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MEASURING THE ARCS OF THE ORBIT OF A ONE-PARAMETER TRANSFORMATION GROUP

I.V. POLIKANOVA

Abstract. In the orbits of the one-parameter transformation group, various geometric structures are introduced by means of bijection with the set of real numbers, in particular, the structure of the Euclidean straight line in the sense used by Hilbert with corresponding concepts: points, half-orbits (an analogue of a ray), an oriented arc (an analogue of a segment with ordered endpoints), the relations of "between" for three points, the equality of oriented arcs and other invariants of the transformation group. It is shown that the measure of arcs, defined as a positive definite additive function that is invariant with respect to the group of transformations, exists and splits into two independent measures which are uniquely defined on the classes of arcs of a similar orientation by setting the standards — one in each class — and coinciding with the measurement results by these standards from different endpoints of the arc. λ-Congruence permits to measure oppositely oriented arcs using a single standard. In this case, the opposite arcs ab and ba (not λ-congruent) have different measure values. This circumstance forces us to question the correctness of the known proofs of the existence and uniqueness of the measure of the length of a line-segment in Euclidean space. With λ-congruence for $\lambda = -1$, the orbit becomes a model of Euclidean straight line.

Keywords: Orbits of a one-parameter group of transformations, midpoint of an arc, equal arcs, length of an arc, invariant with respect to some group of transformations, split-complex numbers (hyperbolic numbers).

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1. Introduction

It is known that the length of a straight line segment can be measured by sequentially laying off a fixed segment (standard) and dividing it in half [1-3]. More precisely, the measurement process consists of assigning a positive real number l, represented by a binary rational fraction $n, i_1 i_2 ... i_k ...$, to any segment $[AB]$ in accordance with the following rule: 1) A sequence of segments $[A_0 A_1]$, $[A_1 A_2]$,..., $[A_k A_{k+1}], ...$, congruent to the standard, is built on the ray $[AB)$ from its vertex $A = A_0$. If for some natural number n the point A_n coincides with B, then we assume that $l = n$, and the measurement process ends. If the point B is between A_n and A_{n+1} for some n, then the integer part of the number l is prescribed a value equal to n, and we proceed to the next step. 2) Let R_0 be the midpoint of the segment $[A_n A_{n+1}]$. If the point B coincides with the point R_0, then assume that $l = n, 1$ and terminate the measurement process. Otherwise, put $l = n, 1$ or $l = n, 0$ depending on whether the point B belongs to the segment $[R_0 A_{n+1}]$ or $[A_n R_0]$, and move to the next step. 3) Plot the midpoint R_1 of the segment $[R_0 A_{n+1}]$, $[A_n R_0]$, which contains the point B, and prescribe $l = n, i_1 1$, if the segment $[A_n B]$ contains, in addition to the i_1 "halves" of the standard, also a "quarter" of the standard, otherwise, $l = n, i_1 0$, etc. The process finishes if the point B coincides with one of the midpoints R_0, $R_1, ...$ of the binary parts of the standard. In this case, the length is written as a final binary fraction. Otherwise, the fraction will be infinite.

The arcs of a circle can be measured using the same method if the measure is required to be invariant with respect to the group of similarities. What other kinds of curves permit the same measurement method?

In [4], the author substantiated that the moments curve (also called enika), given in some affine frame by the parameterization

$$\vec{r} = (t, t^2, t^3, ..., t^n),\ t \in \mathbb{R}, \tag{1}$$

is a model of a straight line in absolute geometry according to Hilbert with an appropriate definitions of the basic concepts, in particular, if we understand the equality of segments as the equivalence of the enika arcs with respect to unimodular affine transformations.

Further searches in this direction led us to curves that are the orbits of one-parameter transformation groups. The theory of measuring their arcs constitutes the content of this paper.

On the orbit of a one-parameter group of transformations, various geometric structures are introduced by means of bijection with the set of real numbers, in particular, the structure of an Euclidean straight line in the sense used by Hilbert with the corresponding concepts: a half-orbit (an analogue of a ray), an oriented arc or, in short, an ord-arc (an analogue of a segment with ordered endpoints), relations of "between" for three points, equality of ord-arcs and other invariants of the transformation group.

It is shown that the measure of the arcs, defined as a positive definite additive function that is invariant with respect to the group of transformations, exists and splits into two independent measures that are uniquely defined on the classes of ord-arcs of a similar orientation by setting the standards — one in each class — and coinciding with the measurement results made by these standards from different ends of the arc. λ-Congruence allows us to measure oppositely oriented ord-arcs using one standard. In this case, the opposite ord-arcs ab and ba (which are not

λ-congruent) have different measure values. This circumstance forces us to question the correctness of the known proofs of the existence and uniqueness of the measure of the length of a line-segment in Euclidean space. With λ-congruence for $\lambda = -1$, the orbit becomes a model of Euclidean straight line.

2. Orbits of a one-parameter transformation group

Suppose that *the action of the additive topological group* $\mathbb{R}$ *of real numbers* on the topological space E is defined, that is, a homomorphism g of the group $\mathbb{R}$ into the group of continuous transformations $\mathbb{G}_E$ of the space E, which assigns to each number $a \in \mathbb{R}$ a transformation g_a, so that:

1) $g_0 = i$ is the identity transformation,

2) $g_{a+b} = g_a g_b$,

3) the function σ of two variables $a \in \mathbb{R}$ and $x \in E$, defined by $\sigma(a, x) = g_a(x)$, is a continuous function $\mathbb{R} \times E \to E$.

The group

$$\mathbb{G} = g(\mathbb{R}) = \{g_a |\ a \in \mathbb{R}\}$$

is called *the one-parameter group of transformations of the space* E.

For a group action and related concepts, see, for example, [5], [6].

The orbit of an element $x \in E$ *under the action of a one-parameter group* G is the set

$$\{g_a(x)|\ a \in \mathbb{R}\}.$$

According to condition 3), the orbit of a one-parameter transformation group is a continuous image of a straight line.

Let γ be the orbit of a fixed element $o^* \in E$. We denote $g_a(o^*) = a^*$. As g_0 is the identity transformation, we have $g_0(o^*) = o^* = 0^*$.

Since

$$g_s(a^*) = g_s g_a(o^*) = g_{s+a}(o^*) = (s + a)^* \tag{2}$$

for all $s \in \mathbb{R}$, the following statement is true: *the orbit of a one-parameter group of transformations is invariant with respect to transformations of this group.* Note that the group $\mathbb{R}$ acts *transitively* on γ: for every $a^* \in \gamma$ and $b^* \in \gamma$ there is an element $s \in \mathbb{R}$, such that $g_s(a^*) = b^*$. Indeed, as seen from (2), such element is the number $s = b - a$. But such element is not necessarily unique. This means that the action of the group $\mathbb{R}$ on γ may not be *effective.*

Below we will consider some examples of orbits of one-parameter transformation groups: of the Euclidean plane in examples 1-3 and of the affine n-dimensional space in examples 4-5.

Example 1. The orbits of the group of rotations of the plane around the center x are concentric circles with the center x and the point x itself.

Examples 2, 3. Let the transformations of the Euclidean plane be specified in the polar coordinate system by the formulas

$$1)\ \bar{r} = re^{ks},\ \ \bar{\phi} = \phi + s,$$

$$2)\ \bar{r} = r + ks,\ \ \bar{\phi} = \phi + s,$$

where (r, ϕ) are the polar coordinates of a point, $(\bar{r}, \bar{\phi})$ are the polar coordinates of its image, and k is a fixed number. Then the orbits are logarithmic spirals

$$r = \left(r_0 e^{-k\phi_0}\right) e^{k\phi}$$

in the first case and Archimedes spirals

$$r = (r_0 - k\phi_0) + k\phi$$

in the second case.

Example 4. The orbits of the group containing all parallel translations to the vectors of the form $t\vec{e}$, where $\vec{e}$ is a fixed nonzero vector, are straight lines parallel to the vector $\vec{e}$.

Example 5. The orbits of the groups of affine transformations defined in some affine coordinate system by formulas

$$\bar{x}_i = \sum_{j=1}^{i} C_i^j s^{i-j} x_j + s^i, \quad i = 1, ..., n \tag{3}$$

(here C_i^j are binomial coefficients and the superscripts at the parameter s denote degrees), are the moments curves (eniks), since they are obtained from (1) by means of affine transformations

$$\bar{y}_i = \sum_{j=1}^{i} C_i^j x_j y_{i-j} + y_i, \quad i = 1, ..., n,$$

with a determinant which is not equal to 0 (it is equal to 1).

The action of the group $\mathbb{R}$ on the orbits in example 1 is not effective, since for every $s \in \mathbb{R}$ it is true that $g_s(o^*) = g_{s+2\pi k}(o^*)$, where k is an integer. In all other examples, the group $\mathbb{R}$ acts effectively on orbits.

3. Invariants of a one-parameter transformation group

From here until Section 11, we assume that the *group* $\mathbb{R}$ *acts effectively on its orbits.* Then the mapping $a \to a^*$ is bijective: each point of the orbit corresponds to a single parameter $a \in \mathbb{R}$.

An oriented arc, or, in short, an *ord-arc*, is an ordered pair of orbital points (a^*, b^*), possibly, $a^* = b^*$; we denote it by a^*b^*. In this case, a^* is considered the *tail* and b^* is the *head* of the ord-arc, and both points are called the *endpoints* of the ord-arc.

The quantity $\langle a^*b^* \rangle = b - a$ will be called the *measure of the ord-arc* a^*b^*.

Proposition 1. *The measure of the ord-arc is an invariant of the group* $g(\mathbb{R})$.

Proof. We agree to consider that the image $f(a^*b^*)$ of the ord-arc under transformation f of the orbit is the ord-arc $a_1^*b_1^*$ such that $f(a^*) = a_1^*$, $f(b^*) = b_1^*$. Then $\langle g_t(a^*b^*) \rangle = \langle (a+t)^*(b+t)^* \rangle = (b+t) - (a+t) = b - a = \langle a^*b^* \rangle$. □

An ord-arc is said to be *zero* one if its measure equals zero, which means that its tail and head coincide.

We call an ord-arc a^*b^* *positively oriented* if $\langle a^*b^*\rangle > 0$, and *negatively oriented* if $\langle a^*b^*\rangle < 0$.

Two nonzero ord-arcs have *a similar orientation* (are *co-directed*) if both of them are positively oriented or both are negatively oriented, otherwise, they are *oppositely oriented.* The relation of a similar orientation of the ord-arcs, which is obviously an equivalence on the set of nonzero ord-arcs, will be denoted by the symbol $\uparrow\uparrow$, while for the relation of the opposite orientation of the ord-arcs, we will use the denotation $\uparrow\downarrow$.

Proposition 1 implies

Proposition 2. *Transformations of the group $g(\mathbb{R})$ map zero ord-arcs into zero ord-arcs and preserve the orientation of nonzero ord-arcs.*

The *midpoint* of the ord-arc a^*b^* is $\left(\frac{a+b}{2}\right)^*$.

Proposition 3. *The midpoint of the ord-arc a^*b^* is an invariant of the group $g(\mathbb{R})$.*

Proof. The midpoint of the ord-arc $g_t(a^*b^*) = (t+a)^*(t+b)^*$ is the point

$$\left(\frac{(t+a)+(t+b)}{2}\right)^* = \left(t+\frac{a+b}{2}\right)^* = g_t\left(\left(\frac{a+b}{2}\right)^*\right).$$

In other words, for every transformation of the group $g(\mathbb{R})$, the midpoint of the ord-arc is mapped to the midpoint of its image. □

Two ord-arcs are called *equal* if their measures are equal:

$$a^*b^* = c^*d^* \quad \Leftrightarrow \quad \langle a^*b^*\rangle = \langle c^*d^*\rangle\,.$$

Proposition 4.
1) *Equality of ord-arcs is an equivalence relation.*
2) $(a+t)^*(b+t)^* = a^*b^*$ *for all* $t \in \mathbb{R}$.
3) $a^*b^* = o^*(b-a)^*$.

Proposition 5. $a^*b^* = c^*d^*$ *if and only if any of the equivalent statements holds:*
1) $b^*a^* = d^*c^*$.
2) $a^*c^* = b^*d^*$.
3) *The midpoints of the ord-arcs a^*d^* and b^*c^* coincide.*
4) *There is a transformation $g_t \in g(\mathbb{R})$ that maps a^*b^* onto c^*d^*.*

Proof.
1) $a^*b^* = c^*d^* \Leftrightarrow b-a = d-c \Leftrightarrow a-b = c-d \Leftrightarrow b^*a^* = d^*c^*$.
2) $a^*b^* = c^*d^* \Leftrightarrow b-a = d-c \Leftrightarrow d-b = c-a \Leftrightarrow a^*c^* = b^*d^*$.
3) $b-a = d-c. \Leftrightarrow \frac{b+c}{2} = \frac{a+d}{2}$. The last equality means the coincidence of the midpoints of the ord-arcs b^*c^* and a^*d^*.

4) If $c^*d^* = g_t(a^*b^*)$, then $c^*d^* = (t+a)^*(t+b)^*$ and by item 2) of Proposition 4, $c^*d^* = a^*b^*$.

Suppose that, on the contrary, $a^*b^* = c^*d^*$. Then $d-c = b-a \Rightarrow c-a = d-b$, and for $t = c-a$ according to (2) we will have that

$$g_t(a^*) = (t+a)^* = (c-a+a)^* = c^*, \quad g_t(b^*) = (t+b)^* = (d-b+b)^* = d^*.$$

Hence, $c^*d^* = g_t(a^*b^*)$. □

Proposition 6. 1) *The midpoints of the ord-arcs a^*b^* and b^*a^* coincide.*

2) $a^*c^* = c^*b^* \quad \Leftrightarrow \quad c^*$ is the midpoint of the ord-arc a^*b^*.

Proof. 1) The proof is trivial.

2) $a^*c^* = c^*b^* \Leftrightarrow c-a = b-c \Leftrightarrow c = \frac{a+b}{2}. \Leftrightarrow c^*$ is the midpoint of the ord-arc a^*b^*. □

We will say that *point b^* lies between a^* and c^* on γ* and denote $a^* - b^* - c^*$, if the ord-arcs a^*b^* and b^*c^* have similar orientation: $a^*b^* \uparrow\uparrow b^*c^*$, in other words, if $a < b < c$ or $c < b < a$. The points lying between a^* and c^* are called *interior points of the ord-arc a^*c^*.*

The sets

$$[a^*b^*) = \{s^* = g_s(o^*) |\ s \in [a,b)\}, \quad (a^*b^*] = \{s^* = g_s(o^*) |\ s \in (a,b]\} \quad \text{as } a < b$$

will be called *half-open arcs.*

Each of the sets

$$\gamma_c^+ = \{d^* |\ d > c\}, \quad \gamma_c^- = \{d^* |\ d < c\}$$

is called *a half-orbit with a vertex c^**: the first half-orbit is called *right*, and the second–*left.* We assume that every positively oriented ord-arc has orientation similar to every right half-orbit, and every negatively oriented ord-arc has orientation similar to every left half-orbit.

Proposition 7. *Every transformation of the group $g(\mathbb{R})$ maps a half-open arc to a half-open one, a half-orbit to a half-orbit, the right one into the right one, and the left one into the left one, in other words, an ord-arc, half-open arc, and half-orbit are invariants of the group $g(\mathbb{R})$.*

Proof. $g_t([a^*b^*)) = [(a+t)^*(b+t)^*), \quad g_t((a^*b^*]) = ((a+t)^*(b+t)^*]$.

$$g_t(\gamma_c^+) = \{g_t(s^*) |\ s > c\} = \{(t+s)^* |\ s > c\} = \{u^* |\ u > t+c\} = \gamma_{t+c}^+.$$

Similarly, $g_t(\gamma_c^-) = \gamma_{t+c}^-$. □

We call each of the ord-arcs a^*c^*, c^*b^* *a half of an ord-arc a^*b^**, if c^* is the midpoint of an ord-arc a^*b^*. It is easy to verify the following

Proposition 8. a) *Each of the halves of the ord-arc a^*b^* is co-directed with a^*b^*.*

b) *An ord-arc r^*t^* is a half of an ord-arc a^*b^* if and only if one of its endpoints coincides with one of the endpoints of the ord-arc a^*b^*, the other endpoint is an interior point of the ord-arc a^*b^*, and $\langle r^*t^* \rangle = \frac{1}{2}\langle a^*b^* \rangle$.*

The equality of ord-arcs, the relation of "between" for three points, the midpoint of an ord-arc, the half of an ord-arc are invariants of the transformation group $g(\mathbb{R})$.

4. Structures on an orbit of a one-parameter transformation group

Theorem 1. *The set of ord-arcs of an orbit of a one-parameter transformation group $g(\mathbb{R})$ can be endowed with a structure of a 1-dimensional real vector space.*

Proof. We define the structure of a real vector space on the orbit γ of the point $o^* \in E$ under the action of the one-parameter transformation group $g(\mathbb{R})$, taking the set of equal ord-arcs of the orbit as the vector. Each of the ord-arcs is considered a representative of the vector; we will denote the vector by any of its representatives. The set of all vectors will be denoted by $V(\gamma)$. The mapping

$$\Psi : a \in \mathbb{R} \to o^*a^* \in V(\gamma)$$

is bijective. We define linear operations on vectors as follows:
1) *addition* $o^*a^* + o^*b^* = o^*(a+b)^*$,
2) *multiplication by a real number* $\lambda\, o^*a^* = o^*(\lambda a)^*$.

Then all the axioms of the vector space for $V(\gamma)$ are satisfied [7, p. 292], and the mapping Ψ is an isomorphism between the vector spaces $\mathbb{R}$ and $V(\gamma)$. In this case, the zero vector is an ord-arc with zero measure, and the vector b^*a^* is opposite to a^*b^*. □

Proposition 9.
1) $a^*b^* + c^*d^* = (a+c)^*(b+d)^*$.
2) $\lambda\, a^*b^* = (\lambda a)^*(\lambda b)^*$.

Proof. According to item 3) of Proposition 4,
1) $a^*b^* + c^*d^* = o^*(b-a)^* + o^*(d-c)^* = o^*((b-a)+(d-c))^* =$
$= o^*((b+d)-(a+c))^* = (a+c)^*(b+d)^*$.
2) $\lambda\, a^*b^* = \lambda\, o^*(b-a)^* = o^*(\lambda(b-a))^* = o^*(\lambda b - \lambda a)^* = (\lambda a)^*(\lambda b)^*$. □

Proposition 10.

$$1)\ \langle a^*b^* + c^*d^*\rangle = \langle a^*b^*\rangle + \langle c^*d^*\rangle .$$
$$2)\ \langle \lambda\, a^*b^*\rangle = \lambda\ \langle a^*b^*\rangle .$$

Proof. The mapping inverse to the isomorphism $\Psi : \mathbb{R} \to V(\gamma)$, defined in Theorem 1, precisely, $\Psi^{-1} : a^*b^* \to \langle a^*b^*\rangle \in \mathbb{R}$, is also an isomorphism. □

Theorem 2. *The set of ord-arcs of an orbit of a one-parameter transformation group $g(\mathbb{R})$ can be endowed with a field structure.*

Proof. Let $V(\gamma)$ be the quotient set of ord-arcs of an orbit γ with respect to an equivalence, which is an equality of ord-arcs. Together with the addition operation defined in Theorem 1, it forms a commutative group. We define the operation of multiplication of the ord-arcs and the inverse division operation in such a way that the mapping $\Psi : a \in \mathbb{R} \to o^*a^* \in V(\gamma)$ is an isomorphism of the fields $\mathbb{R}$ and $V(\gamma)$, namely:

$$1)\ \textit{multiplication}: \quad o^*a^* \cdot o^*b^* = o^*(ab)^*,$$
$$2)\ \textit{division}: \qquad \frac{o^*b^*}{o^*a^*} = o^*\left(\frac{b}{a}\right)^*, \quad a \neq 0.$$

The field axioms are fulfilled [7, p. 49, 50], the unit is the ord-arc $e = 0^*1^*$.

□

Proposition 11.

$$1)\ a^*b^* \cdot c^*d^* = (ad+bc)^*(ac+bd)^*.$$

$$2)\ \frac{a^*b^*}{c^*d^*} = \frac{1}{d^2-c^2}(ad-bc)^*(bd-ac)^* = \left(\frac{ad-bc}{d^2-c^2}\right)^* \left(\frac{bd-ac}{d^2-c^2}\right)^*.$$

Proof.

$$1)\ a^*b^* \cdot c^*d^* = o^*(b-a)^* \cdot o^*(d-c)^* = o^*((b-a)\cdot(d-c))^* =$$
$$= o^*((ac+bd)-(ad+bc))^* = (ad+bc)^*(ac+bd)^*.$$

$$2)\ \frac{a^*b^*}{c^*d^*} = \frac{o^*(b-a)^*}{o^*(d-c)^*} = o^*\left(\frac{b-a}{d-c}\right)^* = o^*\left(\frac{(b-a)(d+c)}{(d-c)(d+c)}\right)^* =$$
$$= o^*\left(\frac{(bd-ac)-(ad-bc)}{d^2-c^2}\right)^* = o^*\left(\frac{bd-ac}{d^2-c^2} - \frac{ad-bc}{d^2-c^2}\right)^* =$$
$$= \left(\frac{ad-bc}{d^2-c^2}\right)^* \left(\frac{bd-ac}{d^2-c^2}\right)^*.$$

□

Proposition 12.

$$1)\ \langle a^*b^* \cdot c^*d^* \rangle = \langle a^*b^* \rangle \cdot \langle c^*d^* \rangle.$$

$$2)\ \left\langle \frac{c^*d^*}{a^*b^*} \right\rangle = \frac{\langle c^*d^* \rangle}{\langle a^*b^* \rangle}.$$

Theorem 3. *The set of ord-arcs of an orbit of the one-parameter transformation group $g(\mathbb{R})$ with respect to the operations of addition, multiplication by real numbers, and multiplication of ord-arcs, is a two-dimensional associative-commutative algebra over the field of real numbers, isomorphic to the algebra of split-complex numbers (hyperbolic or double numbers).*

Proof. The assertion follows from Propositions 9 and 11. The isomorphism is established by the correspondence $a^*b^* \to b + a\mathbf{j}$, where $\mathbf{jj} = 1$. [8]

□

Proposition 13.

$$a^*b^* + b^*c^* = a^*c^*.$$

Proof. According to item 1) of Proposition 10, we have

$$\langle a^*b^* + b^*c^* \rangle = \langle a^*b^* \rangle + \langle b^*c^* \rangle = (b-a) + (c-b) = c - a = \langle a^*c^* \rangle.$$ □

By laying off the ord-arc a^*b^* of the orbit γ from the point $c^* \in \gamma$ we mean finding the point $d^* \in \gamma$ such that $a^*b^* = c^*d^*$.

Proposition 14. *From any point of an orbit, one can lay off an ord-arc equal to the given one in a unique way.*

Proof. Equality of ord-arcs $a^*b^* = c^*d^*$ implies that $b - a = d - c$, whence the parameter $d = b - a + c$ of the required point is uniquely determined.

□

Theorem 4. *The set of ord-arcs of an orbit of the one-parameter transformation group $g(\mathbb{R})$ can be endowed with a structure of a 1-dimensional Euclidean space.*

Proof. Let $V(\gamma)$ be the 1-dimensional real vector space defined in Theorem 1. The mapping $(a^*, b^*) \in \gamma \times \gamma \rightarrow a^*b^* \in V(\gamma)$ satisfies the axioms of the affine space [7, p. 301]: 1) the uniqueness of a vector laid off from a given point (Proposition 14), 2) the triangle axiom (Proposition 13). Therefore, the pair $(\gamma, V(\gamma))$ is an affine line.

The scalar product of vectors a^*b^* and c^*d^* is defined as a measure of their product, if the vectors are interpreted as elements of the field $V(\gamma)$, defined in Theorem 2, i.e. $\langle a^*b^* \cdot c^*d^* \rangle = (b-a)(d-c)$. The properties of the field and the properties of the measure (Propositions 10, 12) imply the fulfillment of the scalar product axioms [7, p. 299]. Hence, the pair $(\gamma, V(\gamma))$ is a Euclidean space of dimension 1. *The modulus of the vector* a^*b^* is determined by the dot product: $|a^*b^*| = \sqrt{\langle a^*b^* \cdot a^*b^* \rangle} = |\langle a^*b^* \rangle| = |b-a|$.

□

5. Archimedean property and its corollaries

We denote by $\mathbb{N}_0$ the set of natural numbers, including zero.

We denote the set of positively oriented ord-arcs of an orbit γ by $Arc^+\gamma$, the set of negatively oriented ord-arcs of γ by $Arc^-\gamma$, and the set of all orbital ord-arcs by $Arc\gamma$. Obviously, $Arc\gamma = Arc^+\gamma \cup Arc^-\gamma$ and $Arc^+\gamma \cap Arc^-\gamma = \emptyset$.

Proposition 15. *For every half-orbit γ_a with a vertex a^* and an ord-arc p^*q^* with a similar orientation, there exists a unique sequence of points $a_0^*, a_1^*, \ldots, a_i^*, \ldots$ such that*

1) $a_0^* = a^*, \quad a_{i-1}^*a_i^* = p^*q^*, \quad i = 1, 2, \ldots;$
2) $a_i^* \in \gamma_a, \quad i = 1, 2, \ldots$

Proof. The sequence a_i^*, corresponding to the parameters $a_i = a + i(q-p)$, is the required one. If we admit the existence of another sequence $b_0^*, b_1^*, \ldots, b_i^*, \ldots$, satisfying condition 1) (and, hence, 2)), then the non-empty set of indices $\{i \in \mathbb{N}_0 \mid a_i \neq b_i\}$ contains the smallest index $n > 0$. Then $a_{n-1} = b_{n-1}$ and $a_n \neq b_n$. We arrive at a contradiction $p^*q^* = a_{n-1}^*a_n^* \neq b_{n-1}^*b_n^* = p^*q^*$, which proves the uniqueness of a sequence satisfying requirement 1).

□

Proposition 16 (an analogue of Archimedean principle for ord-arcs). *For co-directed ord-arcs a^*b^* and p^*q^*, there is a finite set of points $a_0^*, a_1^*, \ldots, a_n^*$ such that:*

a) $a_0^* = a^*, \quad a_{i-1}^*a_i^* = p^*q^*, \quad i = 1, 2, \ldots, n;$
b) $a^* - b^* - a_n^*$.

Proof. By Proposition 15, a sequence satisfying condition a) is a finite subsequence of the sequence $a_i = a + ix$, where $x = \langle p^*q^* \rangle = q - p$. Let $y = \langle a^*b^* \rangle = b - a$.

Case 1: $a^*b^*, p^*q^* \in Arc^+\gamma$. Then $x > 0$ and $y > 0$. By Archimedean principle for real numbers, there is a natural number n such that $y < nx$. Since $0 < y < nx$, then $a < y + a = b < a + nx = a_n$, and this entails b).

Case 2: $a^*b^*, p^*q^* \in Arc^-\gamma$. Then with the same notation we have that $-x > 0$ and $-y > 0$. By Archimedean principle for real numbers, there is a natural number

n such that $-y < n(-x)$. Since $0 < -y < n(-x)$, we have that $nx < y < 0 \Rightarrow a + nx < a + y = b < a, \Rightarrow a_n < b < a$, which implies b).

□

Proposition 17. *For co-directed ord-arcs a^*b^* and p^*q^* there is a unique number $n \in \mathbb{N}_0$ such that*

$$b^* = a_n^* \quad \text{or} \quad a_n^* - b^* - a_{n+1}^*, \tag{4}$$

where the sequence of points a_0^*, a_1^*, ... , a_n^* satisfies requirement 1) *of Proposition 15.*

Proof. By Proposition 16, the set $\left\{i \in \mathbb{N}_0 \mid a^* - b^* - a_{i+1}^*\right\}$ is not empty. It is bounded from below, and therefore has a minimum n. Then $a^* - b^* - a_{n+1}^*$ is true, while $a^* - b^* - a_n^*$ is not. Thus, relations (4) are fulfilled. This proves the existence of the required number $n \in \mathbb{N}_0$.

Assume that there is another number $m \in \mathbb{N}_0$ satisfying condition (4), $m \neq n$. Note that the sequence $a_i = a + ix$, where $x = \langle p^*q^* \rangle = q - p$, is monotonically increasing for $x > 0$ and monotonically decreasing for $x < 0$. We split the proof of uniqueness into two cases.

Case 1: $p^*q^* \in Arc^+\gamma$. Then $x > 0$, and condition (4) for the numbers n and m means that $b \in [a_n, a_{n+1})$ and $b \in [a_m, a_{m+1})$. Hence, $b \in [a, a_{m+1})$, therefore $a^* - b^* - a_{m+1}^* \quad \Rightarrow \quad m > n$. For integer values of m, n, we have that $m \geq n + 1$. Then $a_m \geq a_{n+1}$, and $b \in [a_m, a_{m+1})$ implies that $b \geq a_{n+1}$. On the other hand, the condition $b \in [a_n, a_{n+1})$ entails a contradicting inequality $b < a_{n+1}$.

Case 2: $p^*q^* \in Arc^-\gamma$. Now $x < 0$, and condition (4) for the numbers n and m means that $b \in (a_{n+1}, a_n]$ and $b \in (a_{m+1}, a_m]$. Hence, $b \in (a_{m+1}, a]$, therefore $a^* - b^* - a_{m+1}^* \quad \Rightarrow \quad m > n$. For integer values of m, n, we have that $m \geq n + 1$. Then $a_m \leq a_{n+1}$, and $b \in (a_{m+1}, a_m]$ implies that $b \leq a_{n+1}$. On the other hand, the condition $b \in (a_{n+1}, a_n]$ entails a contradicting inequality $b > a_{n+1}$.

The resulting contradictions prove the uniqueness of the number $n \in \mathbb{N}_0$ for which (4) holds.

□

Definition 1. We say that an ord-arc p^*q^* *fits into an ord-arc a^*b^* n* times, where an integer $n \neq 0$, or that p^*q^* is a $\frac{1}{n}$*-th part of an ord-arc a^*b^**, if there is a sequence of points a_0^*, a_1^*, ... , a_n^*, such that

a) $a_0^* = a^*, \quad a_{i-1}^* a_i^* = p^*q^*, \quad i = 1, 2, ..., n;$

b) $b^* = a_n^*$.

Proposition 18. *An ord-arc p^*q^* fits into an ord-arc a^*b^* n times if and only if any of the conditions is satisfied:*

1) $b = a + n(q - p)$,

2) $a^*b^* = n\, p^*q^*$.

Proof. Suppose that an ord-arc p^*q^* fits into an ord-arc a^*b^* n times. Propositions 15 and 17 imply 1). By the definition of multiplication of an ord-arc by a number and from Proposition 4 follows condition 2):

$$a^*b^* = a^*(a + n(q - p))^* = o^*(n(q - p))^* = n\, o^*(q - p)^* = n\, p^*q^*.$$

Conversely, let condition 1) be satisfied. The sequence $a_i^* = (a + i(q - p))^*$, $i = 0, 1, ..., n$, satisfies the requirements a) and b) of Definition 1. Therefore, p^*q^* fits into a^*b^* exatcly n (integer) times.

Now let condition 2) be satisfied: $n\ p^*q^* = a^*b^*$. Inverting the chain of equalities established above, we obtain $n\ p^*q^* = a^*(a + n(q - p))^*$. So $a^*b^* = a^*(a + n(q - p))^*$. By Proposition 14, we have that $b^* = (a + n(q - p))^*$, which implies 1). By the proven above, p^*q^* fits into a^*b^* exactly n (integer) times.

□

Proposition 18 justifies the notation $p^*q^* = \frac{1}{n}\ a^*b^*$ for the ord-arc p^*q^*, which is the $\frac{1}{n}$-th part of the ord-arc a^*b^*.

Remark 1. Note the different essence of the ord-arcs a^*b^* and p^*q^* in Propositions 15-18. While the first ord-arc is considered as a fixed pair of points, the second is seen as an element of the vector structure since it can be replaced by any ord-arc equal to it.

Proposition 19. *For every* $n \in \mathbb{N}_0$, $n \neq 0$,

$$n\ p^*q^* = \underbrace{p^*q^* + ... + p^*q^*}_{n \text{ terms}}.$$

Proof. Let $a^*b^* = n\ p^*q^*$. By Proposition 18, there is a sequence of points a_0^*, a_1^*, ... , a_n^*, such that $a_0^* = a^*$, $a_{i-1}^*a_i^* = p^*q^*$, $i = 1, 2, ..., n$, and $b^* = a_n^*$. Applying Proposition 13 and mathematical induction on the number of terms, we arrive at the required equality $n\ p^*q^* = a^*b^* = a_0^*a_1^* + a_1^*a_2^* + ... + a_{n-1}^*a_n^* = \underbrace{p^*q^* + p^*q^* + ... + p^*q^*}_{n \text{ terms}}$.

□

Proposition 20. *If an ord-arc p^*q^* fits into the ord-arc a^*b^* n times and an ord-arc r^*s^* fits into an ord-arc p^*q^* m times, then the ord-arc r^*s^* fits into the ord-arc a^*b^* nm times.*

Proof. The fact that the ord-arc p^*q^* fits into the ord-arc a^*b^* n times and the ord-arc r^*s^* fits into the ord-arc p^*q^* m times means, according to Proposition 18, that $a^*b^* = n\ p^*q^*$ and $p^*q^* = m\ r^*s^*$. Bearing in mind the vector nature of the ord-arcs, we can write: $a^*b^* = n\ p^*q^* = n(m\ r^*s^*) = (nm)\ r^*s^*$. By Proposition 18, the ord-arc r^*s^* fits into the the ord-arc a^*b^* nm times.

□

6. Existence and uniqueness theorems for the length of ord-arcs

We define the measurement of ord-arcs of a one-parameter transformation group in a way similar to the one used for line segments in [1, p. 371].

Definition 2. We say that *the measure of the length of the ord-arcs of an orbit* γ is given if there exists a mapping $l : Arc\gamma \to \mathbb{R}$, satisfying the axioms:

1) $l(a^*b^*) > 0$ for all $a^*b^* \in Arc\gamma$,
2) if $a_1^*b_1^* = a^*b^*$, then $l(a_1^*b_1^*) = l(a^*b^*)$,
3) if $a^* - b^* - c^*$, then $l(a^*b^*) + l(b^*c^*) = l(a^*c^*)$,
4) there is an ord-arc p^*q^* (*unit or standard*) such that $l(p^*q^*) = 1$.

As will be shown below, there are many functions $l : Arc\gamma \to \mathbb{R}$, satisfying *the axioms of measure*, which depend on the choice of ord-arcs as units of measurement.

Theorem 5. *For any two oppositely oriented ord-arcs $p_1^*q_1^*$ and $p_2^*q_2^*$ of the orbit γ there exists a mapping $l : Arc\gamma \to \mathbb{R}$, satisfying the axioms of a measure for which the given ord-arcs are unit ones.*

Proof. On the orbit γ, we fix two oppositely oriented ord-arcs, $p_1^*q_1^* \in Arc^+\gamma$ and $p_2^*q_2^* \in Arc^-\gamma$. We assume for $a^*b^* \in Arc^+\gamma$ that

$$l(a^*b^*) = \frac{\langle a^*b^* \rangle}{\langle p_1^*q_1^* \rangle}. \tag{5}$$

And for $a^*b^* \in Arc^-\gamma$ we assume that

$$l(a^*b^*) = \frac{\langle a^*b^* \rangle}{\langle p_2^*q_2^* \rangle}. \tag{6}$$

It is obvious that the 1st, 2nd and 4th axioms of measure are valid. We will check the fulfillment of Axiom 3. If $a^* - b^* - c^*$, then the ord-arcs a^*b^*, b^*c^*, a^*c^* have a similar orientation. Then they are oriented in a way similar to one of the ord-arcs $p_1^*q_1^*$, $p_2^*q_2^*$, suppose that it is $p_i^*q_i^*$, where $i = 1$ or $i = 2$. By item 1) of Proposition 10, and by Proposition 13,

$$l(a^*b^*) + l(b^*c^*) = \frac{\langle a^*b^* \rangle}{\langle p_i^*q_i^* \rangle} + \frac{\langle b^*c^* \rangle}{\langle p_i^*q_i^* \rangle} = \frac{\langle a^*b^* \rangle + \langle b^*c^* \rangle}{\langle p_i^*q_i^* \rangle} = \frac{\langle a^*c^* \rangle}{\langle p_i^*q_i^* \rangle} = l(a^*c^*).$$

The theorem is proved. □

Remark 2. From the proof of the theorem it is clear that the axioms of measure admit the existence of several different units of measurement. In particular, the measure defined above splits into two independent measures: $l_{p_1^*q_1^*}$ defined on $Arc^+\gamma$ and $l_{p_2^*q_2^*}$ defined on $Arc^-\gamma$.

Theorem 6. *For every real number $x > 0$ and for every given unit ord-arc in the orbit there exists an ord-arc of length x measured by the measure l.*

Proof. Let p^*q^* be one of two units of measurement given in the orbit γ. For a given real number $x > 0$, suppose tht $a^*b^* = x\, p^*q^*$. Since $x > 0$, the ord-arcs a^*b^* and p^*q^* have a similar orientation. Then, by item 2) of Proposition 10, we have that

$$l(a^*b^*) = \frac{\langle a^*b^* \rangle}{\langle p^*q^* \rangle} = \frac{\langle x\, p^*q^* \rangle}{\langle p^*q^* \rangle} = \frac{x\, \langle p^*q^* \rangle}{\langle p^*q^* \rangle} = x.$$

□

The proof of the uniqueness theorem for a measure on the set $Arc^*\gamma$ of ord-arcs of a similar orientation for a given standard $p^*q^* \in Arc^*\gamma$ is based on two lemmas and basically repeats the reasoning of a similar theorem from [1, p. 376-377].

Lemma 1. *For every measure of length l, every ord-arc p^*q^*, and every $n \in \mathbb{N}_0$,*

$$l(n \cdot p^*q^*) = n\, l(p^*q^*).$$

Proof. The proof is by induction on n. For $n = 1$, it is trivial. Let the statement hold for $n - 1$. Consider the case when $a^*b^* = n \cdot p^*q^*$. By Propositions 15, 18, there is a sequence of points

$$a_i^* = (a + i(q - p))^*, \quad i = 0, 1, ..., n,$$

that satisfies the requirements a) and b) of Definition 1. It is clear that $a^* - a^*_{n-1} - b^*$. According to Axiom 3,

$$l(a^*b^*) = l(a^*a^*_{n-1}) + l(a^*_{n-1}b^*).$$

By Axiom 2, $l(a^*_{n-1}b^*) = l(p^*q^*)$. The ord-arc p^*q^* fits $n-1$ times in the ord-arc $a^*a^*_{n-1}$. By the induction hypothesis, we have that

$$l(a^*a^*_{n-1}) = (n-1)\, l(p^*q^*),$$

and as a result we obtain

$$l(a^*b^*) = (n-1)\, l(p^*q^*) + l(p^*q^*) = n\, l(p^*q^*).$$

□

Lemma 2. *For every measure of length l, every ord-arc p^*q^*, and every rational number $r > 0$,*

$$l(r \cdot a^*b^*) = r\, l(a^*b^*).$$

Proof. Let a rational number r be represented by an irreducible fraction $\frac{m}{n}$, where m, n are natural numbers. From Lemma 1 it follows that if the ord-arc p^*q^* is the $\frac{1}{n}$-th part of the ord-arc a^*b^*, then

$$l(p^*q^*) = \frac{1}{n}\, l(a^*b^*), \quad \text{otherwise,} \quad l\left(\frac{1}{n}\, a^*b^*\right) = \frac{1}{n}\, l(a^*b^*).$$

Hence, by Lemma 1,

$$l\left(\frac{m}{n}\, a^*b^*\right) = l\left(m \cdot \frac{1}{n}\, a^*b^*\right) = m \cdot l\left(\frac{1}{n}\, a^*b^*\right) = \left(m \cdot \frac{1}{n}\right) l(a^*b^*) = \frac{m}{n}\, l(a^*b^*).$$

□

Remark 3. By virtue of Axiom 2 of measure, the concept of length can be attributed to the entire class of equal ord-arcs. Therefore, the formulations of Lemmas 1 and 2 are correct.

Theorem 7. *On the set $Arc^*\gamma$ of ord-arcs with a similar orientation for a given standard $p^*q^* \in Arc^*\gamma$ there exists at most one mapping $l : Arc^*\gamma \to \mathbb{R}$ satisfying the axioms of measure of an ord-arc.*

Proof. Suppose, on the contrary, that there are two measures l and θ defined on $Arc^*\gamma$, satisfying the axioms 1-3 of measure and taking the value of 1 on some ord-arc $p^*q^* \in Arc^*\gamma$. Since they are different, there is an ord-arc $a^*b^* \in Arc^*\gamma$ such that $l(a^*b^*) \neq \theta(a^*b^*)$. Let r^*s^* be the $\frac{1}{n}$-th part of the ord-arc p^*q^*. Its existence follows from item 1) of Proposition 18. It is clear that $r^*s^* \uparrow\uparrow p^*q^*$, therefore, $r^*s^* \in Arc^*\gamma$. By Proposition 17, for a sequence of points a^*_0, a^*_1, ... , a^*_k, ... such that $a^*_{i-1}a^*_i = r^*s^*$, $i = 1, 2, ...$, there is a uniquely defined integer number $m \geq 0$, for which $b^* = a^*_m$ or $a^*_m - b^* - a^*_{m+1}$.

According to Lemma 2 and the fact that $l(p^*q^*) = 1$, we have that

$$l(a^*a^*_m) = l(m\, r^*s^*) = l\left(m \cdot \frac{1}{n}\, p^*q^*\right) = l\left(\frac{m}{n}\, p^*q^*\right) = \frac{m}{n}\, l(p^*q^*) = \frac{m}{n} \cdot 1 = \frac{m}{n}\ .$$

In a similar way, we obtain that

$$l(a^*a^*_{m+1}) = \frac{m+1}{n}\ .$$

Using a similar reasoning, we have that $\theta(a^*a_m^*) = \frac{m}{n}$, therefore equality $b^* = a_m^*$ is impossible. This means that $a_m^* - b^* - a_{m+1}^*$. By Axiom 1 of measure, $l(a_m^*b^*) > 0$ and $l(b^*a_{m+1}^*) > 0$. By Axiom 3 of measure,

$$l(a^*b^*) = l(a^*a_m^*) + l(a_m^*b^*) > l(a^*a_m^*) = \frac{m}{n} ,$$

$$\frac{m+1}{n} = l(a^*a_{m+1}^*) = l(a^*b^*) + l(b^*a_{m+1}^*) > l(a^*b^*) .$$

Thus, we have a double inequality:

$$\frac{m}{n} < l(a^*b^*) < \frac{m+1}{n} .$$

In the same way, we obtain

$$\frac{m}{n} < \theta(a^*b^*) < \frac{m+1}{n} .$$

Subtracting one from the other, we get

$$|l(a^*b^*) - \theta(a^*b^*)| < \frac{1}{n} .$$

Since we can take an arbitrarily large natural number n, then

$$l(a^*b^*) = \theta(a^*b^*) .$$

We obtain a contradiction, which means that the assumption of the existence of two measures in $Arc^*\gamma$ is false.

□

7. Measurement of ord-arcs by laying off the standard

Another approach to determining the lengths of the orbital ord-arcs goes back to the construction of the mapping $\theta : Arc\gamma \to \mathbb{R}$, as it was done in [3, p. 74] and [1, p. 372] when measuring the lengths of line segments. To do this, we use two constructions: 1) laying off the ord-arc p^*q^* starting from a given point (Proposition 14), 2) dividing the ord-arc in half. The value $\theta(a^*b^*)$ is defined as a binary fraction in the following way. On the orbit γ, starting from the point a^* we construct successively ord-arcs equal to the ord-arc p^*q^*, co-directed with the ord-arc a^*b^*, which must be measured, so that condition 1) of Proposition 15 is satisfied. By Proposition 17, there is a uniquely defined integer $n \geq 0$, for which (4) holds. If a_n^* coincides with b^*, then we set $\theta(a^*b^*) = n$, and complete the measurement process. If $a_n^* - b^* - a_{n+1}^*$, then we assume that $\theta(a^*b^*)$ contains n integers, and the fractional part is determined by the subsequent division of the ord-arc $a_n^*a_{n+1}^*$ in half. Namely, we write $\theta(a^*b^*) = n,1$ (or $n+\frac{1}{2}$), if b^* is the midpoint of the ord-arc $a_n^*a_{n+1}^*$, otherwise, two variants arise: 1) $a_n^* - b^* - s^*$, where s^* is the midpoint of the ord-arc $a_n^*a_{n+1}^*$, 2) $s^* - b^* - a_{n+1}^*$. In the first case, we put zero after the comma, in the second case, one. We denote by $p_1^*q_1^*$ one of the two halves of the ord-arc $a_na_{n+1}^*$ which contains the point b^*, and proceed to dividing the arc $p_1^*q_1^*$ in half. If the ord-arcs $p_k^*q_k^*$, $k = 1, 2, ...m$, are constructed, then when the point b^* coincides with the middle of the ord-arc $p_m^*q_m^*$, we put 1 in the k-th digit after the comma, and the resulting number is considered the measure of the ord-arc a^*b^*. Otherwise, we denote by $p_{m+1}^*q_{m+1}^*$ one of the two halves of the ord-arc $p_m^*q_m^*$, that contains the point b^*. This process will be called *the measurement of the ord-arc.*

As a result, we obtain some positive real number x, possibly expressed by an infinite binary expansion.

Now we will formalize the measurement process described above.

Definition 3. We say that a sequence of orbital points

$$a^* = a_0^*,\ a_1^*, ...,\ a_n^*,\ a_{n+1}^*;\ p_1^*, q_1^*,\ p_2^*, q_2^*, ...,\ p_k^*, q_k^* ... \tag{7}$$

realizes the measurement of the ord-arc a^*b^* *by the ord-arc* p^*q^*, $p^*q^* \uparrow\uparrow a^*b^*$, if

a) $a_0^*a_1^* = a_1^*a_2^* = ... = a_n^*a_{n+1}^* = p^*q^*$,

b) $p_i^*q_i^*$ is the half of the ord-arc $p_{i-1}^*q_{i-1}^*$, $i = 1, 2, ...$, ($p_0^*q_0^* = a_n^*a_{n+1}^*$)

c) $b^* \in [p_i^*q_i^*)$ for all $i = 0, 1, 2, ...$

Then we set $\theta_{p^*q^*}(a^*b^*) = n, j_1 j_2 ... j_k ...$, where

$$j_k = \begin{cases} 0, \text{ if } \ p_k^* = p_{k-1}^* \\ 1, \text{ if } \ q_k^* = q_{k-1}^*. \end{cases} \tag{8}$$

Theorem 8.

$$\begin{array}{ll} 1. & \textit{If } p^*q^* = s^*t^*, \ \textit{ then } \ \theta_{p^*q^*}(a^*b^*) = \theta_{s^*t^*}(a^*b^*). \\ 2. & \textit{If } a^*b^* = c^*d^*, \ \textit{ then } \ \theta_{p^*q^*}(a^*b^*) = \theta_{p^*q^*}(c^*d^*). \\ 3. & \qquad \theta_{p^*q^*}(a^*b^*) = \theta_{q^*p^*}(b^*a^*). \end{array}$$

Proof. 1. The first statement is true due to the transitivity of the equality of ord-arcs (Proposition 4, item 1))

2. According to item 4) of Proposition 5, for equal ord-arcs a^*b^* and c^*d^* there exists a transformation $g_t \in g(R)$ that maps a^*b^* to c^*d^*. Let the sequence (7) of orbital points realize the measurement of an ord-arc a^*b^* by the ord-arc p^*q^* co-directed with it. Then the sequence of points

$$g_t(a_0^*), g_t(a_1^*), ..., g_t(a_n^*), g_t(a_{n+1}^*);\ g_t(p_1^*), g_t(q_1^*), ..., g_t(p_k^*), g_t(q_k^*) ... \tag{9}$$

realizes the measurement of the ord-arc c^*d^* by the ord-arc p^*q^*. Indeed, according to Propositions 5, 7, 8, the transformation g_t maps ord-arcs into ord-arcs equal to them, a half of an ord-arc into a half of its image, a half-open arc into a half-open arc. Therefore, properties a), b), c) for sequence (7) imply the same properties for sequence (9). Taking into account that

$$p_k^* = p_{k-1}^* \ \Rightarrow \ g_t(p_k^*) = g_t(p_{k-1}^*), \qquad q_k^* = q_{k-1}^* \ \Rightarrow \ g_t(q_k^*) = g_t(q_{k-1}^*),$$

and based on the formula (8), we conclude that $\theta_{p^*q^*}(a^*b^*) = \theta_{p^*q^*}(c^*d^*)$.

3. The mapping $\epsilon : \gamma \to \gamma$, defined by formula $\epsilon(a^*) = (-a)^*$, transforms the ord-arc o^*x^* into the ord-arc $o^*(-x)^*$, and the ord-arc o^*e^* into the ord-arc $o^*(-e)^*$. It is easy to ascertain that it changes the orientation of ord-arcs, but preserves their equality, maps half-open arcs into half-open arcs, and a half of an ord-arc into a half of an ord-arc. In a similar way as in item 2 it is obtained that the sequence of orbital points (7), which realizes the measurement of the ord-arc o^*x^* by the ord-arc o^*e^* is translated by the mapping ϵ into a sequence of points that realizes the measurement of the ord-arc $o^*(-x)^*$ by the ord-arc $o^*(-e)^*$. Taking into account that $p_k^* = p_{k-1}^* \ \Rightarrow \ \epsilon(p_k^*) = \epsilon(p_{k-1}^*)$, $q_k^* = q_{k-1}^* \ \Rightarrow \ \epsilon(q_k^*) = \epsilon(q_{k-1}^*)$, based on formula (8), we conclude that

$$\theta_{o^*e^*}(o^*x^*) = \theta_{o^*(-e)^*}(o^*(-x)^*). \tag{10}$$

Because

$$p^*q^* = o^*(q-p)^*,\;\; o^*(p-q)^* = q^*p^*,\;\; a^*b^* = o^*(b-a)^*,\;\; o^*(a-b)^* = b^*a^*,$$

by virtue of the proved statements 1, 2 of the theorem and on the basis of formula (10), we can write:

$$\theta_{p^*q^*}(a^*b^*) = \theta_{o^*(q-p)^*}(a^*b^*) = \theta_{o^*(q-p)^*}(o^*(b-a)^*) =$$
$$= \theta_{o^*(p-q)^*}(o^*(a-b)^*) = \theta_{o^*(p-q)^*}(b^*a^*) = \theta_{q^*p^*}(b^*a^*).$$

The theorem is completely proved.

□

Theorem 9. *For any choice of oppositely oriented unit ord-arcs $p_i^*q_i^*$, $i=1,2$, the value of the measure $l(a^*b^*)$, defined by formulas (5), (6), coincides with the measurement result $\theta(a^*b^*)$ for the ord-arc a^*b^* by the unit ord-arc, co-directed with a^*b^*.*

Proof. Let the ord-arc o^*x^* be measured by the ord-arc o^*e^*, for $0<x$ and $0<e$. We will prove the coincidence of the binary expansions of numbers

$$\theta_{o^*e^*}(o^*x^*) = m, j_1j_2.... \quad \text{and} \quad l_{o^*e^*}(o^*x^*) = \frac{x}{e} = n, i_1i_2...,$$

Here

$$\frac{x}{e} = n + \frac{i_1}{2} + \frac{i_2}{2^2} + ... + \frac{i_k}{2^k} + ..., \tag{11}$$

where n is a non-negative number, $i_k=0$ or $i_k=1$. We assume that in the expansion of the fraction $\frac{x}{e}$, all digits cannot be ones, starting with some digit k, otherwise, $i_{k-1}=0$ and

$$\frac{i_k}{2^k} + \frac{i_{k+1}}{2^{k+1}} + \frac{i_{k+2}}{2^{k+2}} + ... = \frac{1}{2^k} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + ... = \frac{1}{2^{k-1}},$$

and the expansion can be written as a finite fraction with a 1 in the $(k-1)$-th digit and figures preceding it: $n, i_1i_2...i_{k-2}1$ (similarly, if the ones start immediately after the comma, then $n, 11111... = n+1$).

We will construct a sequence of orbital points corresponding to the parameters

$$a_l = le,\;\; l = 0,1,...,n+1, \tag{12}$$

$$p_k = p_{k-1} + \frac{i_k e}{2^k},\quad q_k = p_k + \frac{e}{2^k},\;\; k = 1,2,..., \tag{13}$$

if we assume that $p_0 = a_n$, $q_0 = a_{n+1}$. We now verify that sequence (7), defined by formulas (12), (13), realizes the measurement of the ord-arc o^*x^* by the ord-arc o^*e^*.

We estimate the expansion (11) from below by setting $i_1 = i_2 = ... = i_k = ... = 0$, and from above by setting $i_1 = i_2 = ... = i_k = ... = 1$. Then

$$n \le \frac{x}{e} < n + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^k} + ... = n+1. \Rightarrow$$
$$ne \le x < (n+1)e \;\Rightarrow\; a_n \le x < a_{n+1}.$$

Since $\langle a_k^*a_{k+1}^*\rangle = a_{k+1} - a_k = (k+1)e - ke = e = \langle o^*e^*\rangle$, $k = 0,1,...,n$, then requirements a) and c) for $i=0$ of Definition 3 are satisfied. Hence, sequence (7) realizes the measurement of the integer part of the arc length, and m = n. If $x = a_n$, then the measurement ends, and $l_{o^*e^*}(o^*x^*) = \theta_{o^*e^*}(o^*x^*) = n$.

A refined estimate for the number given by expansion (11) implies the following inequalities:

$$n+\frac{i_1}{2}+\frac{i_2}{2^2}+...+\frac{i_k}{2^k}\le\frac{x}{e}<n+\frac{i_1}{2}+\frac{i_2}{2^2}+...+\frac{i_k}{2^k}+\frac{1}{2^{k+1}}+\frac{1}{2^{k+2}}+...+\frac{1}{2^{k+p}}+...=$$

$$=n+\frac{i_1}{2}+\frac{i_2}{2^2}+...+\frac{i_k}{2^k}+\frac{1}{2^k}.\ \Rightarrow\ p_k\le x<p_k+\frac{e}{2^k}=q_k\ \Rightarrow\ x^*\in[p_k^*q_k^*).$$

Note that

$$p_{k-1}\le p_k\le x<q_k<q_{k-1}\ \text{ for all }\ k=1,2,.... \tag{14}$$

Thus, we have a sequence of nested half-open arcs containing the point x^*. Therefore, requirement c) of Definition 3 is also fulfilled.

For $i_k=0$, we have that

$$p_k=\left(n+\frac{i_1}{2}+\frac{i_2}{2^2}+...+\frac{i_{k-1}}{2^{k-1}}+\frac{0}{2^k}\right)e=p_{k-1}\quad\Rightarrow\quad p_k^*=p_{k-1}^*,$$

and it follows from (14) that $p_{k-1}^*-q_k^*-q_{k-1}^*$. Besides,

$$\langle p_k^*q_k^*\rangle=q_k-p_k=\frac{e}{2^k}=\frac{1}{2}\left\langle p_{k-1}^*q_{k-1}^*\right\rangle.$$

By Proposition 8, the ord-arc $p_k^*q_k^*$ is a half of the ord-arc $p_{k-1}^*q_{k-1}^*$.

For $i_k=1$, we have that

$$q_k=p_k+\frac{e}{2^k}=\left(n+\frac{i_1}{2}+\frac{i_2}{2^2}+...+\frac{i_{k-1}}{2^{k-1}}+\frac{1}{2^k}\right)e+\frac{e}{2^k}=$$

$$=\left(n+\frac{i_1}{2}+\frac{i_2}{2^2}+...+\frac{i_{k-1}}{2^{k-1}}\right)e+\frac{e}{2^k}+\frac{e}{2^k}=p_{k-1}+\frac{e}{2^{k-1}}=q_{k-1}\quad\Rightarrow\quad q_k^*=q_{k-1}^*,$$

and it follows from (13) and (14) that $p_{k-1}^*-p_k^*-q_{k-1}^*$. As it is shown above, $\langle p_k^*q_k^*\rangle=\frac{1}{2}\left\langle p_{k-1}^*q_{k-1}^*\right\rangle$. By Proposition 8, the ord-arc $p_k^*q_k^*$ is a half of the ord-arc $p_{k-1}^*q_{k-1}^*$. So, requirement b) of Definition 3 is also fulfilled. We have proved that the sequence (7) defined by formulas (12) and (13) realizes the measurement of the ord-arc o^*x^* by the ord-arc o^*e^*.

Formula (8) provides us with the following:

$$j_k=\begin{cases}0,\ \text{if}\ \ i_k=0\\ 1,\ \text{if}\ \ i_k=1.\end{cases}=i_k,\quad k=1,2,...$$

As a result,

$$\theta_{o^*e^*}(o^*x^*)=l_{o^*e^*}(o^*x^*). \tag{15}$$

We now proceed to the general case. Let p^*q^* be one of the two unit ord-arcs $p_i^*q_i^*$, $i=1,2$, which is co-directional with the arc a^*b^* to be measured.

Case 1. If $a^*b^*,p^*q^*\in Arc^+\gamma$, then by Theorem 8 and formula (15), we can write that

$$\theta_{p^*q^*}(a^*b^*)=\theta_{o^*(q-p)^*}(o^*(b-a)^*)=l_{o^*(q-p)^*}(o^*(b-a)^*)=\frac{b-a}{q-p}=l_{p^*q^*}(a^*b^*).$$

Case 2. Let now $a^*b^*,p^*q^*\in Arc^-\gamma$. By Theorem 8(3),

$$\theta_{p^*q^*}(a^*b^*)=\theta_{q^*p^*}(b^*a^*).$$

But $b^*a^*,q^*p^*\in Arc^+\gamma$. According to the above proof,

$$\theta_{q^*p^*}(b^*a^*)=l_{q^*p^*}(b^*a^*)=\frac{a-b}{p-q}=\frac{b-a}{q-p}=l_{p^*q^*}(a^*b^*).$$

And we obtain a similar result:

$$\theta_{p^*q^*}(a^*b^*) = l_{p^*q^*}(a^*b^*).$$

The theorem is completely proved.

□

8. When the lengths of opposite ord-arcs are not equal

We will show that a structure of the Euclidean line in the sense used by Hilbert can be introduced on an orbit.

Note that the "cutoff" of the system of axioms of the Euclidean space [1, p. 306-313], performed by means of severance of its sentences of a spatial nature, leads to impoverished geometry of a straight line that does not contain the measurement of segments, since a number of statements regarding the relative position of points and segments are justified by using the whole complex of axioms. For example, when proving the theorem on the existence and uniqueness of the midpoint of a segment, which is an important element used for construction of a length-measuring function, the axiom of congruence of angles and Pasch's axiom are used. Constructions of the axiomatics of the Euclidian line as an independent object goes back to Pasch. Following the same path, V.F. Kagan in [9] provides an extended system of axioms, consisting of 17 axioms characterizing an *oriented straight line* and divided into 4 groups: I - of order (5), II - of structure (3), III - of congruence (7), IV - of continuity (2). Replacing in it the axiom of congruence III_6 with the axiom $III_{6'}$: $a^*b^* = b^*a^*$, he defines a *two-sided line.*

The consistency of the system of axioms of an oriented straight line (and a two-sided line) is justified by constructing a model in the set $\mathbb{R}$ of real numbers with order and congruence relations (equality) of segments, which have a meaning similar to the one that we used in Section 3 when identifying numbers with points: $a^* = a$. Therefore, *an orbit* of a transformation group effectively acting on it, being isomorphic to $\mathbb{R}$, also *represents a model of an oriented line.* Due to this reason, the proof of the existence and uniqueness theorems for a measure (Theorems 5, 7) could be dropped. However, we considered it expedient to present them both for the completeness of presentation and because of the richness of the structures belonging to the orbit allows us to simplify the whole justification greatly. The main result (Theorem 9) which establishes the possibility for the measurement of orbital ord-arcs by laying off the standard sequentially with its subsequent division in half, does not follow from [9], since the measure of the length of a segment in Kagan's theory is introduced through the ratio of segments determined by the Euclidean algorithm, and the term "the midpoint of a segment" does not appear in the work.

V.F. Kagan, having introduced on the basis of the first two groups of axioms the concept of an oriented straight line as a straight line with rays and segments of a similar orientation, further absolutizes one of them, "forgetting" about its "twin".

In this section, we will define the congruence of ord-arcs in a way that allows us to bridge the gap between the two oriented straight lines, establishing a possibility for existence of a single standard that can be used to measure ord-arcs of any orientation.

Let the relation of "between" for three orbital points be defined as in Section 3. Then the axioms of the first two groups of the system of axioms [9] are fulfilled due to the existing order in the set of real numbers, which is easily verified.

Definition 4. We fix an arbitrary real number $\lambda < 0$. Ord-arcs a^*b^* and c^*d^* are called λ-*congruent* if $a^*b^* = \lambda^\delta c^*d^*$, where the degree δ is determined by the following formula:

$$\delta = \begin{cases} 0, & \text{if } a^*b^* \uparrow\uparrow c^*d^* \\ 1, & \text{if } a^*b^* \in Arc^+\gamma,\ c^*d^* \in Arc^-\gamma \\ -1, & \text{if } a^*b^* \in Arc^-\gamma,\ c^*d^* \in Arc^+\gamma. \end{cases} \tag{16}$$

λ-*congruence* is an invariant of the group $g(\mathbb{R})$ in the sense that if $a^*b^* = c^*d^*$, then for any transformation $f \in g(\mathbb{R})$ it is true that $f(a^*b^*) = f(c^*d^*)$. The notation is as follows: $a^*b^* =_\lambda c^*d^*$.

As can be seen from the definition, λ-congruence in the class $Arc^*\gamma$ of codirectional ord-arcs coincides with the equality of ord-arcs, and, therefore, satisfies all the congruence axioms of the 3rd group in the Kagan's system of axioms [9].

Proposition 21. *λ-Congruence of ord-arcs is an equivalence.*

Proof. Reflexivity follows from the fact that $a^*b^* \uparrow\uparrow a^*b^*$ and $a^*b^* = \lambda^0 a^*b^*$. *Symmetry*: $a^*b^* =_\lambda c^*d^*. \Rightarrow a^*b^* = \lambda^\delta c^*d^*, \Rightarrow c^*d^* = \lambda^{-\delta} a^*b^*$, where the degree $-\delta$ is defined by the formula

$$-\delta = \begin{cases} 0, & \text{if } \delta = 0, & \text{i.e. when } c^*d^* \uparrow\uparrow a^*b^* \\ 1, & \text{if } \delta = -1, & \text{i.e. when } c^*d^* \in Arc^+\gamma,\ a^*b^* \in Arc^-\gamma \\ -1, & \text{if } \delta = 1, & \text{i.e. when } c^*d^* \in Arc^-\gamma,\ a^*b^* \in Arc^+\gamma. \end{cases}$$

Consequently, $c^*d^* =_\lambda a^*b^*$.
Transitivity. Suppose that $a^*b^* =_\lambda c^*d^*$ and $c^*d^* =_\lambda p^*q^*$. Then we have that

$$a^*b^* = \lambda^\delta c^*d^*, \quad c^*d^* = \lambda^{\delta'} p^*q^*, \tag{17}$$

where δ is defined by formula (16) and δ' is defined by the following:

$$\delta' = \begin{cases} 0, & \text{if } c^*d^* \uparrow\uparrow p^*q^* \\ 1, & \text{if } c^*d^* \in Arc^+\gamma,\ p^*q^* \in Arc^-\gamma \\ -1, & \text{if } c^*d^* \in Arc^-\gamma,\ p^*q^* \in Arc^+\gamma. \end{cases}$$

Note that the degrees δ and δ' are not independent, in particular, the following cases are possible:

$$\begin{array}{lll} 1.a) \quad \delta = 0, \delta' = 0; & 1.b)\ \delta = 0, \delta' = 1; & 1.c)\ \delta = 0, \delta' = -1; \\ 2.a) \quad \delta = 1, \delta' = 0; & 2.b)\ \delta = 1, \delta' = -1; & \\ 3.a) \quad \delta = -1, \delta' = 0; & 3.b)\ \delta = -1, \delta' = 1. & \end{array}$$

From (17) follows the equality $a^*b^* = \lambda^{\delta+\delta'} p^*q^*$, in which

$$\delta + \delta' = \begin{cases} 0, & \text{in cases } 1.a),\ 2.b),\ 3.b), & \text{when } a^*b^* \uparrow\uparrow p^*q^* \\ 1, & \text{in cases } 1.b),\ 2.a), & \text{when } a^*b^* \in Arc^+\gamma,\ p^*q^* \in Arc^-\gamma \\ -1, & \text{in cases } 1.c),\ 3.a), & \text{when } a^*b^* \in Arc^-\gamma,\ p^*q^* \in Arc^+\gamma. \end{cases}$$

Therefore, $a^*b^* =_\lambda p^*q^*$. Transitivity is proved.

□

We will show that λ-congruence also satisfies congruence axioms $III_1 - III_3$ of the truncated Hilbert axiom system [1].

Proposition 22. (An analogue of Axiom III_1.) *For every ord-arc a^*b^* and every half-orbit $\gamma_{\tilde{a}}$ with the vertex $\tilde{a}^*$, there is a unique point $\tilde{b}^* \in \gamma_{\tilde{a}}$ such that*

$$a^*b^* =_\lambda \tilde{a}^*\tilde{b}^*.$$

In addition, it is required that $a^*b^* =_\lambda b^*a^*$.

Proof. The assertion follows from the uniqueness of the construction of an ord-arc from any point (Proposition 14): from $\tilde{a}^*$ we lay off the ord-arc $\lambda^{-\delta}a^*b^*$, where

$$\delta = \begin{cases} 0, & \text{if } a^*b^* \uparrow\uparrow \gamma_{\tilde{a}}, \\ 1, & \text{if } a^*b^* \uparrow\downarrow \gamma_{\tilde{a}} \text{ and } a^*b^* \in Arc^+\gamma, \\ -1, & \text{if } a^*b^* \uparrow\downarrow \gamma_{\tilde{a}} \text{ and } a^*b^* \in Arc^-\gamma. \end{cases}$$

It is not hard to see that $\tilde{b}^* \in \gamma_{\tilde{a}}$ and $a^*b^* = \lambda^\delta \tilde{a}^*\tilde{b}^*$, and the degree δ is determined by formula (16).

As for the required equality $a^*b^* =_\lambda b^*a^*$, it is fulfilled if and only if $\lambda = -1$. □

Proposition 23. (An analogue of Axiom III_2.) *If $a^*b^* =_\lambda c^*d^*$ and $p^*q^* =_\lambda c^*d^*$, then $a^*b^* =_\lambda p^*q^*$.*

Proof. This follows from symmetry and transitivity of the λ-congruence. □

Proposition 24. (An analogue of Axiom III_3.) If $a^* - b^* - c^*$ and $\tilde{a}^* - \tilde{b}^* - \tilde{c}^*$, and $a^*b^* =_\lambda \tilde{a}^*\tilde{b}^*$, $b^*c^* =_\lambda \tilde{b}^*\tilde{c}^*$, then $a^*c^* =_\lambda \tilde{a}^*\tilde{c}^*$.

Proof. Let the conditions of Axiom III_3 be satisfied.
$a^* - b^* - c^* \Rightarrow a^*b^* \uparrow\uparrow b^*c^* \uparrow\uparrow a^*c^*$.
$\tilde{a}^* - \tilde{b}^* - \tilde{c}^* \Rightarrow \tilde{a}^*\tilde{b}^* \uparrow\uparrow \tilde{b}^*\tilde{c}^* \uparrow\uparrow \tilde{a}^*\tilde{c}^*$.
The following cases are possible:

1. $a^*b^* \uparrow\uparrow \tilde{a}^*\tilde{b}^*$. Then $b^*c^* \uparrow\uparrow \tilde{b}^*\tilde{c}^*$ and $a^*c^* \uparrow\uparrow \tilde{a}^*\tilde{c}^*$. Therefore, the equalities $a^*b^* =_\lambda \tilde{a}^*\tilde{b}^*$, $b^*c^* =_\lambda \tilde{b}^*\tilde{c}^*$ mean that $a^*b^* = \tilde{a}^*\tilde{b}^*$, $b^*c^* = \tilde{b}^*\tilde{c}^*$. Taking into account the vector nature of ord-arcs, we can write that $a^*b^* + b^*c^* = \tilde{a}^*\tilde{b}^* + \tilde{b}^*\tilde{c}^*$. By Proposition 13, $a^*b^* + b^*c^* = a^*c^*$, $\tilde{a}^*\tilde{b}^* + \tilde{b}^*\tilde{c}^* = \tilde{a}^*\tilde{c}^*$. Thereby, $a^*c^* = \tilde{a}^*\tilde{c}^*$. Since $a^*c^* \uparrow\uparrow \tilde{a}^*\tilde{c}^*$, we have that $a^*c^* =_\lambda \tilde{a}^*\tilde{c}^*$.

2. $a^*b^* \uparrow\downarrow \tilde{a}^*\tilde{b}^*$. Then $b^*c^* \uparrow\downarrow \tilde{b}^*\tilde{c}^*$ and $a^*c^* \uparrow\downarrow \tilde{a}^*\tilde{c}^*$. There are two options that stand out here: A) $a^*b^* \in Arc^+\gamma$. B) $a^*b^* \in Arc^-\gamma$. Consider A), and option B) is verified in a similar way. The equalities $a^*b^* =_\lambda \tilde{a}^*\tilde{b}^*$, $b^*c^* =_\lambda \tilde{b}^*\tilde{c}^*$ mean that $a^*b^* = \lambda\, \tilde{a}^*\tilde{b}^*$, $b^*c^* = \lambda\, \tilde{b}^*\tilde{c}^*$. Taking into account the vector nature of ord-arcs, we can write that $a^*b^* + b^*c^* = \lambda\, \tilde{a}^*\tilde{b}^* + \lambda\, \tilde{b}^*\tilde{c}^* = \lambda\, (\tilde{a}^*\tilde{b}^* + \tilde{b}^*\tilde{c}^*)$. By Proposition 13, we obtain $a^*b^* + b^*c^* = a^*c^*$, $\tilde{a}^*\tilde{b}^* + \tilde{b}^*\tilde{c}^* = \tilde{a}^*\tilde{c}^*$. Thereby, $a^*c^* = \lambda\, \tilde{a}^*\tilde{c}^*$. Since $a^*c^* \in Arc^+\gamma$ and $\tilde{a}^*\tilde{c}^* \in Arc^-\gamma$, we have that $a^*c^* =_\lambda \tilde{a}^*\tilde{c}^*$. □

λ-Congruence allows us to specify a single standard on the set $Arc\gamma$ of all orbital ord-arcs. More precisely, we have

Theorem 10. *On the set $Arc\gamma$ of ord-arcs of an orbit γ for a given standard p^*q^* and every real number $\lambda < 0$ there exists a unique mapping $l : Arc\gamma \to \mathbb{R}$, satisfying the axioms of measure of an ord-arc, if in Axiom 2 the congruence is understood as λ-congruence.*

Proof. For any ord-arc p^*q^*, there is a λ-congruent oppositely oriented ord-arc $\tilde{p}^*\tilde{q}^*$. By Theorem 5, there exists a mapping $l : Arc\gamma \to \mathbb{R}$, satisfying the axioms of measure with respect to the usual equality of ord-arcs and such that $l(p^*q^*) = l(\tilde{p}^*\tilde{q}^*) = 1$. We will verify whether it satisfies Axiom 2 also for λ-congruent ord-arcs. Since λ-congruence is an equality for co-directed ord-arcs, it is sufficient to check that Axiom 2 is satisfied for two oppositely oriented λ-congruent ord-arcs.

Suppose that $a^*b^* =_\lambda \tilde{a}^*\tilde{b}^*$ and $a^*b^* \uparrow\downarrow \tilde{a}^*\tilde{b}^*$. Without loss of generality, it can be assumed that $a^*b^* \uparrow\uparrow p^*q^*$. Then $\tilde{a}^*\tilde{b}^* \uparrow\uparrow \tilde{p}^*\tilde{q}^*$, and we can write that $p^*q^* = \lambda^\delta \tilde{p}^*\tilde{q}^*$ and $a^*b^* = \lambda^\delta \tilde{a}^*\tilde{b}^*$, where $\delta = \pm 1$ ($+$ for $p^*q^* \in Arc^+\gamma$ and $-$ for $p^*q^* \in Arc^-\gamma$)). According to formulas (5), (6) and item 2) of Proposition 10,

$$l(a^*b^*) = \frac{\langle a^*b^* \rangle}{\langle p^*q^* \rangle} = \frac{\left\langle \lambda^\delta \tilde{a}^*\tilde{b}^* \right\rangle}{\langle \lambda^\delta \tilde{p}^*\tilde{q}^* \rangle} = \frac{\lambda^\delta \left\langle \tilde{a}^*\tilde{b}^* \right\rangle}{\lambda^\delta \langle \tilde{p}^*\tilde{q}^* \rangle} = \frac{\left\langle \tilde{a}^*\tilde{b}^* \right\rangle}{\langle \tilde{p}^*\tilde{q}^* \rangle} = l(\tilde{a}^*\tilde{b}^*).$$

We will prove the uniqueness by contradiction, assuming the existence of two measures l and θ, defined in $Arc\gamma$, which satisfy the axioms 1-3 of measure and are such that $l(p^*q^*) = \theta(p^*q^*) = 1$. Since congruence is an equality on the set of co-directional ord-arcs, the restriction of the measures l and θ to the classes $Arc^+\gamma$ and $Arc^-\gamma$ ensures the fulfillment of the axioms of measure 1-4 with respect to the congruence of ord-arcs, understood as equality. By Theorem 7, the measures l and θ coincide on $Arc^+\gamma$ and $Arc^-\gamma$. Since $Arc^+\gamma \cup Arc^-\gamma = Arc\gamma$, they also coincide on $Arc\gamma$. We arrive at a contradiction. The theorem is proved.

□

Theorem 11. $l(b^*a^*) = |\lambda|^\delta l(a^*b^*)$, *where*

$$\delta = \begin{cases} 1, & \text{if } a^*b^* \in Arc^+\gamma, \\ -1, & \text{if } a^*b^* \in Arc^-\gamma. \end{cases} \tag{18}$$

Proof. Let p^*q^* and $\tilde{p}^*\tilde{q}^*$ be oppositely oriented unit λ-congruent ord-arcs, and $p^*q^* \in Arc^+\gamma$. Then $p^*q^* = \lambda \tilde{p}^*\tilde{q}^*$.

If $a^*b^* \in Arc^+\gamma$, then $b^*a^* \in Arc^-\gamma$, and by formulas (5), (6) and item 2) of Proposition 10, we have that

$$l(b^*a^*) = \frac{\langle b^*a^* \rangle}{\langle \tilde{p}^*\tilde{q}^* \rangle} = \frac{\langle b^*a^* \rangle}{\langle \lambda^{-1} p^*q^* \rangle} = \frac{-\langle a^*b^* \rangle}{\lambda^{-1} \langle p^*q^* \rangle} = -\lambda \frac{\langle a^*b^* \rangle}{\langle p^*q^* \rangle} = |\lambda| \frac{\langle a^*b^* \rangle}{\langle p^*q^* \rangle} = |\lambda| \, l(a^*b^*).$$

For $a^*b^* \in Arc^-\gamma$, the result follows from the above formula.

□

Remark 4. This fact does not contradict Axiom 2 of measure, since the ord-arcs a^*b^* and b^*a^* are not λ-congruent for $\lambda \neq -1$. Moreover, this can be perfectly confirmed in practice. If the distance between settlements is measured by flight time (in this case, the aircraft speed with respect to the Earth will be the unit), then because of daily rotation of the Earth, the flight times from East to West and in the opposite direction will differ. Likewise, the time to travel a similar distance upstream and downstream will not be the same.

9. Orbital arc measurement

We emphasize that by virtue of Proposition 2, λ-congruent ord-arcs of different orientations cannot be mapped one into another by means of transformations of the group $g(\mathbb{R})$. Therefore, λ-congruence is an overgeometric construction that constitutes *a convention*. However, sometimes (or always?) it is possible to extend the group $g(\mathbb{R})$ to a group containing transformations inverting the ord-arcs, i.e. such that $f(a^*b^*) = b^*a^*$ for some ord-arc a^*b^* of an orbit of the group $g(\mathbb{R})$. For example, for an orbit of the group $\mathbb{R}(O)$ of rotations of the plane about the center O, the minimal group H with this property is obtained by adding a single transformation, namely, the reflection in the point O. For a group containing transformations that reverse the arcs, λ-congruence for $\lambda = -1$ has a geometric meaning, although the extended group is not a one-parameter one. This extension allows us to introduce the measurement of orbital arcs in a completely natural way.

An arc with endpoints a^* *and* b^* can be understood as a pair of ord-arcs $\{a^*b^*, b^*a^*\}$, where $a \neq b$. The designation is as follows: $[a^*b^*]$.

The measure of the arc $[a^*b^*]$ is the quantity $|a^*b^*| = |b - a| = |\langle a^*b^*\rangle|$.

The arcs $[a^*b^*]$ and $[c^*d^*]$ are called *congruent* iff $|a^*b^*| = |c^*d^*|$. The designation is as follows: $[a^*b^*] = [c^*d^*]$. Note that

$$[a^*b^*] = [c^*d^*]. \quad \Leftrightarrow \quad a^*b^* =_{-1} c^*d^*.$$

By virtue of item 1) of Proposition 6, for every arc $[a^*b^*]$ there is a point

$$c^* = \left(\frac{a+b}{2}\right)^*,$$

that is the midpoint of both ord-arcs a^*b^* and b^*a^*. It is natural to call it the *midpoint of the arc* $[a^*b^*]$. In this case, $[a^*c^*] = [c^*b^*]$.

An arc, a measure of an arc, congruence of arcs, the midpoint of an arc are invariants of the extended group H.

The *length of an arc* is defined similarly to the measure of an ord-arc by axioms 1-4, but the word "ord-arc" is replaced by the word "arc". The length of the arc $[a^*b^*]$ will be denoted by $\theta[a^*b^*]$.

Theorem 12. *On the set* $Arc[\gamma]$ *of arcs of the orbit* γ *for a given unit arc* $[p^*q^*]$, *there is only one mapping* $\theta : Arc[\gamma] \to \mathbb{R}$ *that satisfies the axioms of arc length.*

Proof. By Theorem 10, in the set $Arc\gamma$ of ord-arcs of the orbit γ, there exists a measure $l : Arc\gamma \to \mathbb{R}$, with respect to λ-congruence for $\lambda = -1$ such that $l(p^*q^*) = 1$. It is unique and, as follows from the proof of the theorem, is determined by the formula

$$l(a^*b^*) = \begin{cases} \frac{\langle a^*b^*\rangle}{\langle p^*q^*\rangle}, & \text{if } a^*b^* \uparrow\uparrow p^*q^*, \\ \\ \frac{\langle a^*b^*\rangle}{\langle q^*p^*\rangle}, & \text{if } a^*b^* \uparrow\downarrow p^*q^*. \end{cases}$$

Obviously,

$$l(a^*b^*) = \frac{|a^*b^*|}{|p^*q^*|} = l(b^*a^*).$$

Then the function $\theta : Arc[\gamma] \to \mathbb{R}$ is well-defined by the condition

$$\theta[a^*b^*] = l(a^*b^*). \tag{19}$$

The fulfillment of the axioms of measure 1-4 for this function follows from the corresponding properties of the measure l.

We now prove the uniqueness of measure of length in the set $Arc[\gamma]$. Let $\tilde{\theta}$ be an arbitrary measure of length on the set of arcs $Arc[\gamma]$ such that $\tilde{\theta}[p^*q^*] = 1$. Then the function, defined on the set of ord-arcs $Arc\gamma$ by the formula $\tilde{l}(a^*b^*) = \tilde{\theta}[a^*b^*]$, satisfies the axioms 1-3 of measure of ord-arcs with respect to λ-congruence for $\lambda = -1$, moreover, $\tilde{l}(p^*q^*) = 1$. By Theorem 10, it is true that $\tilde{l} = l$, which means that $\tilde{\theta} = \theta$. The theorem is proved.

□

The possibility of measuring an arc by laying off the standard sequentially follows from formula (19) and Theorems 9 and 10.

10. On proofs of existence and uniqueness theorems for the measure of an interval in Euclidean geometry

The proof of the uniqueness of length measure of a segment in textbooks [1-3] actually substantiates the uniqueness of one-sided measure, since the measurement process depends on the endpoint of the measured segment, from which the process of laying off of the standard starts. It is logically possible, as Theorem 11 shows for the case of λ-congruence for $\lambda \neq -1$, that when measuring a segment starting from one of its endpoints, the standard fits into the segment an integer n times, and an integer m times (or not an integer number of times) from the other end.

Therefore, in textbooks [2, 3], the proof of the uniqueness of measure must be completed after verification of fulfillment of axiom 1 of measure (congruent segments have equal measures of length) for the function revealed in the process of justifying the uniqueness of measure (which we informally defined in the introduction). And it should end with a remark that, due to the congruence of the segments $[AB]$ and $[BA]$ and the fulfillment of the first axiom of measure, both one-sided measures determined by the two endpoints of the segment are similar.

The proof of the uniqueness theorem for measure in textbook [1, p. 376, Theorem 1] uses the uniqueness of measure in the class of segments that are multiples of the standard. The uniqueness of measure in the class of segments that are multiples of the standard does not follow from auxiliary Lemma 1, which states that the length of the segment in which the standard (unit segment) fits n times is equal to n, because the uniqueness of the result of sequential laying off the standard in the segment from different endpoints as well as from one endpoint requires, albeit not complicated, but special justification. The reader's attention should be drawn to this, especially since this fact is implicitly exploited in [1] when justifying the existence theorem for measure. Namely, when the standard is deposited from the interior point B of the segment $[AC]$ in the direction of the point A k times and the point A_k is obtained, and then in the direction of the point C s times and the point C_s is obtained. After that it is assumed that the standard deferred from the point A_k in the direction of the point C_s fits into the segment A_kC_s exactly $k + s$ times. This is true if the standard does not change when laid off in different directions (recall the analogy with the time of going up and down the river).

Realizing the impossibility of citing all the reasoning in the educational literature in full, we consider it necessary at least to draw the attention of the readers to the "forks" that may entail the ambiguity of measure.

11. Conclusion

So, for the case of effective action of a one-parameter group on its own orbits, we have shown: 1) the equality of arcs in an orbit of a one-parameter group of transformations, defined internally by bijection of a Euclidean space $\mathbb{R}$ with a modular metric onto it, coincides with the equality of arcs, defined externally as an equivalence with respect to group transformations $g(\mathbb{R})$; 2) an arc measure, also defined internally, agrees with measurement result by external operations of laying off an arc from a given point and dividing it in half.

In the case when the transformation group $g(\mathbb{R})$ of the space E acts ineffectively on its orbits and the transformation $a \to a^*$ is not bijective, all reasoning remains valid if the points of the orbit are considered to be tied to the parameters to which they correspond. Then we will have a "straight line wound onto an orbit," as in the case of number circumference. Let the orbit γ be generated by the point $\tilde{o} \in E$. The "attachment" of the point to a parameter is formalized by the pair $(a, g_a(\tilde{o}))$, which will be denoted by a^* in order to keep the above results in the same notations. Then the mapping $f : \mathbb{R} \to \mathbb{R} \times E$, defined by the formula

$$f(a) = (a, g_a(\tilde{o})) = a^*,$$

is a homomorphism (its inverse is the projection onto the first factor $\mathbb{R}$, which is a continuous mapping). Hence, the set $f(\mathbb{R})$ is a one-dimensional manifold in $\mathbb{R} \times E$. We will call it *the orbit sweep* γ and denote it by $\hat{\gamma}$. The one-parameter transformation group $g(\mathbb{R})$ of the space E generates the action of the additive topological group $\mathbb{R}$ in the space $\mathbb{R} \times E$, that is, the homomorphism $\hat{g}$ of the group $\mathbb{R}$ into the group of transformations of the space $\mathbb{R} \times E$, assigning to each number $a \in \mathbb{R}$ the transformation $\hat{g}_a$, defined by the formula

$$\hat{g}_a(s, \tilde{x}) = (s + a, g_a(\tilde{x})).$$

Indeed,

$$\hat{g}_0(s, \tilde{x}) = (s + 0, g_0(\tilde{x})) = (s, \tilde{x}) \quad \Rightarrow \quad \hat{g}_0 \text{ is the identity transformation.}$$

$$\hat{g}_b \circ \hat{g}_a(s, \tilde{x}) = \hat{g}_b(s + a, g_a(\tilde{x})) = ((s + a) + b, g_b(g_a(\tilde{x}))) =$$
$$= (s + (a + b), g_{a+b}(\tilde{x}))) = \hat{g}_{a+b}(s, \tilde{x}) \quad \Rightarrow \quad \hat{g}_b \circ \hat{g}_a = \hat{g}_{a+b}.$$

Then the sweep $\hat{\gamma}$ of the orbit γ is the orbit of the transformation group $\hat{g}(\mathbb{R})$. Indeed,

$$\hat{\gamma} = \{a^* = (a, g_a(\tilde{o}))|\ a \in \mathbb{R}\}.$$

But $(a, g_a(\tilde{o})) = \hat{g}_a(0, \tilde{o}) = \hat{g}_a(0, g_0(\tilde{o})) = \hat{g}_a(o^*)$. Therefore, $\hat{\gamma} = \{\hat{g}_a(o^*))|\ a \in \mathbb{R}\}$. The group $\hat{g}(\mathbb{R})$ acts effectively in $\hat{\gamma}$. All the results obtained above remain valid for the orbital sweep.

Note the problems related to measuring of arcs of curves using the selected unit of measure:

1. The question on practical methods of arcs measurement for each specific case of a given group of transformations $g(\mathbb{R})$: how and with a help of which tools it is possible to a) lay off an arc equal to a given arc from a given point, b) find the midpoint of the arc.

For example, on enika (1) in an n-dimensional affine space, the equality of arcs is defined as equivalence with respect to the group of affine unimodular transformations (affine transformations in which the absolute value of the determinant is equal to one). In [4], a simpler way of measuring the arcs of an enika is given, rather than sequential laying off of the standard and dividing the arcs in half. It is enough to project the measurable arc a^*b^* and the unit arc p^*q^* onto any straight line of non-asymptotic direction (for example, onto a tangent line to enika) along asymptotic hyperplanes so that the segments $[a_1b_1]$ and $[p_1q_1]$ are their images. Then the length of the arc a^*b^* will be $\left|\frac{\overrightarrow{a_1b_1}}{\overrightarrow{p_1q_1}}\right|$.

2. When considering curves that allow the measurement of arcs by a standard in a space in which the length of the curve is determined in some universal way (for example, in a metric space), there arises a question about the relation of the lengths of arcs for different approaches on their definition. For example, in [4] it was shown that when enika is given by a canonical parameterization, the affine length of its arc coincides with the result of measurement performed using an arc whose endpoints correspond to the parameters 0 and 1.

3. With the help of a more extensive group H of transformations, which includes, in addition to transformations of the group G, the transformations that map one orbit to another, it is possible to transfer the unit of measure from one orbit to all the other ones. Thus, the measurement of arcs that is invariable under the group H is defined on all orbits of the group G. By extending the transformation group, arc measurement can be expanded to an even wider class of curves. So, for example, the measurement of arcs of a circle, which is the orbit of a group of rotations in the Euclidean plane, extends to all the circles concentric to it by means of a group of similarity transformations with only one fixed point (including the identity transformation), and then to all circles by means of similarities. In the latter case, the measure is invariant with respect to the group of similarities; the standard is traditionally a degree or a radian.

Therefore, the third problem is the existence of a transformation group H, containing the one-parameter group $g(\mathbb{R})$ and acting effectively on its orbits in such a way that the equality of arcs, defined through H-equivalence, makes it possible to measure the arcs using a standard.

4. The problem of finding the minimal and maximal groups $H \supset g(\mathbb{R})$, possessing the above property. For example, any circle on the Euclidean plane E^2 can be mapped onto another circle on the same plane as a result of composition of parallel translation and homothety, that is, similarity of the first kind. Therefore, the equality of arcs of circles of the Euclidean plane can be regarded as equivalence relation with respect to the group of similarities of the first kind. This group, apparently, will be minimal among the ones allowing the measurement of arcs of all circles in E^2.

Consideration of a more extensive group of transformations allows us to use new opportunities for constructions. So, instead of the group of affine transformations of the form (3), consideration of arbitrary affine unimodular transformations allows us to obtain equal arcs using symmetries about the coordinate planes passing through the odd-numbered axes of the canonical coordinate systems of an enika [4].

However, the extension of the transformation group is fraught with disruption of the uniqueness of constructing of the arc from a given point and the uniqueness of the midpoint of the arc. For example, as it was established by the author in [10], any two arcs of an enika are affinely equivalent. Therefore, the group of affine unimodular transformations cannot be extended to an affine group with preservation of the possibility of measuring of the arcs using the standard.

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Irina Viktorovna Polikanova
Altai State Pedagogical University,
6(1), Soviet str.,
Barnaul, 656002, Russia
Email address: anirix1@yandex.ru