits_0^T\Delta\overrightarrow{E}(x,\tau)d\tau=0,\quad \mathrm{div}\vec{\omega}=0,\\&\vec{\omega}_n(x,t)\big|_{x\in S}=\left[-\sigma\mu\vec{h}_{0n}+\int\limits_0^T\mathrm{rot}_n\mathrm{rot}\vec{h}(x,\tau)d\tau\right]\big|_{x\in S}.\end{aligned} \tag{36}$$

But since $\mathrm{rot}_n\vec{\varphi}$ depends only on $\vec{\varphi}_\tau\big|_{x\in S}$ (here under $\vec{\varphi}$ is $\mathrm{rot}\vec{h}(x,t)$) using (34) we get

$$\begin{aligned}&\vec{\omega}_n\big|_{x\in S}=-\sigma\mu\vec{h}_{0n}+\int\limits_0^t\mathrm{rot}_n\mathrm{rot}\vec{h}_0d\tau+\int\limits_0^t\mathrm{rot}_n\vec{j}d\tau=\\&=\sigma\mu\overrightarrow{H}_{0n}-\int\limits_0^t\left(\sigma\mu\frac{\partial h_{0n}}{\partial\tau}+\mathrm{rot}_n\mathrm{rot}\vec{h}_0-\mathrm{rot}_n\vec{j}\right)=0.\end{aligned} \tag{37}$$

It follows from (36) and (37) that $\vec{\omega}=0$; the difference of the vector (35) by t, we obtain the desired equation (33).

Due to issues: (27), (29), (34) and (37) the vector $\overrightarrow{H}(x,t)=\vec{h}_0+\vec{h}$ is a solution to problem (2).

We estimate the vectors $\vec{h}_0(x,t)$, $\vec{h}(x,t)$ and $\overrightarrow{H}(x,t)$. To solve the Cauchy problem (27), the following estimate is known

$$\begin{aligned}&\left\|\vec{h}_0(x,t)\right\|_{W_p^{2,1}(\Pi_T)}\le C\left(\left\|\overrightarrow{H}_0\right\|_{W_p^{2-\frac{2}{p}}(R^3)}+\|\vec{g}\|_{L_p(\Pi_T)}\right)\le\\&\le C\left(\left\|\overrightarrow{H}_0\right\|_{W_p^{2-\frac{2}{p}}(\Omega)}+\left\|\mathrm{rot}\vec{j}\right\|_{L_p(Q_T)}\right).\end{aligned} \tag{38}$$

The next step is to evaluate the potential of a simple layer in $W_p^{2,1}(Q_T)$:

$$\begin{aligned}&\left\|\overrightarrow{A}(x,t)\right\|_{W_p^{2,1}(Q_T)}\le\\&\le C\left(\left\|\mathrm{rot}\vec{h}_0\right\|_{W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(S_T)}+\left\|\vec{j}_\tau\right\|_{W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(S_T)}\right)\le\\&\le C\left(\left\|\vec{h}_0\right\|_{W_p^{2,1}(Q_T)}+\left\|\vec{j}_\tau\right\|_{W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(S_T)}\right),\quad p\neq 3;\\&\|\mathrm{A}\|_{W_3^{2,1}(Q_T)}\le\left(\left\|\vec{h}_0\right\|_{W_3^{2,1}(Q_T)}+\left\|\vec{j}_\tau\right\|_{W_3^{\frac{2}{3},\frac{1}{3}}(S_T)}+\right.\end{aligned} \tag{39}$$

$$+\left(\int\limits_0^T\int\limits_S\int\limits_\Omega \frac{\left|\mathrm{rot}\overrightarrow{H}_0(y)-\vec{j}_\tau(x,t)\right|^3}{(|x-y|^2+t)^{\frac{5}{2}}}dS_xdydt\right)^{\frac{1}{3}}\Bigg),\quad p=3.$$

Evaluate grad$\vec{\varphi}$. Since $\vec{\varphi}(x,t)$ is the solution of problem (33), as shown in [6, 13] and in the theorem on estimates of the thermal potential of a simple layer of the third order, we have

$$\|\mathrm{grad}\vec{\varphi}\|_{W_p^2(\Omega)}\le\|\vec{\varphi}\|_{W_p^3(\Omega)}\le \tag{40}$$

$$\le C\left(\left\|\overrightarrow{A}\cdot\vec{n}\right\|_{W_p^{2-\frac{1}{p}}(S)}+\left\|\vec{h}_0\cdot\vec{n}\right\|_{W_p^{2-\frac{2}{p}}(S)}\right)\le C\left(\left\|\overrightarrow{A}\right\|_{W_p^2(\Omega)}+\left\|\vec{h}_0\right\|_{W_p^2(\Omega)}\right).$$

To evaluate grad$\vec{\varphi}(x,t)$, we present $\vec{\varphi}_t(x,t)$ the sum $\vec{\varphi}_t(x,t)=\vec{\varphi}'_t(x,t)+\vec{\varphi}''_t(x,t)$, where $\vec{\varphi}'_t$ and $\vec{\varphi}''_t$ are the solution of the problems:

$$\Delta\varphi'_t=\mathrm{div}\overrightarrow{A}_t,\quad \frac{\partial\varphi'_t}{\partial t}\Big|_{x\in S}=-\overrightarrow{A}_t\cdot\vec{n}\Big|_{x\in S}, \tag{41}$$

$$\Delta\varphi''_t=0,\quad \frac{\partial\varphi''_t}{\partial t}\Big|_{x\in S}=-\vec{h}_{0t}\cdot\vec{n}\Big|_{x\in S}=\mathrm{rot}_n\mathrm{rot}\vec{h}_0\Big|_{x\in S}.$$

It is known [14] that every smooth $\vec{a}(x)$ can be represented as the sum of two orthogonal $L_2(\Omega)$ summands

$$\vec{a}(x)=\mathrm{grad}b(x)+\vec{c}(x),$$

where $\vec{c}(\vec{x})$ is the vector satisfying div$\vec{c}(x)=0$, $\quad\vec{c}\cdot\vec{n}\big|_S=0$, then $\vec{b}(x)$ defined from the Neumann problem:

$$\Delta\vec{b}(x)=\mathrm{div}\vec{a}(x),\quad \frac{\partial b}{\partial n}\Big|_{x\in S'}=\vec{a}\cdot\vec{n}\Big|_{x\in S'} \tag{42}$$

in which $\vec{n}(x)$ is the vector of the external normal to S'.

To evaluate the solutions to problem (41) , we will use the following results obtained in [8, 15]:

Fair assessment

$$\|\mathrm{grad}b\|_{L_p(\Omega)}+\|\vec{c}\|_{L_p(\Omega)}\le C\,\|\vec{a}\|_{L_p(\Omega)}, \tag{43}$$

which, when proved in [15], the solution of problem (42) is used in the form $b=b_1+b_2$, where

$$b_1=\mathrm{div}\overrightarrow{A}(x),\quad \overrightarrow{A}(x)=\int\limits_\Omega G(x-y)\vec{a}(y)dy,$$

b_2 is solving the problem

$$\Delta b_2=0,\quad \frac{\partial b_2}{\partial n}\Big|_{x\in S'}=(\vec{a}\cdot\vec{n}-\mathrm{grad}b_1\cdot\vec{n})\Big|_{x\in S'},$$

and to solve these problems, estimates are set

$$\|\mathrm{grad}b_1\|_{L_p(\Omega)}\le C\,\|\vec{a}\|_{L_p(\Omega)},$$

$$\|\mathrm{grad}b_2\|_{L_p(\Omega)}\le C\left\|\mathrm{rot}\overrightarrow{A}\right\|_{W_p^{1-\frac{1}{p}}(S)}\le C\left\|\overrightarrow{A}\right\|_{W_p^2(\Omega)}\le C\,\|\vec{a}\|_{L_p(\Omega)}.$$

Using all these results, we obtain the following estimates for solving problems (41):

$$\|\mathrm{grad}\varphi'_t\|_{L_p(\Omega)}\le C\left\|\overrightarrow{A}_t\right\|_{L_p(\Omega)},$$

$$\|\mathrm{grad}\varphi_t''\|_{L_p(\Omega)} \leq C \left\|\vec{h}_0\right\|_{W_p^2(\Omega)}.$$

From these estimates and from (40) it follows that

$$\|\mathrm{grad}\varphi\|_{W_p^{2,1}(Q_T)} \leq C \left(\left\|\overrightarrow{A}\right\|_{W_p^{2,1}(Q_T)} + \left\|\vec{h}_0\right\|_{W_p^{2,1}(Q_T)} \right),$$

instead of (38) and (39), it leads to inequality (25)–(26).

The constructed solution is unique, since $\vec{\omega}(x,t)$ the solution of problem (2) with $\vec{j}(x,t) = 0$, $\overrightarrow{H}_0(x) = 0$ is simultaneously the solution of a parabolic initial-boundary value problem

$$\frac{\partial \vec{\omega}}{\partial t} - \Delta \vec{\omega}(x,t) = 0,$$

$$\vec{\omega}(x,t)\big|_{t=0} = 0, \quad \vec{\omega} \cdot \vec{n}\big|_{x\in S} = 0, \quad \mathrm{rot}_\tau \vec{\omega}\big|_{x\in S} = 0.$$

(The complementarity condition is easily checked; it is sufficient to check it for a half-space $x_3 \geq 0$). As shown in [16] $\vec{\omega}(x,t) = 0$. The theorem is proved.

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