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THE MODULUS OF A FAMILY OF CURVES ON AN ABSTRACT SURFACE OVER A SPHERICAL RING

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Abstract. We obtain a two-sided estimate for the modulus of the family of all locally rectifiable curves joining two concentric spheres on a so-called abstract surface. The last notion means that, for a given curve and a point on it, the length element of the curve at this point depends on the direction of movement along the curve; in addition, the volume element is generated by some weight function.

Keywords: abstract surface, modulus of a family of curves, spherical ring.

1. Introduction

In [1], the notion of an abstract surface, which is used for studying conformal maps of irregular surfaces in Euclidean space, is introduced. The same concept, although in a slightly different form, is used in [2] and [3] for dealing with other problems. Moreover, V. M. Miklyukov (the author of all aforementioned books) believes that abstract surfaces have some applications: in particular, they can be used to model anisortopic media and media with dislocations. He adds that numerous examples of physically meaningful abstract metrics can be found in [4], which is devoted to gravitation, and also in [5], where the mathematical theory of black holes is exposed.

In [1], as it usually takes place in quasiconformal analysis, the main tools are the notions of the modulus of a family of curves and of the capacity of a condenser. Thus, there arises a natural problem of computing, or at least estimating, moduli

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of curve families on abstract surfaces defined over standard domains such as a cylindrical ring, a surface of revolution, etc. One example of a double inequality for the modulus of a family of curves on an abstract surface over a plane spherical ring can be found in Theorem 1.7.1 of [1, page. 24]. Other domains are studied in [6] and [7]. In this paper we will consider another important domain — a spherical ring in the n-dimensional Euclidean space.

As always, for $n \geqslant 1$, the set of ordered n-tuples $x = (x_1, \ldots, x_n)$ of real numbers is denoted by $\mathbb{R}^n$ (naturally, $\mathbb{R}^1 = \mathbb{R}$). With operations of addition and scalar multiplication defined in a coordinatewise fashion, $\mathbb{R}^n$ becomes an n-dimensional vector space over $\mathbb{R}$. For x and y in $\mathbb{R}^n$, the inner product $\langle x, y\rangle$ and the length $|x|$ are defined by the formulae

$$\langle x, y\rangle = x_1 y_1 + \ldots + x_n y_n, \quad |x| = \sqrt{\langle x, x\rangle}.$$

Definition 1. Let D be a domain in Euclidean space $\mathbb{R}^n$. Suppose given a locally summable function $\omega\colon D \to (0, \infty)$ and a continuous function $H\colon D \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following requirements[1] :

a) $H(x, \xi) \geqslant 0$ for all $x \in D$ and $\xi \in \mathbb{R}^n$;

b) for every $x \in D$, the set

$$\Xi(x) = \{\xi \in \mathbb{R}^n \colon H(x, \xi) < 1\} \tag{1}$$

is convex;

c) at every point $x \in D$, for all $\alpha \geqslant 0$ and $\xi \in \mathbb{R}^n$

$$H(x, \alpha\xi) = \alpha H(x, \xi).$$

Then the triple $\mathfrak{S} = (D, H, \omega)$ is called *an abstract surface* over the domain D.

On an abstract surface $\mathfrak{S}$, there appear the length element $ds_{\mathfrak{S}}$ generated by the function H and the volume element $\omega\, dx$, where dx stands for the Lebesgue measure in $\mathbb{R}^n$. Let us specify $ds_{\mathfrak{S}}$. For a Borel function $\rho\colon D \to [0, \infty]$ and a rectifiable[2] curve $\gamma\colon [a, b] \to D$, we define the integral of ρ along γ with respect to $ds_{\mathfrak{S}}$ by setting

$$\int_\gamma \rho\, ds_{\mathfrak{S}} = \int_0^{\ell(\gamma)} \rho(\gamma^\circ(s)) H(\gamma^\circ(s), \dot{\gamma}^\circ(s))\, ds,$$

where $\gamma^\circ\colon [0, \ell(\gamma)] \to D$ is a parametrization of the curve γ by the natural parameter s and $\ell(\gamma)$ is the length of γ. The vector $\dot{\gamma}^\circ(s)$ exists for almost all $s \in [0, \ell(\gamma)]$ and, therefore, the indeterminacy of the quantity $H(\gamma^\circ(s), \dot{\gamma}^\circ(s))$ at the remaining points does not affect the value of the integral. It is useful to note that if $\gamma\colon [a, b] \ni t \mapsto \gamma(t)$ is an absolutely continuous curve (of course, here the parameter t is not necessarily natural), then the formula for the integral remains the same, i.e.

$$\int_\gamma \rho\, ds_{\mathfrak{S}} = \int_a^b \rho(\gamma(t)) H(\gamma(t), \dot{\gamma}(t))\, dt.$$

Following Miklyukov, we define the modulus of a family of curves on an abstract surface. Let $\mathfrak{S} = (D, H, \omega)$ be an abstract surface over a domain $D \subset \mathbb{R}^n$, and let

[1] By satisfying conditions a), b) and c) for all $x \in D$ a function $H(x, \cdot)\colon \mathbb{R}^n \to \mathbb{R}$ partially resembles a norm. Indeed, condition a) ensures this function is non-negative, b) implies that it satisfies the triangle inequality, and c) guarantees positive (but not absolute) homogenity.

[2] Throughout, “rectifiability” and “local rectifiability” are understood with respect to the usual Euclidean metric.

$p \in (1,\infty)$. Suppose given a family Γ of locally rectifiable curves in D. A Borel function $\rho\colon D \to [0,\infty]$ is said to be *admissible* for Γ (we write $\rho \in \mathrm{Adm}(\Gamma)$) if $\int_\gamma \rho\, ds_{\mathfrak{S}} \geqslant 1$ for every $\gamma \in \Gamma$. The quantity

$$\mathrm{mod}_p(\Gamma;\mathfrak{S}) = \inf_{\rho\in\mathrm{Adm}(\Gamma)} \int_D \rho^p(x)\omega(x)\,dx. \tag{2}$$

is called the *p-modulus* of the curve family Γ on the abstract surface $\mathfrak{S}$.

We will also need the notion of capacity. Let E and F be disjoint compact sets in $\overline{D}$. We say that the triple $(E,F;\mathfrak{S})$ is a *condenser* on an abstract surface $\mathfrak{S} = (D,H,\omega)$. For $p \in (1,\infty)$, the *p-capacity* of this condenser is the number

$$\mathrm{cap}_p(E,F;\mathfrak{S}) = \inf_{\varphi\in\mathrm{Adm}(E,F;\mathfrak{S})} \int_D G^p(x,\nabla\varphi(x))\omega(x)\,dx, \tag{3}$$

where the family $\mathrm{Adm}(E,F;\mathfrak{S})$ of admissible functions for the condenser $(E,F;\mathfrak{S})$ consists of all continuous functions φ in $D\cup E\cup F$ that belong to the class ACL in D and satisfy the equalities $\varphi|_E = 0$ and $\varphi|_F = 1$. The function $G\colon D\times\mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ can be computed by the following formula

$$G(x,\eta) = \sup_{\xi\in\Xi(x)} \langle \xi,\eta\rangle, \tag{4}$$

where the set $\Xi(x)$ is defined by (1). It can be easily verified that conditions a),b) and c) of Definition 1 hold if we substitute[3] G for H. The function G is said to be *dual* to H.

It was shown in [1, Theorem 1.8.1] that if Γ is the family of all locally rectifiable curves in D joining the sets E and F, then

$$\mathrm{cap}_p(E,F;\mathfrak{S}) = \mathrm{mod}_p(\Gamma;\mathfrak{S}). \tag{5}$$

Recall that $\gamma\colon (a,b) \to \mathbb{R}^n$ *joins* sets E and F (in this very order) if $\lim\limits_{t\to a+0}\gamma(t) \in E$ and $\lim\limits_{t\to b-0}\gamma(t) \in F$, provided these limits exist. We also mention that the coincidence of p-modulus and p-capacity in Finsler and sub-Finsler spaces was proved by Dymchenko [8], [9].

2. Results

Hereinafter, by S^{n-1} we denote the unit sphere in $\mathbb{R}^n$ centered at zero, i.e., $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$; the symbol $\mathcal{H}^{n-1}$ designates the $(n-1)$–dimensional Hausdorff measure in $\mathbb{R}^n$.

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[3]We will check this for b), i.e., we will show that the set

$$\Theta(x) = \{\eta \in \mathbb{R}^n : G(x,\eta) < 1\}$$

is convex. To this end, fix a real number $\lambda \in [0,1]$ and vectors $\eta_1 \in \Theta(x)$ и $\eta_2 \in \Theta(x)$. Then it follows from well-known properties of the supremum that

$$\begin{aligned} G(x,\lambda\eta_1+(1-\lambda)\eta_2) &= \sup_{\xi\in\Xi(x)} (\lambda\langle\xi,\eta_1\rangle + (1-\lambda)\langle\xi,\eta_2\rangle) \leqslant \lambda \sup_{\xi\in\Xi(x)}\langle\xi,\eta_1\rangle + (1-\lambda)\sup_{\xi\in\Xi(x)}\langle\xi,\eta_2\rangle \\ &= \lambda G(x,\eta_1) + (1-\lambda)G(x,\eta_2) < \lambda + 1 - \lambda = 1, \end{aligned}$$

whence $\lambda\eta_1 + (1-\lambda)\eta_2 \in \Theta(x)$.

Theorem 1. *Suppose given a point $x_0 \in \mathbb{R}^n$ and real numbers a and b such that $0 < a < b < \infty$. We introduce the following sets*

$$E = \{x \in \mathbb{R}^n \colon |x - x_0| = a\}, \quad F = \{x \in \mathbb{R}^n \colon |x - x_0| = b\},$$
$$D = \{x \in \mathbb{R}^n \colon a < |x - x_0| < b\}.$$

Assume that an abstract surface $\mathfrak{S} = (D, H, \omega)$, described in Definition 1, is defined over D. If Γ is the family of all locally rectifiable curves joining spheres E and F in D, then for $1 < p < \infty$

$$\int_{S^{n-1}} \left[\int_a^b H^{\frac{p}{p-1}}(ru + x_0, u) \omega^{\frac{1}{1-p}}(ru + x_0) r^{\frac{1-n}{p-1}} \, dr \right]^{1-p} d\mathcal{H}^{n-1}(u) \leqslant \operatorname{mod}_p(\Gamma; \mathfrak{S})$$
$$\leqslant \frac{1}{\left(\int_a^b r^{\frac{1-n}{p-1}} \, dr \right)^p} \int_{S^{n-1}} \left[\int_a^b G^p\,(ru + x_0, u)\, \omega(ru + x_0) r^{\frac{p-n}{p-1}} \, dr \right] d\mathcal{H}^{n-1}(u),$$

where the function G is defined by equality (4).

Proof. We will first prove the lower bound. It is clear that for all $u \in S^{n-1}$ a curve $\gamma_u \colon (a, b) \to \mathbb{R}^n$ defined by the formula $\gamma_u(r) = ru + x_0$ lies in Γ. Consider an arbitrary point $u \in S^{n-1}$ and any function $\rho \in \operatorname{Adm}(\Gamma)$. Making use of Hölder's inequality we obtain that

$$1 \leqslant \int_{\gamma_u} \rho \, ds_{\mathfrak{S}} = \int_a^b \rho(ru + x_0) H(ru + x_0, u) \, dr$$
$$= \int_a^b \rho(ru + x_0) \omega^{\frac{1}{p}}(ru + x_0) r^{\frac{n-1}{p}} H(ru + x_0, u) \omega^{-\frac{1}{p}}(ru + x_0) r^{\frac{1-n}{p}} \, dr$$
$$\leqslant \left(\int_a^b \rho^p(ru + x_0) \omega(ru + x_0) r^{n-1} \, dr \right)^{\frac{1}{p}} C^{\frac{p-1}{p}}(u),$$

where

$$C(u) = \int_a^b H^{\frac{p}{p-1}}(ru + x_0, u) \omega^{\frac{1}{1-p}}(ru + x_0) r^{\frac{1-n}{p-1}} \, dr. \tag{6}$$

Consequently,

$$\int_a^b \rho^p(ru + x_0) \omega(ru + x_0) r^{n-1} \, dr \geqslant C^{1-p}(u). \tag{7}$$

This inequality holds for all $u \in S^{n-1}$, therefore, we can integrate it over S^{n-1}. In virtue of the coarea formula and Fubini's theorem, we get

$$\int_D \rho^p(x) \omega(x) \, dx = \int_a^b \left[\int_{S^{n-1}} \rho^p(ru + x_0) \omega(ru + x_0) d\mathcal{H}^{n-1}(u) \right] r^{n-1} \, dr$$
$$= \int_{S^{n-1}} \left[\int_a^b \rho^p(ru + x_0) \omega(ru + x_0) r^{n-1} \, dr \right] d\mathcal{H}^{n-1}(u).$$

Taking into account (7), (6) and the definition of the modulus (2), we obtain the required lower bound.

Now we will prove the upper bound. In view of (5), we can estimate the p-capacity instead of the p-modulus. Note that the function

$$\varphi(x) = \frac{\int_a^{|x-x_0|} r^{\frac{1-n}{p-1}}\,dr}{\int_a^b r^{\frac{1-n}{p-1}}\,dr}, \quad a \leqslant |x-x_0| \leqslant b,$$

is admissible for the condenser $(E, F; \mathfrak{S})$, since $\varphi|_E = 0$, $\varphi|_F = 1$. For every $i = 1, \dots, n$

$$\frac{\partial}{\partial x_i}\int_a^{|x-x_0|} r^{\frac{1-n}{p-1}}\,dr = |x-x_0|^{\frac{1-n}{p-1}}\frac{\partial|x-x_0|}{\partial x_i} = |x-x_0|^{\frac{1-n}{p-1}}\frac{x_i-(x_0)_i}{|x-x_0|},$$

whence

$$\nabla\varphi(x) = \frac{|x-x_0|^{\frac{1-n}{p-1}}}{\int_a^b r^{\frac{1-n}{p-1}}\,dr}\frac{x-x_0}{|x-x_0|}, \quad a \leqslant |x-x_0| \leqslant b.$$

For each point $x \in D$ there exist $r \in (a, b)$ and $u \in S^{n-1}$ such that $x - x_0 = ru$. By using the coarea formula, we obtain the following equality:

$$\int_D G^p(x, \nabla\varphi)\omega(x)\,dx$$
$$= \int_a^b \left[\int_{S^{n-1}} G^p\left(ru + x_0, \frac{r^{\frac{1-n}{p-1}}}{\int_a^b r^{\frac{1-n}{p-1}}\,dr}u\right)\omega(ru+x_0)\,d\mathcal{H}^{n-1}(u)\right] r^{n-1}\,dr.$$

Taking into account (3) and changing the order of integration in the last relation by Fubini's theorem, we obtain the required upper bound. □

Our result has a visible drawback: we use the function G instead of the originally given function H when expressing the upper estimate. We have to do so, because the modulus is usually estimated from above by presenting some admissible (ideally – extreme) function. In our case it is very hard to do so, since the length element $ds_{\mathfrak{S}}$ can be very intricate. In [1, Page 25] an upper and a lower estimate for the conformal modulus of the family of curves joining two concentric circles on the plane is obtained in terms of the function H, however, this function in [1] has some specific properties and the proof of the estimates involves a number of nontrivial results.

We now present some examples where our results are used for various H and ω.

Example 1. Under the assumptions of Theorem 1, set

$$H(x, \xi) = \lambda(x)|\xi|, \quad \omega(x) = \lambda^p(x),$$

where λ is a positive continuous function in D (if $p = n$, it is said that the length element $ds_{\mathfrak{S}}$ and the volume element $\omega\,dx$ are expressed in "isothermal coordinates"). Then $G(x, \eta) = |\eta|/\lambda(x)$. The two-sided estimate from Theorem 1 leads to the following equality:

$$\operatorname{mod}_p(\Gamma; \mathfrak{S}) = \sigma_{n-1}\left(\int_a^b r^{\frac{1-n}{p-1}}\,dr\right)^{1-p},$$

where $\sigma_{n-1} = \mathcal{H}^{n-1}(S^{n-1})$. In particular, if $p = n$, then

$$\operatorname{mod}_n(\Gamma; \mathfrak{S}) = \sigma_{n-1}\left(\ln\frac{b}{a}\right)^{1-n}.$$

The obtained expressions for the modulus coincide with those which are widely known in the standard weightless Euclidean space, i.e., when $\lambda \equiv 1$ (see, for instance, [10, Theorem 4.2.4]).

In the next examples we will use the following

Lemma 1. *Let e be a vector of length 1 in $\mathbb{R}^n$. If the function H from Definition 1 is defined by the formula $H(x,\xi) = |\langle e,\xi\rangle|$, then one can compute its dual function as follows:*

$$G(x,\eta) = \begin{cases} |\langle e,\eta\rangle|, & \textit{if vectors } e \textit{ u } \eta \textit{ are collinear}, \\ \infty & \textit{otherwise.} \end{cases}$$

Proof. Fix a vector n such that $|n| = 1$ and $\langle n,e\rangle = 0$. For every $\xi \in \mathbb{R}^n$, introduce the following quantities:

$$\xi^{\|} = \langle e,\xi\rangle, \quad \xi^{\perp} = \langle n,\xi\rangle.$$

Then $\xi = \xi^{\|}e + \xi^{\perp}n$ and $\langle e,\xi\rangle = \xi^{\|}$. Hence, by (4),

$$G(x,\eta) = \sup_{\{\xi\in\mathbb{R}^n:\ |\xi^{\|}|<1\}} \left(\xi^{\|}\langle e,\eta\rangle + \xi^{\perp}\langle n,\eta\rangle\right). \tag{8}$$

If vectors η and e are collinear, i.e., $\langle n,\eta\rangle = 0$, then

$$G(x,\eta) = \sup_{-1<\xi^{\|}<1} \xi^{\|}\langle e,\eta\rangle = |\langle e,\eta\rangle|.$$

If $\langle n,\eta\rangle \neq 0$, then by unboundedly increasing the absolute value of $\xi^{\perp}$ in (8), we discover that $G(x,\eta) = \infty$. □

Example 2. Under the assumptions of Theorem 1, let

$$H(x,\xi) = \left|\left\langle \frac{x-x_0}{|x-x_0|},\xi\right\rangle\right|, \quad \omega \equiv 1.$$

Using Lemma 1, we deduce that the two-sided estimate from Theorem 1 yields the same expression for $\mathrm{mod}_p(\Gamma;\mathfrak{S})$ as in Example 1.

Example 3. Under assumptions of Theorem 1, for every $x \in D$, consider a vector $n(x)$, which satisfies the following conditions:

$$\left\langle \frac{x-x_0}{|x-x_0|}, n(x)\right\rangle = 0, \quad |n(x)| = 1.$$

Set

$$H(x,\xi) = |\langle n(x),\xi\rangle|, \quad \omega \equiv 1.$$

Applying Lemma 1, we infer that the two-sided estimate from Theorem 1 yields $\mathrm{mod}_p(\Gamma;\mathfrak{S}) = \infty$. This result can be explained by the fact that the radial segments have zero length with respect to $ds_{\mathfrak{S}}$ generated by H. Therefore, the family of these segments alone, which is a subfamily of Γ, has infinite p-modulus.

References

[1] V.M. Miklyukov, *Conformal Mapping of Irregular Surfaces and Their Applications*, Volgograd. Gos. Univ., Volgograd, 2005 [in Russian].
[2] V.M. Miklyukov, *Geometric Analysis. Differential Forms, Almost-Solutions, Almost Quasiconformal Mappings*, Volgograd. Gos. Univ., Volgograd, 2007 [in Russian].
[3] V.M. Miklyukov, *Functions of Sobolev Weight Classes, Anisotropic Metrics, and Degenerating Quasiconformal Mappings*, Volgograd Gos. Univ., Volgograd, 2010 [in Russian].
[4] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, W. H. Freeman and Co., San Francisco, Calif., 1973. MR0418833
[5] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press, Oxford, 1998. MR1647491
[6] M.V. Tryamkin, *An Estimate of the Modulus of a Family of Curves on an Abstract Surface over a Cylinder*, Math. Notes, **107**:1 (2020), 177–181. MR4045694
[7] M.V. Tryamkin, *Modulus Estimates on Abstract Surfaces over a Domain of Revolution and a Cylindrical Ring*, Math. Notes, **108**:2 (2020), 297–301. MR4133422
[8] Yu.V. Dymchenko, *Equality of the capacity and modulus of a condenser in Finsler spaces*, Math. Notes, **85**:4 (2009), 566–573. MR2549420
[9] Yu.V. Dymchenko, *Equality of the capacity and module of a condenser in a sub-Finsler space*, J. Math. Sci. (N.Y.), **225**:6 (2017), 883–892. MR3580132
[10] F.W. Gehring, G.J. Martin, B.P. Palka, *An Introduction to the Theory of Higher-Dimensional Quasiconformal Mappings*, Math. Surveys Monogr., 216, Amer. Math. Soc., Providence, RI, 2017. MR3642872

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