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SOFT 3-STARS IN SPARSE PLANE GRAPHS

O.V. BORODIN, A.O. IVANOVA

Abstract. We consider plane graphs with large enough girth g, minimum degree δ at least 2 and no $(k+1)$-paths consisting of vertices of degree 2, where $k \geq 1$. In 2016, Hudák, Maceková, Madaras, and Široczki studied the case $k = 1$, which means that no two 2-vertices are adjacent, and proved, in particular, that there is a 3-vertex whose all three neighbors have degree 2 (called a soft 3-star), provided that $g \geq 10$, which bound on g is sharp. For the first open case $k = 2$ it was known that a soft 3-star exists if $g \geq 14$ but may not exist if $g \leq 12$. In this paper, we settle the case $k = 2$ by presenting a construction with $g = 13$ and no soft 3-star. For all $k \geq 3$, we prove that soft 3-stars exist if $g \geq 4k+6$ but, as follows from our construction, possibly not exist if $g \leq 3k+7$. We conjecture that in fact soft 3-stars exist whenever $g \geq 3k+8$.

Keywords: plane graph, structure properties, girth, tight description, weight, height, 3-star, soft 3-star.

1. Introduction

In what follows, G is a finite plane graph. The degree of a vertex v or a face f in G, that is the number of edges incident with v or f, is denoted by $d(v)$ or $d(f)$, respectively. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \geq k$, etc.

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Let $\delta(G)$ be the minimum vertex degree and $g(G)$ be the girth (the length of a shortest cycle) in G. A k-star $S_k(v)$ in G consists of a central vertex v and k its neighbor vertices.

The height $h(S_k(v))$ and weight $w(S_k(v))$ of $S_k(v)$ is the maximum degree and degree-sum of its vertices, respectively. The height $h_k(G)$ and weight $w_k(G)$ of G is the maximum height and weight of its k-stars. We will often drop the argument when it is clear from context.

An edge uv, that is an $S_1(u)$ or $S_1(v)$, is an (i,j)-*edge* if $d(u) \le i$ and $d(v) \le j$. More generally, a path $v_1v_2v_3$ (which is an $S_2(v_2)$), is a *path of type* (i_1, i_2, i_3), or an (i_1, i_2, i_3)-*path* if $d(v_j) \le i_j$ whenever $1 \le j \le 3$. The types of higher stars are defined similarly.

Already in 1904, Wernicke [31] proved that every G with $\delta = 5$ satisfies $w_2 \le 11$, and Franklin [20] strengthened this in 1922 to the existence of a $(6,5,6)$-path, which description is tight. In [9], we proved that there is another tight description, "a $(5,6,6)$-path" and that no other tight descriptions exist.

In [7], we gave a tight description of 3^--stars in arbitrary plane graphs with $\delta \ge 3$ and $g \ge 3$ by proving that there is either a $(3,10)$-edge, or a $(5,4,9)$-path, or a $(6,4,8)$-path, or a $(7,4,7)$-path, or a $(5;4,5,5)$-star, or a $(5;5,b,c)$-star with $5 \le b \le 6$ and $5 \le c \le 7$, or else a $(5;6,6,6)$-star. This extends or strengthens several previously known results by Balogh, Jendrol', Harant, Kochol, Madaras, Van den Heuvel, Yu and others [21, 26, 30] and disproves a conjecture in Harant, Jendrol' [21].

In 1940, Lebesgue [28] gave an approximate description of 5-stars centered at 5-vertices for the case $\delta = 5$ and $g \ge 3$. Recently, we obtained several tight results on the height, weight and structure of such 5-stars assuming the absence of 6^+-vertices from certain degree-sets, see [8, 11, 14, 15, 17–19].

Also, Lebesgue [28] proved that every G with $\delta \ge 3$ and $g = 5$ satisfies $h_2 = 3$ and $w_2 = 9$. In 2004, Madaras [29] refined this by showing that there is a 3-star with $h_3 = 4$ and $w_3 = 13$, which is tight. Recently, we gave [13] another tight description of 3-stars for $g = 5$ in terms of degree of their vertices and showed that there are only these two tight descriptions of 3-stars.

There exist many results on the height, weight and structure of 2^--stars when $\delta = 2$, see, for example, [1–4, 10, 16, 22–25] and also surveys by Jendrol', Voss [27] and Borodin, Ivanova [12].

In 2016, Hudák, Maceková, Madaras and Široczki [22] considered the class of plane graphs with $\delta = 2$ in which no two vertices of degree 2 are adjacent. They proved that $h_3 = w_3 = \infty$ if $g \le 6$, $h_3 = 5$ if $g = 7$, $h_3 = 3$ if $g \ge 8$, $w_3 = 10$ if $8 \le g \le 9$ and $w_3 = 9$ if $g \ge 10$. For $g = 7$, Hudák et al. [22] proved $11 \le w_3 \le 20$, and we recently proved [6] that in fact $w_3 = 12$.

In the present paper, we deal with the class of plane graphs with large enough girth g, minimum degree δ at least 2 and no $(k+1)$-paths consisting of vertices of degree 2, where $k \ge 1$.

Hudák et al. [22] studied the case $k = 1$ and proved, in particular, that there is a 3-vertex whose all neighbors have degree 2 (such a vertex is also called a *soft 3-star*), provided that $g \ge 10$, which bound on g is sharp.

For the first open case $k = 2$ concerning a soft 3-star, it was known that it exists if $g \ge 14$ but may not exist if $g \le 12$.

The main purpose of our paper is to settle the case $k = 2$ by proving that a soft 3-star may not exist even if $g \leq 13$ (put $k = 2$ in Theorem 2 below). The other purpose is to establish lower and upper bounds on g that ensure the existence of a soft 3-star whenever $k \geq 2$.

It is not hard to prove the following fact.

Theorem 1. *Every plane graph with $\delta = 2$, $g \geq 4k + 6$ and no $(k + 1)$-paths consisting of vertices of degree 2, where $k \geq 2$, has a soft 3-vertex, where $k \geq 2$.*

Our main result is as follows.

Theorem 2. *For all $k \geq 2$, there is a plane graph with $\delta = 2$, $g \leq 3k + 7$, no $(k + 1)$-paths consisting of vertices of degree 2, and no soft 3-stars.*

The two above theorems resolve the case $k = 2$ as follows.

Corollary 1. *Every plane graph with $\delta = 2$, $g \geq 14$ and no 3-paths consisting of vertices of degree 2 has a soft 3-vertex, where the bound 14 is best possible.*

We believe that the restriction on g in Theorem 2 is sharp whenever $k \geq 3$.

Conjecture 1. *Every plane graph with $\delta = 2$, $g \geq 3k + 8$ and no $(k + 1)$-paths consisting of vertices of degree 2, where $k \geq 3$, has a soft 3-vertex for all $k \geq 2$.*

2. Proof of Theorem 1

Let G be a counterexample to Theorem 1 by having $\delta(G) \geq 2$, $g(G) = g \geq 4k+6$ with $k \geq 2$, and no soft 3-vertices. Without loss of generality, we can assume that G is connected.

Let V, E, and F be the sets of vertices, edges and faces of G, respectively. Euler's formula $|V| - |E| + |F| = 2$ for G may be rewritten as

$$\sum_{v \in V} \left(\frac{g-2}{2} \times d(v) - g\right) + \sum_{f \in F} (d(f) - g) = -2g. \tag{1}$$

Each vertex v contributes the *charge* $\mu(v) = \frac{g-2}{2} \times d(v) - g$ to (1), and each face f contributes the non-negative *charge* $\mu(f) = d(f) - g$. This implies

$$\sum_{v \in V} \mu(v) < 0. \tag{2}$$

Note that if $d(v) = 2$ then $\mu(v) = \frac{g-2}{2} \times 2 - g = -2$, and if $d(v) \geq 3$ then

$$\mu(v) = \frac{g-2}{2} \times d(v) - g = g\left(\frac{d(v)}{2} - 1\right) - d(v) \geq (4k+6)\left(\frac{d(v)}{2} - 1\right) - d(v) =$$
$$= (2k+2)d(v) - 4k - 6.$$

In particular, for $d(v) = 3$ we have

$$\mu(v) \geq (2k+2)3 - 4k - 6 = 2k,$$

while $d(v) \geq 4$ implies

$$\mu(v) \geq (2k+2)d(v) - 4k - 6 > 2kd(v) - 4k \geq kd(v).$$

We now define a local redistribution of $\mu(v)$'s, preserving their sum, such that the *new charge* $\mu'(v)$ is non-negative for all $v \in V$. Namely, $\mu'(v)$ obeys the following rule:

R. Each 2-vertex receives 1 along its maximal path P consisting of 2-vertices from each of the two end-vertices of P.

Using the above-mentioned estimations on $\mu(v)$, we immediately obtain $\mu'(v) \geq 0$ for all $v \in V$. Namely, if $d(v) = 2$ then $\mu'(v) = \mu(v) + 2 \times 1 = 0$. For $d(v) = 3$ we have $\mu'(v) \geq \mu(v) - 2 \times k = 0$ since G has no soft 3-vertices. Finally, $d(v) \geq 4$ implies $\mu'(v) \geq \mu(v) - d(v) \times k = 0$.

Now a contradiction with (2) completes the proof:

$$0 \leq \sum_{v \in V} \mu'(v) = \sum_{v \in V} \mu(v) < 0.$$

3. Proof of Theorem 2

In Fig. 1, we see a bit more than a quarter of a plane graph F_k that produces a plane graph H_k with required properties by putting k vertices of degree 2 on each edge of F_k not labeled by a star. The labeled edges of F_k become normal edges of H_k.

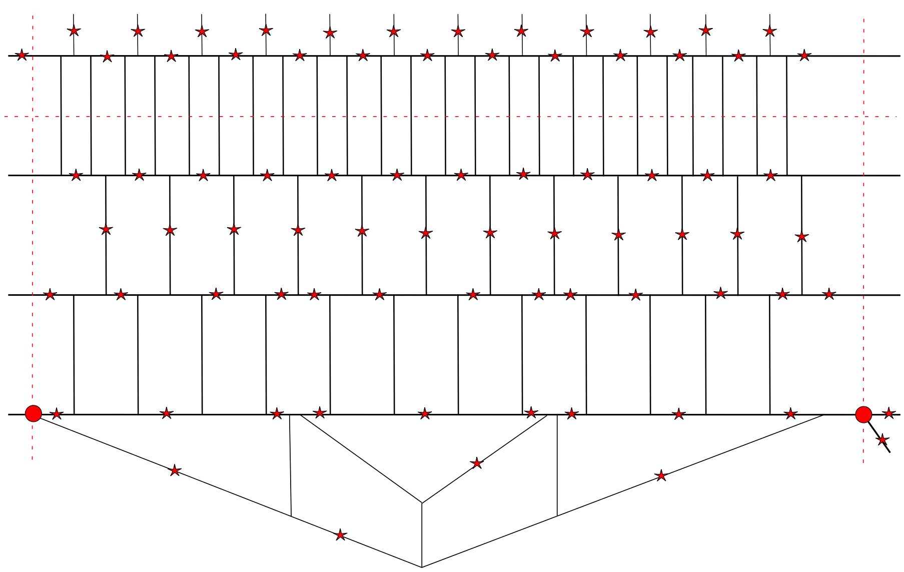

FIG. 1. Dashed lines bound a quarter of a framework F_k of a desired plane graph H_k with $g(H_k) = 3k + 7$ and no soft 3-stars.

More specifically, a quarter of F_k is bounded by one horizontal and two vertical dashed lines. Now a half of F_k is obtained by gluing two quarters along the two vertical dashed lines. To obtain the whole F_k, we glue lower and upper halves along the horizontal "equator" in Fig. 1.

It is not hard to see that F_k consists of 5-faces incident with four non-labelled edges, 7-faces with three non-labelled edges in the boundary, and also two 10-faces (an internal and external) each incident with four non-labelled edges.

Note that the resulting graph H_k has $\delta(H_k) = 2$, no soft 3-vertices, no $(k+1)$-paths consisting of 2-vertices, while each face in H_k has degree either $10 + 4k$, or $5+4k$, or else $7+3k$. Thus $g(H_k) \geq 3k+7$ whenever $k \geq 2$, as required in Theorem 2.

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Oleg Veniaminovich Borodin
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Email address: brdnoleg@math.nsc.ru

Anna Olegovna Ivanova
Ammosov North-Eastern Federal University,
48, Kulakovskogo str.,
Yakutsk, 677000, Russia
Email address: shmgnanna@mail.ru