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REPETITION-FREE AND INFINITARY ANALYTIC CALCULI FOR FIRST-ORDER RATIONAL PAVELKA LOGIC

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Abstract. We present an analytic hypersequent calculus $G^3Ł\forall$ for first-order infinite-valued Łukasiewicz logic $Ł\forall$ and for an extension of it, first-order rational Pavelka logic $RPL\forall$; the calculus is intended for bottom-up proof search. In $G^3Ł\forall$, there are no structural rules, all the rules are invertible, and designations of multisets of formulas are not repeated in any premise of the rules. The calculus $G^3Ł\forall$ proves any sentence that is provable in at least one of the previously known analytic calculi for $Ł\forall$ or $RPL\forall$, including Baaz and Metcalfe's hypersequent calculus $GŁ\forall$ for $Ł\forall$. We study proof-theoretic properties of $G^3Ł\forall$ and thereby provide foundations for proof search algorithms. We also give the first correct proof of the completeness of the $GŁ\forall$-based infinitary calculus for prenex $Ł\forall$-sentences, and establish the completeness of a $G^3Ł\forall$-based infinitary calculus for prenex $RPL\forall$-sentences.

Keywords: many-valued logic, mathematical fuzzy logic, first-order infinite-valued Łukasiewicz logic, first-order rational Pavelka logic, proof theory, hypersequent calculus, proof search, infinitary calculus.

1. Introduction

First-order infinite-valued Łukasiewicz logic $Ł\forall$ and an extension of it by rational truth constants, first-order rational Pavelka logic $RPL\forall$, are among the fundamental fuzzy logics [1, 2, 3] and are considered in the given paper mainly from the standpoint of proof search.

Hilbert-type calculi for the logics under consideration are widely used (see, e.g., [1, 2]), but such calculi are unfit for bottom-up proof search. For $Ł\forall$, we also know

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an analytic hypersequent calculus GŁ∀ [4, 5] with structural rules, which make it unsuitable for bottom-up proof search.

On the basis of the calculus GŁ∀ from [4, 5] and tableau calculi from [6], in [7] we introduced analytic hypersequent calculi G^1Ł∀ and G^2Ł∀ for the logic RPL∀ and hence for Ł∀. The calculi G^1Ł∀ and G^2Ł∀ do not have structural rules; the latter is a noncumulative variant of the former, which is cumulative, i.e., preserves the conclusion of each inference rule in its premises. Any GŁ∀- or G^2Ł∀-provable sentence is provable in G^1Ł∀; any prenex RPL∀-sentence is G^1Ł∀-provable iff it is G^2Ł∀-provable; and any prenex Ł∀-sentence is provable or unprovable in GŁ∀, G^1Ł∀, and G^2Ł∀ simultaneously. Also in [7], a family of proof search algorihms is described; given a prenex G^2Ł∀-provable sentence, such an algorithm constructs some proof for it in a tableau modification of the calculus G^2Ł∀.

A defect of G^2Ł∀ (which does not appear in proving prenex sentences) is that designations of multisets of formulas are repeated in each premise of two quantifier rules. The defect causes repeating much work (on decomposing the same formulas from such multisets) during bottom-up proof search, and prevented us from establishing desirable proof-theoretic properties for the calculus G^2Ł∀, in particular, invertibility of one of its rules.

In the present paper, we introduce an analytic hypersequent calculus G^3Ł∀ for the logic RPL∀. There are no structural rules in the calculus; and designations of multisets of formulas are not repeated in any premise of its rules. The last feature of the calculus allows us to call it and each of its rules *repetition-free*.

Let us provide further background on the logics Ł∀ and RPL∀. The set of all valid Ł∀-formulas is not recursively enumerable [9]; more precisely, the set is Π_2-complete [10]; the same holds for RPL∀, see [1, Section 6.3]. So for Ł∀, there is no sound and complete calculus in the ordinary (finitary) sense; the same for RPL∀. To be sound and complete for Ł∀ or RPL∀, a calculus needs to be infinitary. In [11], [12], and [1, Section 5.4], sound and complete Hilbert-type calculi with an infinitary rule are presented for Ł∀; and in [1, Section 5.4], such an infinitary calculus for RPL∀. In [4], it is claimed that the hypersequent calculus GŁ∀ extended with an infinitary rule is complete for Ł∀; however, the completeness of this GŁ∀-based infinitary calculus is still an open question, according to our findings, which G. Metcalfe, an author of [4], acknowledges. Also in [4], the calculus GŁ∀ extended with the cut rule is shown to prove exactly the same Ł∀-sentences as a Hilbert-type calculus for Ł∀ from [1]; the latter calculus is known to be complete with respect to an algebraic semantics over so-called MV-chains, see [1, 4] for further details. Finally, in [5, Theorem 8.48] and [13, Theorem 6.1.11], erroneous proofs are given that the aforementioned GŁ∀-based infinitary calculus is complete for prenex Ł∀-sentences.

In the present paper, we are interested in foundations for bottom-up proof search algorithms for Ł∀ and RPL∀; so we mainly deal with calculi that are finitary, analytic (hence cut-free), and of course, sound for the given logics. However, in this paper, we also give the first correct proof of the completeness of the GŁ∀-based infinitary calculus for prenex Ł∀-sentences, and extend this result to a G^3Ł∀-based infinitary calculus and prenex RPL∀-sentences. Apart from this, we focus on studying proof-theoretic properties of G^3Ł∀ and leave out studying how G^3Ł∀ can be used to establish properties of Ł∀ and RPL∀.

The paper is organized as follows. In Section 2, we define the syntax and semantics of the logics Ł∀ and RPL∀, and formulate the calculi GŁ∀, G^1Ł∀, and

$G^2Ł\forall$. In Section 3, we introduce the calculus $G^3Ł\forall$ and prove its soundness. In Section 4, we establish the invertibility of all the rules of $G^3Ł\forall$ and show that $G^3Ł\forall$ proves any sentence provable in $GŁ\forall$, or $G^1Ł\forall$, or $G^2Ł\forall$. In Section 5, we investigate transformations of $G^3Ł\forall$-proofs according to proof search tactics and thereby provide foundations for various proof search algorithms. In Section 6, we prove the mid-hypersequent theorem for $G^3Ł\forall$; show that any prenex RPL$\forall$-sentence is provable or unprovable in $G^1Ł\forall$, $G^2Ł\forall$, and $G^3Ł\forall$ simultaneously; and establish the undecidability of $G^3Ł\forall$. In Section 7, we show that in [5, 13] the proofs of the completeness of the $GŁ\forall$-based infinitary calculus for prenex Ł$\forall$-sentences are incorrect; we give a new correct proof of the claim, and extend this proof to the $G^3Ł\forall$-based infinitary calculus and prenex RPL$\forall$-sentences.

2. Preliminaries

Let us describe the syntax and semantics of the logics Ł$\forall$ and RPL$\forall$. We fix an arbitrary signature, which may contain predicate and function symbols of any nonnegative arities.

Terms are defined in the standard manner. *Atomic* Ł$\forall$- and RPL$\forall$-*formulas* are predicate symbols with argument terms, as well as truth constants: in Ł$\forall$, the only truth constant $\bar{0}$; and in RPL$\forall$, truth constants $\bar{r}$ for all rational numbers $r \in [0,1]$ (where $[0,1]$ is an interval of real numbers). Ł$\forall$- and RPL$\forall$-*formulas* are built as usual from atomic Ł$\forall$- and RPL$\forall$-formulas, respectively, using the *logical symbols*: the binary connective $\to$ and the quantifiers $\forall, \exists$.

The notion of an *interpretation* $\langle \mathcal{D}, \mu \rangle$ differs from the classical notion of the same name only in that the map μ takes each n-ary predicate symbol R to a predicate $\mu(R) : \mathcal{D}^n \to [0,1]$. Given an interpretation $\langle \mathcal{D}, \mu \rangle$, a *valuation* is a map of the set of all (individual) variables to the domain $\mathcal{D}$ of the interpretation. For a valuation ν, a variable x, and $d \in \mathcal{D}$, by $\nu[x \mapsto d]$ we denote the valuation that may differ from ν only on x and meets the condition $\nu[x \mapsto d](x) = d$.

The *value* $|t|_{M,\nu}$ *of a term* t under an interpretation M and a valuation ν (appropriate for M) is defined as usual. The *truth value* $|C|_{M,\nu}$ *of an* RPL$\forall$-*formula* C under an interpretation $M = \langle \mathcal{D}, \mu \rangle$ and a valuation ν is defined as follows:

(1) $|\bar{r}|_{M,\nu} = r$;

(2) $|R(t_1, \dots, t_n)|_{M,\nu} = \mu(R)(|t_1|_{M,\nu}, \dots, |t_n|_{M,\nu})$ for an n-ary predicate symbol R and terms $t_1, \dots, t_n$;

(3) $|A \to B|_{M,\nu} = \min(1 - |A|_{M,\nu} + |B|_{M,\nu}, 1)$;

(4) $|\forall x A|_{M,\nu} = \inf_{d \in \mathcal{D}} |A|_{M,\nu[x \mapsto d]}$;

(5) $|\exists x A|_{M,\nu} = \sup_{d \in \mathcal{D}} |A|_{M,\nu[x \mapsto d]}$.

An RPL$\forall$-formula (in particular, an Ł$\forall$-formula) C is called *valid*, written $\vDash C$, if $|C|_{M,\nu} = 1$ for every interpretation M and every valuation ν.

Note that the logic RPL$\forall$ allows us to express partial truth of statements in the following way [1, Section 3.3]. Given a rational number $r \in [0,1]$ and an RPL$\forall$-formula A, we have: (a) for a fixed interpretation M and a fixed valuation ν: $r \leqslant |A|_{M,\nu}$ iff $|\bar{r} \to A|_{M,\nu} = 1$; (b) $r \leqslant |A|_{M,\nu}$ for every interpretation M and every valuation ν iff $\vDash (\bar{r} \to A)$.

In what follows, we work with a fixed signature that includes a countably infinite set of nullary function symbols called *parameters*. The result of substituting a term t for all free occurrences of a variable x in an RPL$\forall$-formula A is denoted by $[A]^x_t$.

Let us formulate the calculus GŁ∀ [4, 5], using parameters instead of free variables, which are syntactically distinct from bound variables in [4, 5].

A GŁ∀-*sequent* is written $\Gamma \Rightarrow \Delta$ and is an ordered pair of finite multisets Γ and Δ consisting of Ł∀-formulas. A GŁ∀-*hypersequent* is a finite multiset of GŁ∀-sequents and is written $\Gamma_1 \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Delta_n$.

The axiom schemes of GŁ∀ are: $A \Rightarrow A$ (id), $\Rightarrow (\Lambda)$, $\bar{0} \Rightarrow A$ $(\bar{0} \Rightarrow)$, where A is an Ł∀-formula.

The inference rules of GŁ∀ are:

$$\frac{\mathcal{G}}{\mathcal{G}\,|\,\Gamma \Rightarrow \Delta}\ (\mathrm{ew}),\qquad \frac{\mathcal{G}\,|\,\Gamma \Rightarrow \Delta\,|\,\Gamma \Rightarrow \Delta}{\mathcal{G}\,|\,\Gamma \Rightarrow \Delta}\ (\mathrm{ec}),\qquad \frac{\mathcal{G}\,|\,\Gamma \Rightarrow \Delta}{\mathcal{G}\,|\,\Gamma, A \Rightarrow \Delta}\ (\mathrm{wl}),$$

$$\frac{\mathcal{G}\,|\,\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G}\,|\,\Gamma_1 \Rightarrow \Delta_1\,|\,\Gamma_2 \Rightarrow \Delta_2}\ (\mathrm{split}),\qquad \frac{\mathcal{G}\,|\,\Gamma_1 \Rightarrow \Delta_1;\quad \mathcal{G}\,|\,\Gamma_2 \Rightarrow \Delta_2}{\mathcal{G}\,|\,\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\mathrm{mix}),$$

$$\frac{\mathcal{G}\,|\,\Gamma, B \Rightarrow A, \Delta}{\mathcal{G}\,|\,\Gamma, A \to B \Rightarrow \Delta}\ ({\to}{\Rightarrow}),\qquad \frac{\mathcal{G}\,|\,\Gamma \Rightarrow \Delta;\quad \mathcal{G}\,|\,\Gamma, A \Rightarrow B, \Delta}{\mathcal{G}\,|\,\Gamma \Rightarrow A \to B, \Delta}\ ({\Rightarrow}{\to}),$$

$$\frac{\mathcal{G}\,|\,\Gamma, [A]^x_t \Rightarrow \Delta}{\mathcal{G}\,|\,\Gamma, \forall x A \Rightarrow \Delta}\ (\forall \Rightarrow),\qquad \frac{\mathcal{G}\,|\,\Gamma \Rightarrow [A]^x_a, \Delta}{\mathcal{G}\,|\,\Gamma \Rightarrow \forall x A, \Delta}\ (\Rightarrow \forall),$$

$$\frac{\mathcal{G}\,|\,\Gamma \Rightarrow [A]^x_t, \Delta}{\mathcal{G}\,|\,\Gamma \Rightarrow \exists x A, \Delta}\ (\Rightarrow \exists),\qquad \frac{\mathcal{G}\,|\,\Gamma, [A]^x_a \Rightarrow \Delta}{\mathcal{G}\,|\,\Gamma, \exists x A \Rightarrow \Delta}\ (\exists \Rightarrow),$$

where $\mathcal{G}$ is a GŁ∀-hypersequent; $\Gamma, \Gamma_i, \Delta, \Delta_i$ $(i = 1, 2)$ are finite multisets of Ł∀-formulas; A, B are Ł∀-formulas; t is a closed term; and a is a parameter not occurring in the conslusion of $(\Rightarrow \forall)$ or $(\exists \Rightarrow)$.

Now we formulate the calculi $\mathrm{G^1}$Ł∀ and $\mathrm{G^2}$Ł∀ [7], and define related notions.

We fix a countably infinite set of new words, call them *semipropositional variables*, and denote by $\mathfrak{p}, \mathfrak{p}_0, \mathfrak{p}_1, \ldots$. An *atom* is an atomic RPL∀-formula or a semipropositional variable. A *formula* is an RPL∀-formula or a semipropositional variable. A *sequent* is written $\Gamma \Rightarrow \Delta$ and is an ordered pair of finite multisets Γ and Δ consisting of formulas. A *hypersequent* is written $\Gamma_1 \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Delta_n$ and is a finite multiset of sequents. A *member* of a sequent $\Gamma \Rightarrow \Delta$ is any element of Γ or Δ. A sequent is *atomic* if its every member is an atom.

An *hs-interpretation* is defined as an interpretation $\langle \mathcal{D}, \mu \rangle$ in which the map μ additionally takes each semipropositional variable to a real number from $(-\infty, 1]$.

For a finite multiset Γ of formulas, a sequent $\Gamma \Rightarrow \Delta$, an hs-interpretation M, and a valuation ν, we put

$$|\Gamma|_{M,\nu} = \sum_{A \in \Gamma} (|A|_{M,\nu} - 1) \qquad \text{and} \qquad |\Gamma \Rightarrow \Delta|_{M,\nu} = |\Delta|_{M,\nu} - |\Gamma|_{M,\nu},$$

where summing is carried out taking multiplicities of multiset elements into account, and $\sum_{A \in \varnothing}(\ldots) = 0$. Following [4, Definition 1], a hypersequent $\mathcal{H}$ is called *valid* (written $\vDash \mathcal{H}$) if, for every hs-interpretation M and every valuation ν, there exists a sequent S in $\mathcal{H}$ such that $|S|_{M,\nu} \geqslant 0$.

A hypersequent $\mathcal{H}$ is an *axiom* of the calculus $\mathrm{G^1}$Ł∀ if the hypersequent consisting of all atomic sequents of $\mathcal{H}$ is valid.

The inference rules of the calculus $\mathrm{G^1}$Ł∀ are:

$$\frac{\mathcal{G}\,|\,\Gamma, A \to B \Rightarrow \Delta\,|\,\Gamma, \mathfrak{p} \Rightarrow \Delta\,|\,B \Rightarrow \mathfrak{p}, A}{\mathcal{G}\,|\,\Gamma, A \to B \Rightarrow \Delta}\ ({\to}{\Rightarrow})^1,$$

$$\frac{\mathcal{G}\,|\,\Gamma \Rightarrow A \to B, \Delta\,|\,\Gamma \Rightarrow \Delta;\quad \mathcal{G}\,|\,\Gamma \Rightarrow A \to B, \Delta\,|\,\Gamma, A \Rightarrow B, \Delta}{\mathcal{G}\,|\,\Gamma \Rightarrow A \to B, \Delta}\ (\Rightarrow\to)^1,$$

$$\frac{\mathcal{G}\,|\,\Gamma, \forall x A \Rightarrow \Delta\,|\,\Gamma, [A]^x_t \Rightarrow \Delta}{\mathcal{G}\,|\,\Gamma, \forall x A \Rightarrow \Delta}\ (\forall\Rightarrow)^1,\qquad \frac{\mathcal{G}\,|\,\Gamma \Rightarrow \forall x A, \Delta\,|\,\Gamma \Rightarrow [A]^x_a, \Delta}{\mathcal{G}\,|\,\Gamma \Rightarrow \forall x A, \Delta}\ (\Rightarrow\forall)^1,$$

$$\frac{\mathcal{G}\,|\,\Gamma \Rightarrow \exists x A, \Delta\,|\,\Gamma \Rightarrow [A]^x_t, \Delta}{\mathcal{G}\,|\,\Gamma \Rightarrow \exists x A, \Delta}\ (\Rightarrow\exists)^1,\qquad \frac{\mathcal{G}\,|\,\Gamma, \exists x A \Rightarrow \Delta\,|\,\Gamma, [A]^x_a \Rightarrow \Delta}{\mathcal{G}\,|\,\Gamma, \exists x A \Rightarrow \Delta}\ (\exists\Rightarrow)^1,$$

where $\mathcal{G}$ is a hypersequent; Γ, Δ are finite multisets of formulas; A, B are RPL∀-formulas; $\mathfrak{p}$ is a semipropositional variable not occurring in the conclusion of $(\to\Rightarrow)^1$; t is a closed term; a is a parameter not occurring in the conslusion of $(\Rightarrow\forall)^1$ or $(\exists\Rightarrow)^1$.

The calculus G^2Ł∀ is defined as G^1Ł∀, except that the inference rules of G^2Ł∀ are:

$$\frac{\mathcal{G}\,|\,\Gamma, \mathfrak{p} \Rightarrow \Delta\,|\,B \Rightarrow \mathfrak{p}, A}{\mathcal{G}\,|\,\Gamma, A \to B \Rightarrow \Delta}\ (\to\Rightarrow)^2,\qquad \frac{\mathcal{G}\,|\,\Gamma \Rightarrow \Delta;\quad \mathcal{G}\,|\,\Gamma, A \Rightarrow B, \Delta}{\mathcal{G}\,|\,\Gamma \Rightarrow A \to B, \Delta}\ (\Rightarrow\to)^2,$$

$$\frac{\mathcal{G}\,|\,\Gamma, \forall x A \Rightarrow \Delta\,|\,\Gamma, [A]^x_t \Rightarrow \Delta}{\mathcal{G}\,|\,\Gamma, \forall x A \Rightarrow \Delta}\ (\forall\Rightarrow)^2,\qquad \frac{\mathcal{G}\,|\,\Gamma \Rightarrow [A]^x_a, \Delta}{\mathcal{G}\,|\,\Gamma \Rightarrow \forall x A, \Delta}\ (\Rightarrow\forall)^2,$$

$$\frac{\mathcal{G}\,|\,\Gamma \Rightarrow \exists x A, \Delta\,|\,\Gamma \Rightarrow [A]^x_t, \Delta}{\mathcal{G}\,|\,\Gamma \Rightarrow \exists x A, \Delta}\ (\Rightarrow\exists)^2,\qquad \frac{\mathcal{G}\,|\,\Gamma, [A]^x_a \Rightarrow \Delta}{\mathcal{G}\,|\,\Gamma, \exists x A \Rightarrow \Delta}\ (\exists\Rightarrow)^2,$$

where $\mathcal{G}$, Γ, Δ, A, B, $\mathfrak{p}$, t, and a obey the same conditions as in the formulation of the inference rules of G^1Ł∀.

The provability of an object $\mathcal{H}$ in a calculus $\mathfrak{C}$ is denoted by $\vdash_{\mathfrak{C}} \mathcal{H}$. By a proof in a calculus, we understand a proof tree. In depicting a proof tree D, if we place a designation over a node N of D and do not separate the designation from N by a horizontal line, then we regard the designation as one for the proof tree whose root is N and that is a subtree of D. A *proof search tree* is defined as a proof tree, except that its leaves are not necessarily axioms of the calculus under consideration.

Let $\mathcal{R}$ be a rule of a calculus; then by a *backward application* of $\mathcal{R}$ (to an object $\mathcal{H}$) as well as by an *application* of $\mathcal{R}$ (to objects $\mathcal{H}_1, \ldots, \mathcal{H}_k$), we mean the figure $\dfrac{\mathcal{H}_1;\ \ldots;\ \mathcal{H}_k}{\mathcal{H}}$, where $\langle \mathcal{H}_1, \ldots, \mathcal{H}_k, \mathcal{H}\rangle \in \mathcal{R}$.

In a hypersequent calculus mentioned in this paper, a *proof of* (*for*) *an* RPL∀-*formula* A is a proof of the hypersequent $\Rightarrow A$.

3. The repetition-free calculus G^3Ł∀ and its soundness

Basically, we obtain G^3Ł∀ from the calculus G^2Ł∀ by replacing its rules $(\forall\Rightarrow)^2$ and $(\Rightarrow\exists)^2$ with repetition-free ones. This passage to the repetition-free calculus is advantageous to proof search (as we know, e.g., from Section 1) and will allow us to establish for G^3Ł∀ proof-theoretic properties that are not established for G^2Ł∀.

Semipropositional variables defined in Section 2 are now called *semipropositional variables of type* 1 and are denoted by $\mathfrak{p}, \mathfrak{p}_0, \mathfrak{p}_1, \ldots$. In addition to them, we introduce a countably infinite set of new words called *semipropositional variables of type* 0 and denoted by $\mathfrak{q}, \mathfrak{q}_0, \mathfrak{q}_1, \ldots$.

The way of obtaining the new repetition-free rules and the role of semipropositional variables used in them will be revealed in the proofs of Lemmas 1 and 2 as well as in Remark 2 below.

We define an *hs-interpretation* as an interpretation $\langle \mathcal{D}, \mu \rangle$ in which the map μ additionally takes each semipropositional variable of type 0 to a real number from $[0, +\infty)$ and each semipropositional variable of type 1 to a real number from $(-\infty, 1]$.

Taking into account that by semipropositional variables we now mean semipropositional variables of both types, the following definitions and abbreviations from Section 2 preserve their forms: the definitions of an *atom*, a *formula*, a *sequent*, a *hypersequent*, a *member* of a sequent, and an *atomic* sequent; the abbreviations $|\Gamma|_{M,\nu}$ and $|\Gamma \Rightarrow \Delta|_{M,\nu}$ (for a finite multiset Γ of formulas, a sequent $\Gamma \Rightarrow \Delta$, an hs-interpretation M, and a valuation ν); and the definition of a *valid* hypersequent (with the abbreviation $\vDash \mathcal{H}$ for such a hypersequent $\mathcal{H}$).

In the sequel, let the letters A, B, and C denote any RPL∀-formulas; F any formula; Γ, Δ, Π, and Σ any finite multisets of formulas; S any sequent; $\mathcal{G}$ and $\mathcal{H}$ any hypersequents; x any (individual) variable; t any closed term; a any parameter; all these letters may have subscripts.

The inference rules of the calculus G^3Ł∀ are:

$$\frac{\mathcal{G} \,|\, \Gamma, \mathfrak{p} \Rightarrow \Delta \,|\, B \Rightarrow \mathfrak{p}, A}{\mathcal{G} \,|\, \Gamma, A \to B \Rightarrow \Delta}\ (\to\Rightarrow)^3, \qquad \frac{\mathcal{G} \,|\, \Gamma \Rightarrow \Delta; \quad \mathcal{G} \,|\, \Gamma, A \Rightarrow B, \Delta}{\mathcal{G} \,|\, \Gamma \Rightarrow A \to B, \Delta}\ (\Rightarrow\to)^3,$$

$$\frac{\mathcal{G} \,|\, \Gamma, \mathfrak{p} \Rightarrow \Delta \,|\, \forall x A \Rightarrow \mathfrak{p} \,|\, [A]^x_t \Rightarrow \mathfrak{p}}{\mathcal{G} \,|\, \Gamma, \forall x A \Rightarrow \Delta}\ (\forall\Rightarrow)^3, \qquad \frac{\mathcal{G} \,|\, \Gamma \Rightarrow [A]^x_a, \Delta}{\mathcal{G} \,|\, \Gamma \Rightarrow \forall x A, \Delta}\ (\Rightarrow\forall)^3,$$

$$\frac{\mathcal{G} \,|\, \Gamma \Rightarrow \mathfrak{q}, \Delta \,|\, \mathfrak{q} \Rightarrow \exists x A \,|\, \mathfrak{q} \Rightarrow [A]^x_t}{\mathcal{G} \,|\, \Gamma \Rightarrow \exists x A, \Delta}\ (\Rightarrow\exists)^3, \qquad \frac{\mathcal{G} \,|\, \Gamma, [A]^x_a \Rightarrow \Delta}{\mathcal{G} \,|\, \Gamma, \exists x A \Rightarrow \Delta}\ (\exists\Rightarrow)^3,$$

where $\mathfrak{p}$ (resp. $\mathfrak{q}$) does not occur in the conclusion of $(\to\Rightarrow)^3$ or $(\forall\Rightarrow)^3$ (resp. $(\Rightarrow\exists)^3$) and is called the *proper semipropositional variable* of an application of the corresponding rule; t is called the *proper term* of an application of $(\forall\Rightarrow)^3$ or $(\Rightarrow\exists)^3$; a does not occur in the conclusion of $(\Rightarrow\forall)^3$ or $(\exists\Rightarrow)^3$ and is called the *proper parameter* of an application of the corresponding rule.

An *axiom* of the calculus G^3Ł∀ is an arbitrary hypersequent such that the hypersequent consisting of all its atomic sequents is valid. Note that axioms of G^3Ł∀ can be recognized in much the same way as described in [7, Section 4.2], by a polynomial time algorithm. Briefly, recognizing whether a hypersequent is an axiom of G^3Ł∀ reduces to checking the inconsistency of a system of strict and nonstrict linear inequalities with rational coefficients and rational-valued variables; the latter can be done in polynomial time by [14, Theorem 2].

Remark 1. Although proper semipropositional variables distinguished in the premises of the rules $(\to\Rightarrow)^3$, $(\forall\Rightarrow)^3$, and $(\Rightarrow\exists)^3$ do not occur in their conclusions, we need not guess such semipropositional variables in applying these rules backward, because any new semipropositional variable (of the respective type) will do. In this respect, proper semipropositional variables are, of course, similar to proper parameters of the rules $(\Rightarrow\forall)^3$ and $(\exists\Rightarrow)^3$. This, as well as the forms of all the rules of G^3Ł∀, accounts for regarding G^3Ł∀ as an analytic calculus.

Let us make it precise what a sequent or formula occurrence in a hypersequent means. Toward this end, every multiset of formulas, sequent, and hypersequent $\mathcal{E}$ is represented by a word $\omega(\mathcal{E})$ as follows. Fix a linear order on the alphabet consisting of the symbols "⇒" and "," and all the symbols used in the formulas of the signature under consideration. This order induces the lexicographic order on the set of all words over this alphabet. A multiset Γ consisting of formulas $A_1, \ldots, A_n$ (where some of the A_i's may be repeated) is represented by a word $\omega(\Gamma)$ that is comprised of all the A_i's listed in the lexicographic order and separated by ",". A hypersequent (in partucular, a sequent) $\mathcal{H} = (\Gamma_1 \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Delta_n)$ is represented by a word $\omega(\mathcal{H})$ that consists of all the words $\omega(\Gamma_i) \Rightarrow \omega(\Delta_i)$ ordered lexicographically and separated by "|". Now by an *occurrence of a sequent S* (resp. *formula F*) *in a hypersequent $\mathcal{H}$*, we mean an occurrence of the word $\omega(S)$ (resp. F) in the word $\omega(\mathcal{H})$.

For an application (or a backward application) of an inference rule of the calculus $\mathrm{G^3Ł}\forall$, the *principal formula occurrence* and the *principal sequent occurrence* are defined in much the same manner as in [15, § 49] and [16, items 3.1.1 and 3.5.1]. By a backward application of an inference rule of $\mathrm{G^3Ł}\forall$ to a formula (resp. sequent) occurrence $\mathcal{O}$ in a hypersequent $\mathcal{H}$, we mean a backward application of the rule to $\mathcal{H}$, where $\mathcal{O}$ is the principal formula (resp. sequent) occurrence.

For a $\mathrm{G^3Ł}\forall$-proof search tree D, a sequent occurrence $\mathcal{S}$ in D, and a formula occurrence $\mathcal{F}$ (as a sequent member) in D, an *ancestor* of $\mathcal{S}$ and an *ancestor* of $\mathcal{F}$ are defined much as in [15, § 49]. In more detail, suppose there is an application or a backward application

$$\frac{\mathcal{G} \,|\, \Gamma, \mathfrak{p} \Rightarrow \Delta \,|\, \forall x A \Rightarrow \mathfrak{p} \,|\, [A]^x_t \Rightarrow \mathfrak{p}}{\mathcal{G} \,|\, \Gamma, \forall x A \Rightarrow \Delta}$$

of the rule $(\forall \Rightarrow)^3$ in D (the other rules of $\mathrm{G^3Ł}\forall$ are considered analogously). Then in an obvious way, the three distinguished sequent occurrences in the premise originate from the principal sequent occurrence, i.e., the distinguished occurrence of $\Gamma, \forall x A \Rightarrow \Delta$ in the conclusion. Likewise, the distinguished occurrences of the formulas $\mathfrak{p}$, $\forall x A$, and $[A]^x_t$ in the premise originate from the principal formula occurrence, i.e., the distinguished occurrence of $\forall x A$ in the conclusion. Any other occurrence of a sequent (resp. formula) in the premise naturally originates from an occurrence of the same sequent (resp. formula) in the conclusion. To avoid ambiguity, we assume that the proof search tree D is supplied with an additional information that, for each rule application in D, indicates (1) its principal sequent and formula occurrences, (2) the occurrences (in the premises) that originate from them, and (3) which remaining occurrences in the premises originate from which ones in the conclusion. Now an *ancestor* of $\mathcal{S}$ is defined inductively to be either $\mathcal{S}$ or an ancestor of a sequent occurrence that originates from $\mathcal{S}$; an *ancestor* of $\mathcal{F}$ is defined very similarly.

A rule $\mathcal{R}$ whose premises and conclusion are hypersequents is called *sound* if the validity of all the premises of $\mathcal{R}$ implies the validity of the conclusion of $\mathcal{R}$; and $\mathcal{R}$ is called *semantically invertible* if the converse holds.

Lemma 1. *Each inference rules of the calculus* $\mathrm{G^3Ł}\forall$ *is sound and semantically invertible.*

Proof. From assertions (1)–(4) of Lemma 2 stated below, it follows that the rules $(\to\Rightarrow)^3$, $(\Rightarrow\to)^3$, $(\Rightarrow \forall)^3$, and $(\exists \Rightarrow)^3$ are sound and semantically invertible.

Any application of the rule $(\forall \Rightarrow)^3$ can be represented as two applications of the rules

$$\frac{\mathcal{G}_0 \,|\, \Gamma_0, \forall x A \Rightarrow \Delta_0 \,|\, \Gamma_0, [A]^x_t \Rightarrow \Delta_0}{\mathcal{G}_0 \,|\, \Gamma_0, \forall x A \Rightarrow \Delta_0} \ (\forall \Rightarrow)^2_0 \quad \text{and} \quad \frac{\mathcal{G} \,|\, \Gamma, \mathfrak{p} \Rightarrow \Delta \,|\, B \Rightarrow \mathfrak{p}}{\mathcal{G} \,|\, \Gamma, B \Rightarrow \Delta} \ (\mathrm{den}_1),$$

where $\mathfrak{p}$ does not occur in the conclusion of the last rule, as follows:

$$\cfrac{\cfrac{\mathcal{G} \mid \Gamma, \mathfrak{p} \Rightarrow \Delta \mid \forall x A \Rightarrow \mathfrak{p} \mid [A]^x_t \Rightarrow \mathfrak{p}}{\mathcal{G} \mid \Gamma, \mathfrak{p} \Rightarrow \Delta \mid \forall x A \Rightarrow \mathfrak{p}}\,(\forall \Rightarrow)^2_0}{\mathcal{G} \mid \Gamma, \forall x A \Rightarrow \Delta}\,(\mathrm{den}_1).$$

By assertion (5) of Lemma 2, the rule $(\forall \Rightarrow)^2_0$ is sound; and it is semantically invertible, since its premise includes its conclusion. By assertion $(\widetilde{5})$ of Lemma 2, the rule (den_1) is sound and semantically invertible. So $(\forall \Rightarrow)^3$ is sound and semantically invertible.

Any application of the rule $(\Rightarrow \exists)^3$ can be represented as two applications of the rules

$$\frac{\mathcal{G}_0 \mid \Gamma_0 \Rightarrow \exists x A, \Delta_0 \mid \Gamma_0 \Rightarrow [A]^x_t, \Delta_0}{\mathcal{G}_0 \mid \Gamma_0 \Rightarrow \exists x A, \Delta_0}\,(\Rightarrow \exists)^2_0 \quad \text{and} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \mathfrak{q}, \Delta \mid \mathfrak{q} \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow B, \Delta}\,(\mathrm{den}_0),$$

where $\mathfrak{q}$ does not occur in the conclusion of the last rule, thus:

$$\cfrac{\cfrac{\mathcal{G} \mid \Gamma \Rightarrow \mathfrak{q}, \Delta \mid \mathfrak{q} \Rightarrow \exists x A \mid \mathfrak{q} \Rightarrow [A]^x_t}{\mathcal{G} \mid \Gamma \Rightarrow \mathfrak{q}, \Delta \mid \mathfrak{q} \Rightarrow \exists x A}\,(\Rightarrow \exists)^2_0}{\mathcal{G} \mid \Gamma \Rightarrow \exists x A, \Delta}\,(\mathrm{den}_0).$$

Then from assertions (6) and $(\widetilde{6})$ of Lemma 2, it follows that $(\Rightarrow \exists)^3$ is sound and semantically invertible. □

Remark 2. We obtained the repetition-free rules $(\forall \Rightarrow)^3$ and $(\Rightarrow \exists)^3$ as shown in the above proof of Lemma 1. The rule $(\forall \Rightarrow)^2_0$ (resp. $(\Rightarrow \exists)^2_0$) differs from the rule $(\forall \Rightarrow)^2$ (resp. $(\Rightarrow \exists)^2$) of the calculus $\mathrm{G^2Ł\forall}$ only in that semipropositional variables of type 0 may occur in the premise and conclusion of $(\forall \Rightarrow)^2_0$ (resp. $(\Rightarrow \exists)^2_0$). The rules (den_1) and (den_0) are our nonstandard variants of the density rule.

The density rule (see [5, Section 4.5]) is:

$$\frac{\mathcal{G} \mid \Gamma, p \Rightarrow \Delta \mid \Pi \Rightarrow p, \Sigma}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta, \Sigma}\,(\mathrm{den}),$$

where p is a propositional variable not occurring in the conclusion. It is not hard to verify that (den) is unsound; and if $\mathfrak{p}$ (resp. $\mathfrak{q}$) were a propositional variable, then (den_1) (resp. (den_0)) would be unsound (in fact, $\mathfrak{p}$ and $\mathfrak{q}$ are semipropositional variables).

An idea of introducing new propositional variables (like in the density rule) is used in [5, Section 7.1.2] to avoid repetition of formulas in the premises of inference rules for some propositional calculi. However, those rules include (besides nonstrict inequalities) strict inequalities, which do not preserve in passing to inf or sup. Recall that the quantifiers $\forall$ and $\exists$ are interpreted as inf and sup. So such propositional calculi do not extend to first-order ones. We use a somewhat similar but rather different idea involving semipropositional variables and (as seen from the definition of a valid hypersequent) only nonstrict inequalities. (*End of Remark 2.*)

For an hs-interpretation M, a semipropositional variable $\mathfrak{r}$ of type 0 (resp. type 1), and a real number $r \in [0, +\infty)$ (resp. $r \in (-\infty, 1]$), by $M[\mathfrak{r} \mapsto r]$ we denote the hs-interpretation that interprets $\mathfrak{r}$ by r and does not differ from M in any other respect.

Lemma 2. *Let* Γ *and* Δ *be finite multisets of formulas;* A *and* B *be* RPL∀*-formulas;* y *be a variable not occurring in* Γ, Δ, A*;* $\mathfrak{p}$ *and* $\mathfrak{q}$ *be semipropositional variables* (*of type 1 and type 0, respectively*) *not occurring in* Γ, Δ, A, B*;* M *be an hs-interpretaton with a domain* $\mathcal{D}$*; and* ν *be a valuation. Then:*

(1) $|\Gamma, A \to B \Rightarrow \Delta|_{M,\nu} \geqslant 0$ *iff, for every* $r \in (-\infty, 1]$*, at least one of the inequalities* $|\Gamma, \mathfrak{p} \Rightarrow \Delta|_{M[\mathfrak{p}\mapsto r],\,\nu} \geqslant 0$ *or* $|B \Rightarrow \mathfrak{p}, A|_{M[\mathfrak{p}\mapsto r],\,\nu} \geqslant 0$ *holds;*

(2) $|\Gamma \Rightarrow A \to B, \Delta|_{M,\nu} \geqslant 0$ *iff* $|\Gamma \Rightarrow \Delta|_{M,\nu} \geqslant 0$ *and* $|\Gamma, A \Rightarrow B, \Delta|_{M,\nu} \geqslant 0$*;*

(3) $|\Gamma \Rightarrow \forall x A, \Delta|_{M,\nu} \geqslant 0$ *iff* $|\Gamma \Rightarrow [A]^x_y, \Delta|_{M,\nu[y\mapsto d]} \geqslant 0$ *for every* $d \in \mathcal{D}$*;*

(4) $|\Gamma, \exists x A \Rightarrow \Delta|_{M,\nu} \geqslant 0$ *iff* $|\Gamma, [A]^x_y \Rightarrow \Delta|_{M,\nu[y\mapsto d]} \geqslant 0$ *for every* $d \in \mathcal{D}$*;*

(5) $|\Gamma, \forall x A \Rightarrow \Delta|_{M,\nu} \geqslant 0$ *if* $|\Gamma, [A]^x_y \Rightarrow \Delta|_{M,\nu[y\mapsto d]} \geqslant 0$ *for some* $d \in \mathcal{D}$*;*

(6) $|\Gamma \Rightarrow \exists x A, \Delta|_{M,\nu} \geqslant 0$ *if* $|\Gamma \Rightarrow [A]^x_y, \Delta|_{M,\nu[y\mapsto d]} \geqslant 0$ *for some* $d \in \mathcal{D}$*;*

$(\widetilde{5})$ $|\Gamma, B \Rightarrow \Delta|_{M,\nu} \geqslant 0$ *iff, for every* $r \in (-\infty, 1]$*, at least one of the inequalities* $|\Gamma, \mathfrak{p} \Rightarrow \Delta|_{M[\mathfrak{p}\mapsto r],\,\nu} \geqslant 0$ *or* $|B \Rightarrow \mathfrak{p}|_{M[\mathfrak{p}\mapsto r],\,\nu} \geqslant 0$ *holds;*

$(\widetilde{6})$ $|\Gamma \Rightarrow B, \Delta|_{M,\nu} \geqslant 0$ *iff, for every* $r \in [0, +\infty)$*, at least one of the inequalities* $|\Gamma \Rightarrow \mathfrak{q}, \Delta|_{M[\mathfrak{q}\mapsto r],\,\nu} \geqslant 0$ *or* $|\mathfrak{q} \Rightarrow B|_{M[\mathfrak{q}\mapsto r],\,\nu} \geqslant 0$ *holds.*

Proof. Assertions (1)–(6) stated above are proved similarly to assertions (1)–(6) in [7, Lemma 2].

Let us prove assertions $(\widetilde{5})$ and $(\widetilde{6})$. Denote $|\Gamma|_{M,\nu}$, $|\Delta|_{M,\nu}$, and $|B|_{M,\nu}$ by γ, δ, and b, respectively; and notice that $0 \leqslant b \leqslant 1$.

Assertion $(\widetilde{5})$ is equivalent to the following:

$$(\widetilde{5}')\ \delta - \gamma + 1 < b \iff (\widetilde{5}'')\ \delta - \gamma + 1 < r < b \text{ for some } r \leqslant 1.$$

It is clear that $(\widetilde{5}'')$ implies $(\widetilde{5}')$. If $(\widetilde{5}')$ holds, then by the density of the set of all real numbers, both inequalities from $(\widetilde{5}'')$ hold for some $r < b \leqslant 1$. Thus $(\widetilde{5})$ holds.

Assertion $(\widetilde{6})$ is equivalent to the following:

$$b < \gamma - \delta + 1 \iff b < r < \gamma - \delta + 1 \text{ for some } r \geqslant 0.$$

By the density of the set of all real numbers, the last equivalence holds and so does $(\widetilde{6})$. □

Theorem 1 (soundness of G^{3}Ł∀)**.** *If* $\vdash_{\mathrm{G^3Ł\forall}} \mathcal{H}$*, then* $\vDash \mathcal{H}$*.*

Proof. All axioms of G^{3}Ł∀ are obviously valid, and all the inference rules of G^{3}Ł∀ are sound by Lemma 1. □

Using the semantical invertibility of the propositional rules of G^{3}Ł∀ (see Lemma 1), we can easily prove

Proposition 1. *Let* $\mathcal{H}$ *be a quantifier-free hypersequent. If* $\vDash \mathcal{H}$*, then* $\vdash_{\mathrm{G^3Ł\forall}} \mathcal{H}$*.*

Note that Proposition 1 with an arbitrary hypersequent (or even an arbitrary Ł∀-formula) instead of a quantifier-free hypersequent $\mathcal{H}$ cannot hold; for otherwise, the set of all valid Ł∀-formulas would be recursively enumerable, which is not the case as mentioned in Section 1.

4. Invertibility of the rules of the calculus G^{3}Ł∀ and its relationship to the calculi GŁ∀, G^{1}Ł∀, and G^{2}Ł∀

Suppose $\mathfrak{C}$ is a calculus. By $h(D)$ we denote the height of a (tree-like) $\mathfrak{C}$-proof D. Let us recall some definitions (cf., e.g., [16, Section 3.4]).

A rule is called *admissible* for $\mathfrak{C}$ if, for its every application $\dfrac{\mathcal{H}_1;\ \ldots;\ \mathcal{H}_k}{\mathcal{H}}$ and every $\mathfrak{C}$-proof D_1 of $\mathcal{H}_1, \ldots,$ $\mathfrak{C}$-proof D_k of $\mathcal{H}_k$, there exists a $\mathfrak{C}$-proof D of $\mathcal{H}$. This rule is called *hp-admissible*, or *height-preserving admissible*, for $\mathfrak{C}$ if, in addition, the condition $h(D) \leqslant \max\{h(D_1), \ldots, h(D_k)\}$ holds. Everywhere in the sequel, the existense of such a proof D means that it can be constructed if such proofs $D_1, \ldots,$ D_k are given.

A k-premise rule $\mathcal{R}$ is called *invertible* (resp. *hp-invertible*, or *height-preserving invertible*) in $\mathfrak{C}$ if, for each $i = 1, \ldots, k,$ the rule $\{\langle \mathcal{H}, \mathcal{H}_i \rangle \mid \langle \mathcal{H}_1, \ldots, \mathcal{H}_k, \mathcal{H} \rangle \in \mathcal{R}\}$ is admissible (resp. hp-admissible) for $\mathfrak{C}$.

Lemma 3. *The following rules are hp-admissible for the calculus* $\mathrm{G}^3\text{Ł}\forall$*:*

$$\frac{\mathcal{G}}{\mathcal{G} \mid S}\ (\mathrm{ew})^3, \quad \frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (\mathrm{split})^3, \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, P \Rightarrow P, \Delta}\ (\mathrm{at}{\Rightarrow}\mathrm{at})^3,$$

where P is an atom.

Proof. 1. The rule $(\mathrm{ew})^3$ is hp-admissible. Indeed, given a proof D of $\mathcal{G}$ and a sequent S, we first obtain a proof D' by renaming in D all proper semipropositional variables occurring in S and all proper parameters occurring in S to new distinct ones. Then we get the required proof of $\mathcal{G} \mid S$ by appending "$\mid S$" to each node hypersequent of D'.

2. Let us establish the hp-admissibility of the rule $(\mathrm{split})^3$.

Suppose $\mathcal{S}_0$ is a sequent occurrence in the root of a proof search tree $\mathcal{D}_0$; then we say that an ancestor $\mathcal{S}$ of the occurrence $\mathcal{S}_0$ is *augmentable* unless $\mathcal{S}$ is an ancestor of an occurrence $\mathcal{S}'$ of a sequent S' such that:

(i) S' has the form (a) $B \Rightarrow \mathfrak{p}, A$, (b) $\forall x A \Rightarrow \mathfrak{p}$ or $[A]^x_t \Rightarrow \mathfrak{p}$, or (c) $\mathfrak{q} \Rightarrow \exists x A$ or $\mathfrak{q} \Rightarrow [A]^x_t$;

(ii) in $\mathcal{D}_0$, $\mathcal{S}'$ is a sequent occurrence in the premise of an application of the rule (a) $({\to}{\Rightarrow})^3$, (b) $(\forall\Rightarrow)^3$, or (c) $(\Rightarrow\exists)^3$, respectively; and

(iii) $\mathcal{S}'$ is distinguished in the formulation of this rule.

Let D_0 be a proof tree for the premise $\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ of the rule $(\mathrm{split})^3$. A tree D is constructed from D_0 as follows: each occurrence $\mathcal{S}$ of a sequent S of the form $\Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2$ such that

(α) $\mathcal{S}$ is an augmentable ancestor of the distinguished occurrence of the sequent $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ in the root of D_0, and

(β) for each $i = 1, 2$ and each formula occurrence $\mathcal{F}$ (as a sequent member) in S, if $\mathcal{F}$ is contained in the distinguished occurrence Π_i or Σ_i in S, then $\mathcal{F}$ is an ancestor of some formula occurrence contained in the distinguished occurrence Γ_i or Δ_i in the root of D_0,

is replaced by $\Pi_1 \Rightarrow \Sigma_1 \mid \Pi_2 \Rightarrow \Sigma_2$.

The rules of $\mathrm{G}^3\text{Ł}\forall$ guarantee that, in a premise of a rule application, there is exactly one augmentable ancestor of the principal sequent occurrence. Therefore, when the tree D is constructed, exactly one sequent occurrence in each node hypersequent of the proof D_0 is split into two sequents. Then it is easy to see that each application of a rule in D_0 is turned into an application of the same rule. Clearly, the hypersequent $\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2$ is in the root of the tree D. Hence D is a proof search tree for the conclusion of the rule $(\mathrm{split})^3$.

Let $\mathcal{L}_0$ be a leaf of the proof tree D_0. Let $\mathcal{S}$ be an occurrence of an atomic sequent S in $\mathcal{L}_0$ such that S has the form $\Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2$, and $\mathcal{S}$ with S meet conditions (α) and (β) above. Then the leaf $\mathcal{L}$ of D obtained from $\mathcal{L}_0$ contains the atomic sequents $\Pi_1 \Rightarrow \Sigma_1$ and $\Pi_2 \Rightarrow \Sigma_2$. Since $\mathcal{L}_0$ is an axiom, it follows easily that $\mathcal{L}$ is an axiom. Consequently, D is a proof.

It remains to note that $h(D) = h(D_0)$.

3. The rule $(\mathrm{at}{\Rightarrow}\mathrm{at})^3$ is hp-admissible, since, given a proof D for $\mathcal{G} \,|\, \Gamma \Rightarrow \Delta$ and an atom P, we can construct a proof for $\mathcal{G} \,|\, \Gamma, P \Rightarrow P, \Delta$ thus. First, in D, rename all proper parameters and proper semipropositional variables that occur in P to new distinct ones. Next, in the resulting proof for $\mathcal{G} \,|\, \Gamma \Rightarrow \Delta$, add the atom P to the antecedent and succedent of each augmentable ancestor of the distinguished occurrence of the sequent $\Gamma \Rightarrow \Delta$ in the root. □

Lemma 4. *All the inference rules of the calculus* $\mathrm{G^3Ł}\forall$ *are hp-invertible in it.*

Proof. The rule $(\forall \Rightarrow)^3$ is hp-invertible, since we can obtain its premise from its conclusion using rules, which are hp-admissible (by Lemma 3):

$$\dfrac{\dfrac{\dfrac{\mathcal{G} \,|\, \Gamma, \forall x A \Rightarrow \Delta}{\mathcal{G} \,|\, \Gamma, \forall x A, \mathfrak{p} \Rightarrow \mathfrak{p}, \Delta}\,(\mathrm{at}{\Rightarrow}\mathrm{at})^3}{\mathcal{G} \,|\, \Gamma, \mathfrak{p} \Rightarrow \Delta \,|\, \forall x A \Rightarrow \mathfrak{p}}\,(\mathrm{split})^3}{\mathcal{G} \,|\, \Gamma, \mathfrak{p} \Rightarrow \Delta \,|\, \forall x A \Rightarrow \mathfrak{p} \,|\, [A]^x_t \Rightarrow \mathfrak{p}}\,(\mathrm{ew})^3.$$

The hp-invertibility of the rule $(\Rightarrow \exists)^3$ is established very similarly.

The fact that all the inference rules of $\mathrm{G^3Ł}\forall$ are repetition-free makes it possible for us to demonstrate the hp-invertibility of the rules $({\to}{\Rightarrow})^3$, $({\Rightarrow}{\to})^3$, $(\Rightarrow \forall)^3$, and $(\exists \Rightarrow)^3$ according to the scheme used to show the hp-invertibility of the rules of some (invertible variants of) classical sequent calculi; cf., e.g., [16, item 3.5.4]. (Note that this possibility was not obvious before these demonstrations were completed; besides, it provides a clue to a useful generalization, as we will mention in Section 8.) We give these demonstrations in full because later[1] we will need to check that formal proofs constructed in them enjoy some properties.

I. Let us demonstrate that the rule $({\to}{\Rightarrow})^3$ is hp-invertible. Toward this end, we show that, given a proof D for a hypersequent of the form $\mathcal{G} \,|\, \Gamma, A \to B \Rightarrow \Delta$ and a semipropositional variable $\mathfrak{p}$ not occurring in the hypersequent, we can construct a proof D' for $\mathcal{G} \,|\, \Gamma, \mathfrak{p} \Rightarrow \Delta \,|\, B \Rightarrow \mathfrak{p}, A$ with $h(D') \leqslant h(D)$. We proceed by induction on $h(D)$.

We can assume that $\mathfrak{p}$ does not occur in D (otherwise replace all occurrences of $\mathfrak{p}$ in D by a semipropositional variable of type 1 not occurring in D).

1. If $h(D) = 0$ (i.e., D consists of a single axiom), then $\mathcal{G}$ is an axiom, hence so is $\mathcal{G} \,|\, \Gamma, \mathfrak{p} \Rightarrow \Delta \,|\, B \Rightarrow \mathfrak{p}, A$.

2. Let the root hypersequent $\mathcal{G} \,|\, \Gamma, A \to B \Rightarrow \Delta$ in D be the conclusion of an application R of a rule $\mathcal{R}$.

2.1. Suppose the principal formula occurrence in R is the distinguished occurrence of $A \to B$. By D_1 denote the subtree of the root of D; D_1 is a proof for the premise of R. The premise has the form $\mathcal{G} \,|\, \Gamma, \mathfrak{p}_1 \Rightarrow \Delta \,|\, B \Rightarrow \mathfrak{p}_1, A$. Then replacing all occurrences of $\mathfrak{p}_1$ in D_1 by $\mathfrak{p}$ yields a proof D' for $\mathcal{G} \,|\, \Gamma, \mathfrak{p} \Rightarrow \Delta \,|\, B \Rightarrow \mathfrak{p}, A$ with $h(D') < h(D)$.

2.2. Now suppose the principal formula occurrence in R is not the distinguished occurrence of $A \to B$.

[1] See the proofs of Lemma 7 and Theorem 4.

2.2.1. If $\mathcal{R}$ is a one-premise rule, the proof D looks like this:

$$\dfrac{\begin{array}{c} D_1 \\ \mathcal{G}_1 \,|\, \Gamma_1, A \to B \Rightarrow \Delta_1 \end{array}}{\mathcal{G} \,|\, \Gamma, A \to B \Rightarrow \Delta}\ \mathcal{R}.$$

By applying the induction hypothesis to the proof D_1, we construct a proof D_1' for $\mathcal{G}_1 \,|\, \Gamma_1, \mathfrak{p} \Rightarrow \Delta_1 \,|\, B \Rightarrow \mathfrak{p}, A$ with $h(D_1') \leqslant h(D_1)$. By applying $\mathcal{R}$ to the root hypersequent of the proof D_1', we obtain a proof D' for $\mathcal{G} \,|\, \Gamma, \mathfrak{p} \Rightarrow \Delta \,|\, B \Rightarrow \mathfrak{p}, A$ such that $h(D') \leqslant h(D)$.

2.2.2. If $\mathcal{R}$ is a two-premise rule, i.e., the rule $(\Rightarrow\to)^3$, then the proof D looks like this:

$$\dfrac{\begin{array}{c} D_1 \\ \mathcal{G}_1 \,|\, \Gamma_1, A \to B \Rightarrow \Delta_1 \end{array} \qquad \begin{array}{c} D_2 \\ \mathcal{G}_2 \,|\, \Gamma_2, A \to B \Rightarrow \Delta_2 \end{array}}{\mathcal{G} \,|\, \Gamma, A \to B \Rightarrow \Delta}\ \mathcal{R}.$$

For each $i = 1, 2$, by the induction hypothesis applied to the proof D_i, we construct a proof D_i' for $\mathcal{G}_i \,|\, \Gamma_i, \mathfrak{p} \Rightarrow \Delta_i \,|\, B \Rightarrow \mathfrak{p}, A$ with $h(D_i') \leqslant h(D_i)$.

By applying $\mathcal{R}$ to the root hypersequents of the proofs D_1' and D_2', we get a proof D' for $\mathcal{G} \,|\, \Gamma, \mathfrak{p} \Rightarrow \Delta \,|\, B \Rightarrow \mathfrak{p}, A$ with $h(D') \leqslant h(D)$.

II. In order to establish the hp-invertibility of the rule $(\Rightarrow\to)^3$, we show that, given a proof D for a hypersequent of the form $\mathcal{G} \,|\, \Gamma \Rightarrow A \to B, \Delta$, we can construct a proof D' for $\mathcal{G} \,|\, \Gamma \Rightarrow \Delta$ and a proof D'' for $\mathcal{G} \,|\, \Gamma, A \Rightarrow B, \Delta$ such that $h(D') \leqslant h(D)$ and $h(D'') \leqslant h(D)$. We use induction on $h(D)$.

1. If $h(D) = 0$, then $\mathcal{G}$ is an axiom, and so are $\mathcal{G} \,|\, \Gamma \Rightarrow \Delta$ and $\mathcal{G} \,|\, \Gamma, A \Rightarrow B, \Delta$.

2. Let the root hypersequent $\mathcal{G} \,|\, \Gamma \Rightarrow A \to B, \Delta$ in D be the conclusion of an application R of a rule $\mathcal{R}$.

2.1. If the principal formula occurrence in R is the distinguished occurrence of $A \to B$, then the subtrees of the root of D are the desired proofs.

2.2. Suppose the principal formula occurrence in R is not the distinguished occurrence of $A \to B$.

2.2.1. In the case the rule $\mathcal{R}$ is one-premise, the proof D looks like this:

$$\dfrac{\begin{array}{c} D_1 \\ \mathcal{G}_1 \,|\, \Gamma_1 \Rightarrow A \to B, \Delta_1 \end{array}}{\mathcal{G} \,|\, \Gamma \Rightarrow A \to B, \Delta}\ \mathcal{R}.$$

Using the induction hypothesis, from the proof D_1, we construct a proof D_1' for $\mathcal{G}_1 \,|\, \Gamma_1 \Rightarrow \Delta_1$ and a proof D_1'' for $\mathcal{G}_1 \,|\, \Gamma_1, A \Rightarrow B, \Delta_1$ such that $h(D_1') \leqslant h(D_1)$ and $h(D_1'') \leqslant h(D_1)$.

Applying $\mathcal{R}$ to the root hypersequent of the proof D_1' gives a proof D' for $\mathcal{G} \,|\, \Gamma \Rightarrow \Delta$ with $h(D') \leqslant h(D)$, and applying $\mathcal{R}$ to the root hypersequent of the proof D_1'' gives a proof D'' for $\mathcal{G} \,|\, \Gamma, A \Rightarrow B, \Delta$ with $h(D'') \leqslant h(D)$.

2.2.2. In the case the rule $\mathcal{R}$ is two-premise, the proof D looks like this:

$$\dfrac{\begin{array}{c} D_1 \\ \mathcal{G}_1 \,|\, \Gamma_1 \Rightarrow A \to B, \Delta_1 \end{array} \qquad \begin{array}{c} D_2 \\ \mathcal{G}_2 \,|\, \Gamma_2 \Rightarrow A \to B, \Delta_2 \end{array}}{\mathcal{G} \,|\, \Gamma \Rightarrow A \to B, \Delta}\ \mathcal{R}.$$

For each $i = 1, 2$, by the induction hypothesis applied to the proof D_i, we construct a proof D'_i for $\mathcal{G}_i \,|\, \Gamma_i \Rightarrow \Delta_i$ and a proof D''_i for $\mathcal{G}_i \,|\, \Gamma_i, A \Rightarrow B, \Delta_i$ such that $h(D'_i) \leqslant h(D_i)$ and $h(D''_i) \leqslant h(D_i)$.

Next, by applying $\mathcal{R}$ to the root hypersequents of the proofs D'_1 and D'_2, we obtain a proof D' for $\mathcal{G} \,|\, \Gamma \Rightarrow \Delta$ with $h(D') \leqslant h(D)$.

Finally, applying $\mathcal{R}$ to the root hypersequents of the proofs D''_1 and D''_2 yields a proof D'' for $\mathcal{G} \,|\, \Gamma, A \Rightarrow B, \Delta$ with $h(D'') \leqslant h(D)$.

III. To establish the hp-invertibility of the rule $(\Rightarrow \forall)^3$, we show that, given a proof D for a hypersequent of the form $\mathcal{G} \,|\, \Gamma \Rightarrow \forall x A, \Delta$ and a parameter a not occurring in the hypersequent, we can construct a proof D' for $\mathcal{G} \,|\, \Gamma \Rightarrow [A]^x_a, \Delta$ with $h(D') \leqslant h(D)$. This is done by induction on $h(D)$.

We can assume that a does not occur in D (otherwise replace all occurrences of a in D by a parameter not occurring in D).

1. If $h(D) = 0$, then $\mathcal{G}$ is an axiom and so is $\mathcal{G} \,|\, \Gamma \Rightarrow [A]^x_a, \Delta$.

2. Let the root hypersequent $\mathcal{G} \,|\, \Gamma \Rightarrow \forall x A, \Delta$ in D be the conclusion of an application R of a rule $\mathcal{R}$.

2.1. Suppose the principal formula occurrence in R is the distinguished occurrence of $\forall x A$. By D_1 denote the subtree of the root of D; D_1 is a proof for the premise of R. The premise has the form $\mathcal{G} \,|\, \Gamma \Rightarrow [A]^x_{a_1}, \Delta$. By replacing all occurrences of a_1 in D_1 by a, we get a proof D' for $\mathcal{G} \,|\, \Gamma \Rightarrow [A]^x_a, \Delta$ with $h(D') < h(D)$.

2.2. Next, suppose the principal formula occurrence in R is not the distinguished occurrence of $\forall x A$.

2.2.1. If $\mathcal{R}$ is one-premise, the proof D looks like this:

$$\dfrac{\begin{array}{c} D_1 \\ \mathcal{G}_1 \,|\, \Gamma_1 \Rightarrow \forall x A, \Delta_1 \end{array}}{\mathcal{G} \,|\, \Gamma \Rightarrow \forall x A, \Delta}\ \mathcal{R}.$$

Using the induction hypothesis, we transform the proof D_1 into a proof D'_1 for $\mathcal{G}_1 \,|\, \Gamma_1 \Rightarrow [A]^x_a, \Delta_1$ such that $h(D'_1) \leqslant h(D_1)$. By applying $\mathcal{R}$ to the root hypersequent of the proof D'_1, we have a proof D' for $\mathcal{G} \,|\, \Gamma \Rightarrow [A]^x_a, \Delta$ with $h(D') \leqslant h(D)$.

2.2.2. If $\mathcal{R}$ is two-premise, the proof D looks like this:

$$\dfrac{\begin{array}{c} D_1 \\ \mathcal{G}_1 \,|\, \Gamma_1 \Rightarrow \forall x A, \Delta_1 \end{array} \qquad \begin{array}{c} D_2 \\ \mathcal{G}_2 \,|\, \Gamma_2 \Rightarrow \forall x A, \Delta_2 \end{array}}{\mathcal{G} \,|\, \Gamma \Rightarrow \forall x A, \Delta}\ \mathcal{R}.$$

For each $i = 1, 2$, by the induction hypothesis, we transform the proof D_i into a proof D'_i for $\mathcal{G}_i \,|\, \Gamma_i \Rightarrow [A]^x_a, \Delta_i$ such that $h(D'_i) \leqslant h(D_i)$.

By applying $\mathcal{R}$ to the root hypersequents of the proofs D'_1 and D'_2, we obtain a proof D' for $\mathcal{G} \,|\, \Gamma \Rightarrow [A]^x_a, \Delta$ with $h(D') \leqslant h(D)$.

IV. The hp-invertibility of the rule $(\exists \Rightarrow)^3$ is established very similarly to the hp-invertibility of the rule $(\Rightarrow \forall)^3$, see item III. □

Remark 3. We know the following about whether the inference rules of the calculus $\mathrm{G^2Ł\forall}$ are invertible in it. The rules $(\forall \Rightarrow)^2$ and $(\Rightarrow \exists)^2$ are hp-invertible because, for each of them, its premise includes its conclusion. Using arguments like those given in the proof of Lemma 4, we can establish the hp-invertibility of the rules $(\Rightarrow\to)^2$, $(\Rightarrow \forall)^2$, and $(\exists \Rightarrow)^2$. However, we do not know whether the rule $(\to\Rightarrow)^2$ is invertible in $\mathrm{G^2Ł\forall}$.

Lemma 5. *The following rule is hp-admissible for the calculus* $\mathrm{G}^3\text{Ł}\forall$*:*

$$\frac{\mathcal{G}\,|\,S\,|\,S}{\mathcal{G}\,|\,S}\ (\mathrm{ec})^3.$$

Proof. We show that a proof D for a hypersequent of the form $\mathcal{G}\,|\,S\,|\,S$ can be transformed into a proof $\widehat{D}$ for $\mathcal{G}\,|\,S$ with $h(\widehat{D}) \leqslant h(D)$. We proceed by induction on $h(D)$.

1. If $h(D) = 0$, then the hypersequents $\mathcal{G}\,|\,S\,|\,S$ and $\mathcal{G}\,|\,S$ are axioms.

2. Let the root hypersequent $\mathcal{G}\,|\,S\,|\,S$ in D be the conclusion of an application R of a rule $\mathcal{R}$.

2.1. If the principal sequent occurrence in R is not one of the two occurrences of S distinguished in $\mathcal{G}\,|\,S\,|\,S$, then we apply the induction hypothesis to the proof for each premise of R and next use $\mathcal{R}$ to obtain the desired proof for $\mathcal{G}\,|\,S$.

2.2. Otherwise, we are to treat each inference rule of $\mathrm{G}^3\text{Ł}\forall$ as $\mathcal{R}$. However, all these cases are similar to one another. So we treat only the case where $\mathcal{R}$ is $(\forall\Rightarrow)^3$. Then the proof D has the form:

$$\frac{\begin{array}{c}D_1\\ \mathcal{G}\,|\,\Gamma,\mathfrak{p}\Rightarrow\Delta\,|\,\forall xA\Rightarrow\mathfrak{p}\,|\,[A]^x_t\Rightarrow\mathfrak{p}\,|\,\Gamma,\forall xA\Rightarrow\Delta\end{array}}{\mathcal{G}\,|\,\Gamma,\forall xA\Rightarrow\Delta\,|\,\Gamma,\forall xA\Rightarrow\Delta}\ (\forall\Rightarrow)^3.$$

Since the rule $(\forall\Rightarrow)^3$ is hp-invertible (see Lemma 4), given the proof D_1, we can find a proof D_1' for

$$\mathcal{G}\,|\,\Gamma,\mathfrak{p}\Rightarrow\Delta\,|\,\forall xA\Rightarrow\mathfrak{p}\,|\,[A]^x_t\Rightarrow\mathfrak{p}\,|\,\Gamma,\mathfrak{p}_1\Rightarrow\Delta\,|\,\forall xA\Rightarrow\mathfrak{p}_1\,|\,[A]^x_t\Rightarrow\mathfrak{p}_1,$$

where $\mathfrak{p}_1$ does not occur in the root hypersequent of D_1 and $h(D_1') \leqslant h(D_1)$.

Replacing all occurrences of $\mathfrak{p}_1$ in D_1' by $\mathfrak{p}$ yields a proof D_1'' for

$$\mathcal{G}\,|\,\Gamma,\mathfrak{p}\Rightarrow\Delta\,|\,\forall xA\Rightarrow\mathfrak{p}\,|\,[A]^x_t\Rightarrow\mathfrak{p}\,|\,\Gamma,\mathfrak{p}\Rightarrow\Delta\,|\,\forall xA\Rightarrow\mathfrak{p}\,|\,[A]^x_t\Rightarrow\mathfrak{p}\,;$$

whence using the induction hypothesis three times, we get a proof $\widetilde{D}_1$ for

$$\mathcal{G}\,|\ \Gamma,\mathfrak{p}\Rightarrow\Delta\,|\,\forall xA\Rightarrow\mathfrak{p}\,|\,[A]^x_t\Rightarrow\mathfrak{p}$$

such that $h(\widetilde{D}_1) \leqslant h(D_1'') \leqslant h(D_1)$.

Finally, by applying $(\forall\Rightarrow)^3$ to the root hypersequent of the proof $\widetilde{D}_1$, we obtain the desired proof $\widehat{D}$ for $\mathcal{G}\,|\,\Gamma,\forall xA\Rightarrow\Delta$ with $h(\widehat{D}) \leqslant h(D)$. □

Lemma 6. *Each inference rule of the calculus* $\mathrm{G}^1\text{Ł}\forall$ *is admissible for the calculus* $\mathrm{G}^3\text{Ł}\forall$*.*

Proof. An application of the rule $(\to\Rightarrow)^1$, $(\Rightarrow\to)^1$, $(\Rightarrow\forall)^1$, or $(\exists\Rightarrow)^1$ of $\mathrm{G}^1\text{Ł}\forall$ can be represented as an application of the corresponding rule of $\mathrm{G}^3\text{Ł}\forall$ followed by an appplication of the rule $(\mathrm{ec})^3$. E.g., an application of $(\Rightarrow\to)^1$ is represented thus:

$$\frac{\dfrac{\mathcal{G}\,|\,\Gamma\Rightarrow A\to B,\Delta\,|\,\Gamma\Rightarrow\Delta;\quad \mathcal{G}\,|\,\Gamma\Rightarrow A\to B,\Delta\,|\,\Gamma,A\Rightarrow B,\Delta}{\mathcal{G}\,|\,\Gamma\Rightarrow A\to B,\Delta\,|\,\Gamma\Rightarrow A\to B,\Delta}\ (\Rightarrow\to)^3}{\mathcal{G}\,|\,\Gamma\Rightarrow A\to B,\Delta}\ (\mathrm{ec})^3.$$

By Lemma 5, the rule $(\mathrm{ec})^3$ is admissible for $\mathrm{G}^3\text{Ł}\forall$. So these four rules of $\mathrm{G}^1\text{Ł}\forall$ are admissible for $\mathrm{G}^3\text{Ł}\forall$.

The rule $(\forall\Rightarrow)^1$ is admissible for $\mathrm{G}^3\text{Ł}\forall$, since an application of it can be represented as several applications of rules, which are admissible for $\mathrm{G}^3\text{Ł}\forall$ (by Lemmas 3 and 5), as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\mathcal{G}\,|\,\Gamma,\forall xA\Rightarrow\Delta\,|\,\Gamma,[A]^x_t\Rightarrow\Delta}{\mathcal{G}\,|\,\Gamma,\forall xA\Rightarrow\Delta\,|\,\Gamma,[A]^x_t,\mathfrak{p}\Rightarrow\mathfrak{p},\Delta}\,(\text{at}{\Rightarrow}\text{at})^3}{\mathcal{G}\,|\,\Gamma,\forall xA\Rightarrow\Delta\,|\,\Gamma,\mathfrak{p}\Rightarrow\Delta\,|\,[A]^x_t\Rightarrow\mathfrak{p}}\,(\text{split})^3}{\mathcal{G}\,|\,\Gamma,\forall xA\Rightarrow\Delta\,|\,\Gamma,\mathfrak{p}\Rightarrow\Delta\,|\,\forall xA\Rightarrow\mathfrak{p}\,|\,[A]^x_t\Rightarrow\mathfrak{p}}\,(\text{ew})^3}{\mathcal{G}\,|\,\Gamma,\forall xA\Rightarrow\Delta\,|\,\Gamma,\forall xA\Rightarrow\Delta}\,(\forall\Rightarrow)^3}{\mathcal{G}\,|\,\Gamma,\forall xA\Rightarrow\Delta}\,(\text{ec})^3,$$

where $\mathfrak{p}$ does not occur in the top hypersequent.

The rule $(\Rightarrow\exists)^1$ is treated similarly to $(\forall\Rightarrow)^1$. □

Theorem 2. *The calculus* $\mathrm{G^3Ł\forall}$ *proves any* $\mathrm{G^1Ł\forall}$*-provable hypersequent.*

Proof. All axioms of $\mathrm{G^1Ł\forall}$ are axioms of $\mathrm{G^3Ł\forall}$, and all the inference rules of $\mathrm{G^1Ł\forall}$ are admissible for $\mathrm{G^3Ł\forall}$ by Lemma 6. □

Let us recall that hypersequents of the calculus $\mathrm{G^1Ł\forall}$ may not contain semipropositional variables of type 0, whereas hypersequents (of the calculus $\mathrm{G^3Ł\forall}$) may. So the converse of Theorem 2 cannot hold. However, it is a problem for investigation whether $\mathrm{G^1Ł\forall}$ proves any $\mathrm{G^3Ł\forall}$-provable hypersequent without semipropositional variables of type 0.

Corollary 1. *The calculus* $\mathrm{G^3Ł\forall}$ *proves any hypersequent provable in* $\mathrm{GŁ\forall}$*, or* $\mathrm{G^1Ł\forall}$*, or* $\mathrm{G^2Ł\forall}$*.*

Proof. Theorem 4 and Proposition 11 both given in [7] say that $\mathrm{G^1Ł\forall}$ proves any hypersequent provable in $\mathrm{GŁ\forall}$ and $\mathrm{G^2Ł\forall}$, respectively. It remains to make use of Theorem 2 above. □

Corollary 2. *For any* $\mathrm{G^1Ł\forall}$*- or* $\mathrm{G^2Ł\forall}$*-proof* D*, a* $\mathrm{G^3Ł\forall}$*-proof* D' *of the same hypersequent can be constructed such that* $h(D')\leqslant h(D)$ *and the number of nodes in* D' *is no greater than the number of nodes in* D*.*

Proof. We merely inspect the proofs of Lemmas 3–6 and Theorem 2 to see that any $\mathrm{G^1Ł\forall}$-proof D can be transformed into the desired $\mathrm{G^3Ł\forall}$-proof D'. Next, any $\mathrm{G^2Ł\forall}$-proof D_2 can be easily transformed into a $\mathrm{G^1Ł\forall}$-proof D_1 of the same hypersequent, with the same height and the same number of nodes, using only the external weakening rule similar to the rule $(\text{ew})^3$ in Lemma 3. □

We define the *size* of a hypersequent to be the number of formula occurrences (as sequent members) in it. The following proposition makes it explicit that the repetition-free calculus $\mathrm{G^3Ł\forall}$ is more efficient than the calculi $\mathrm{G^1Ł\forall}$ and $\mathrm{G^2Ł\forall}$ in terms of the size of hypersequents in formal proofs.

Proposition 2. *Let* n *and* d *be positive integers.*

1. *There exist a hypersequent* $\mathcal{G}$ *of size* n *and a* $\mathrm{G^1Ł\forall}$*-proof* (*resp.* $\mathrm{G^2Ł\forall}$*-proof*) *search tree* D *of height* d *for* $\mathcal{G}$ *such that the size of some leaf hypersequent in* D *is equal to* $n\cdot d$*.*

2. *For any hypersequent* $\mathcal{G}$ *of size* n*, in any* $\mathrm{G^3Ł\forall}$*-proof search tree of height* d *for* $\mathcal{G}$*, the size of any hypersequent is no greater than* $n+4\cdot d$*.*

Proof. 1. Take a hypersequent $\Gamma\Rightarrow\exists xA,\Delta$ of size n without semipropositional variables of type 0 and apply the rule $(\Rightarrow\exists)^1$ or $(\Rightarrow\exists)^2$ backward d times. The only leaf hypersequent in the resulting proof search tree (of height d) has the size $n\cdot d$.

2. The result follows from the observation that, for each inference rule of $\mathrm{G^3Ł\forall}$, the size of its every premise is no greater than the size of its conclusion plus 4. $\square$

5. Transforming $\mathrm{G^3Ł\forall}$-proofs according to tactics

To organize bottom-up $\mathrm{G^3Ł\forall}$-proof search (cf. [7, Section 4.3]), we can use an auxiliary algorithm $\mathfrak{t}$, called a (*proof search*) *tactic*, that takes a proof search tree D as input and returns either

(a) the message $\mathfrak{t}(D)$ indicating that no leaf hypersequent of D contains any logical symbol, or

(b) a non-atomic RPL∀-formula occurrence $\mathfrak{t}(D)$ (as a sequent member) in a leaf hypersequent of D.

By a result of a backward rule application to a proof search tree D *according to a tactic* $\mathfrak{t}$, we mean D if $\mathfrak{t}(D)$ is not a formula occurrence; otherwise, a proof search tree obtained from D by a backward application of a (uniquely determined) rule of $\mathrm{G^3Ł\forall}$ to the occurrence $\mathfrak{t}(D)$. We say that a proof search tree (in particular, a proof) D for $\mathcal{H}$ can be constructed *according to a tactic* $\mathfrak{t}$ if D can be obtained from $\mathcal{H}$ by a finite number of backward rule applications according to $\mathfrak{t}$.

For a tactic $\mathfrak{t}$ and a hypersequent $\mathcal{H}$, let $\mathcal{D}^{\mathfrak{t}}_{\mathcal{H}}$ be a tree obtained from $\mathcal{H}$ by an infinite number of backward applications according to $\mathfrak{t}$. (Obviously, if a proof search tree D for $\mathcal{H}$ is constructed according to $\mathfrak{t}$, then D is a finite subtree of $\mathcal{D}^{\mathfrak{t}}_{\mathcal{H}}$.) Call a tactic $\mathfrak{t}$ *fair* if, for each hypersequent $\mathcal{H}$, each branch $\mathcal{B}$ of the tree $\mathcal{D}^{\mathfrak{t}}_{\mathcal{H}}$, and each non-atomic RPL∀-formula occurrence $\mathcal{F}$ (as a sequent member) on $\mathcal{B}$, there is a backward application to some ancestor of $\mathcal{F}$ on $\mathcal{B}$.

Now we state a theorem that allows us to justify the use of any fair tactic for bottom-up $\mathrm{G^3Ł\forall}$-proof search.

Theorem 3. *Suppose $\mathcal{G}$ is a $\mathrm{G^3Ł\forall}$-provable hypersequent, and $\mathfrak{t}$ is a fair tactic. Then some $\mathrm{G^3Ł\forall}$-proof of $\mathcal{G}$ can be constructed according to $\mathfrak{t}$.*

Before proving this theorem, we establish the following lemma, which is almost immediate from the hp-invertibility of the rules of $\mathrm{G^3Ł\forall}$. Next, to prove Theorem 3, we transform any $\mathrm{G^3Ł\forall}$-proof into one (of the same hypersequent) constructed according to any given fair tactic by means of repeated applications of the lemma in line with the tactic.

Lemma 7. *Suppose D is a $\mathrm{G^3Ł\forall}$-proof for $\mathcal{H}$, and $\mathcal{F}$ is a non-atomic RPL∀-formula occurrence (as a sequent member) in $\mathcal{H}$. Then a $\mathrm{G^3Ł\forall}$-proof $\widehat{D}$ of the form*

$$\dfrac{\begin{matrix}\widehat{D}_1\\ \widehat{\mathcal{H}}_1\end{matrix}}{\mathcal{H}} \quad \text{or} \quad \dfrac{\begin{matrix}\widehat{D}_1\\ \widehat{\mathcal{H}}_1\end{matrix} \quad \begin{matrix}\widehat{D}_2\\ \widehat{\mathcal{H}}_2\end{matrix}}{\mathcal{H}}$$

can be constructed such that:

(1) $\mathcal{F}$ is the principal formula occurrence in the lowest backward application in $\widehat{D}$, and $h(\widehat{D}_i) \leqslant h(D)$ for each i;

(2) if $\mathcal{F}$ is the principal formula occurrence in the lowest backward application in D, then $\widehat{D}$ is the same as D;

(3) *if* $h(D) > 0$ *and the principal formula occurrence* $\mathcal{F}_0$ *in the lowest backward application in* D *differs from* $\mathcal{F}$*, then, for each* i*, the ancestor of* $\mathcal{F}_0$ *in* $\widehat{\mathcal{H}}_i$ *is the princilal formula occurrence in the lowest backward application in* $\widehat{D}_i$*.*[2]

Proof. If $\mathcal{F}$ is the principal formula occurrence in the lowest backward application in D, then we immediately take D as $\widehat{D}$, and assertions (1)–(3) of the lemma clearly hold.

Suppose $\mathcal{F}$ is not the principal formula occurrence in the lowest backward application in D. Then assertion (2) of the lemma is trivially true. Let $\mathcal{R}$ be the only inference rule that can be applied backward to the occurrence $\mathcal{F}$ in $\mathcal{H}$.

Using the construction in the proof of the hp-invertibility of $\mathcal{R}$ (see Lemma 4), from the proof D for $\mathcal{H}$, we construct proofs $\widehat{D}_i$ ($i = 1$ or $i = 1, 2$) for all the premises of a backward application of $\mathcal{R}$ to the occurrence $\mathcal{F}$ in $\mathcal{H}$, and we have $h(\widehat{D}_i) \leqslant h(D)$. In the case $\mathcal{R}$ is $(\forall \Rightarrow)^3$ or $(\Rightarrow \exists)^3$, any closed term can be taken as the proper term of this application.

Now, by applying $\mathcal{R}$ to the root hypersequents of the proofs $\widehat{D}_i$, we obtain a proof $\widehat{D}$ of $\mathcal{H}$ for which assertion (1) of the lemma holds.

After examining the construction in the proof of the hp-invertibility of $\mathcal{R}$ (see Lemma 4), we are sure that $\widehat{D}$ satisfies assertion (3) of the lemma being proved. □

Proof of Theorem 3. Fix a $\mathrm{G}^3\mathrm{Ł}\forall$-proof D_0 for $\mathcal{G}$ and transform it according to $\mathfrak{t}$ in stages. The result of each stage will be some $\mathrm{G}^3\mathrm{Ł}\forall$-proof D for $\mathcal{G}$ consisting of

(a) a proof search tree $D^{\mathfrak{t}}$ that has the common root with D and is constructed according to $\mathfrak{t}$, and which is called the *transformed part* of D, as well as

(b) a finite number of proof trees whose roots are leaves of $D^{\mathfrak{t}}$, and each of which is called a *nontransformed part* of D.

Define the transformed part of the initial proof D_0 to be its root, and the only nontransformed part of it to be D_0 itself.

We use induction on the maximal height $H(D)$ of the nontransformed parts of the current proof D being transformed.

1. If $H(D) = 0$, then D is the required proof.

2. Suppose that $H(D) > 0$ and $D^{\mathfrak{t}}$ is the transformed part of D.

2.1. To obtain a proof $\widetilde{D}$ (with its transformed part $\widetilde{D}^{\mathfrak{t}}$) as a result of the stage, we carry out some finite number N of backward applications to the transformed part of the current proof (which is D initially) according to the fair tactic $\mathfrak{t}$. We choose such a number N so that, for each branch $\mathcal{B}$ of $\widetilde{D}^{\mathfrak{t}}$ and each non-atomic RPL$\forall$-formula occurrence $\mathcal{F}$ (as a sequent member) in the node of $\widetilde{D}^{\mathfrak{t}}$ that was a leaf of $D^{\mathfrak{t}}$ and is on $\mathcal{B}$ now, there is a backward application to some ancestor of $\mathcal{F}$ on $\mathcal{B}$.

2.2. We carry out each backward application to a formula occurrence $\mathcal{F}$ (chosen by $\mathfrak{t}$) in a leaf of the transformed part $\mathcal{D}^{\mathfrak{t}}$ of the current proof $\mathcal{D}$ for $\mathcal{G}$ as follows. Let $\mathcal{D}^{\mathfrak{n}}$ be the nontransformed part of $\mathcal{D}$ whose root is this leaf, and $\mathcal{H}$ be the root hypersequent of $\mathcal{D}^{\mathfrak{n}}$. By Lemma 7, given the proof $\mathcal{D}^{\mathfrak{n}}$ and the occurrence $\mathcal{F}$ in $\mathcal{H}$, we construct a proof $\widehat{\mathcal{D}}^{\mathfrak{n}}$ of the form

[2]Roughly speaking, the lowest backward application in D goes one level up in $\widehat{D}$.

$$\begin{array}{c} \widehat{\mathcal{D}}_1^{\mathfrak{n}} \\ \\ \dfrac{\widehat{\mathcal{H}}_1}{\mathcal{H}} \end{array} \qquad \text{or} \qquad \begin{array}{cc} \widehat{\mathcal{D}}_1^{\mathfrak{n}} & \widehat{\mathcal{D}}_2^{\mathfrak{n}} \\ \\ \multicolumn{2}{c}{\dfrac{\widehat{\mathcal{H}}_1 \qquad \widehat{\mathcal{H}}_2}{\mathcal{H}}} \end{array}$$

such that:

(1^7) $\mathcal{F}$ is the principal formula occurrence in the lowest backward application in $\widehat{\mathcal{D}}^{\mathfrak{n}}$, and $h(\widehat{\mathcal{D}}_i^{\mathfrak{n}}) \leqslant h(\mathcal{D}^{\mathfrak{n}})$ for each i;

(2^7) if $\mathcal{F}$ is the principal formula occurrence in the lowest backward application in $\mathcal{D}^{\mathfrak{n}}$, then $\widehat{\mathcal{D}}^{\mathfrak{n}}$ is the same as $\mathcal{D}^{\mathfrak{n}}$, and hence $h(\widehat{\mathcal{D}}_i^{\mathfrak{n}}) < h(\mathcal{D}^{\mathfrak{n}})$ for each i;

(3^7) if $h(\mathcal{D}^{\mathfrak{n}}) > 0$ and the principal formula occurrence $\mathcal{F}_0$ in the lowest backward application in $\mathcal{D}^{\mathfrak{n}}$ differs from $\mathcal{F}$, then, for each i, the ancestor of $\mathcal{F}_0$ in $\widehat{\mathcal{H}}_i$ is the princilal formula occurrence in the lowest backward application in $\widehat{\mathcal{D}}_i^{\mathfrak{n}}$.

Next, we replace the subtree $\mathcal{D}^{\mathfrak{n}}$ in $\mathcal{D}$ by $\widehat{\mathcal{D}}^{\mathfrak{n}}$. Finally, the lowest backward application in $\widehat{\mathcal{D}}^{\mathfrak{n}}$ is included in the transformed part of the resulting proof for $\mathcal{G}$. Thereby from $\mathcal{D}^{\mathfrak{n}}$ we obtain one or two new nontransformed parts: $\widehat{\mathcal{D}}_1^{\mathfrak{n}}$ or $\widehat{\mathcal{D}}_1^{\mathfrak{n}}$ and $\widehat{\mathcal{D}}_2^{\mathfrak{n}}$.

2.3. Clearly, under the given transformation of D into $\widetilde{D}$, each nontransformed part $\widetilde{D}^{\mathfrak{n}}$ of $\widetilde{D}$ is obtaned from some nontransformed part $D^{\mathfrak{n}}$ of D. If $h(D^{\mathfrak{n}}) = 0$, then it is obvious that $h(\widetilde{D}^{\mathfrak{n}}) = 0$.

Suppose $h(D^{\mathfrak{n}}) > 0$. By item 2.1 and assertion (3^7), when we transform D into $\widetilde{D}$, we carry out so many backward applications that the premise of assertion (2^7) holds for at least one backward application performed in the passage from $D^{\mathfrak{n}}$ to $\widetilde{D}^{\mathfrak{n}}$. Therefore $h(\widetilde{D}^{\mathfrak{n}}) < h(D^{\mathfrak{n}})$.

Thus $H(\widetilde{D}) < H(D)$. By the induction hypothesis applied to $\widetilde{D}$, we construct a proof of $\mathcal{G}$ according to $\mathfrak{t}$. □

Remark 4. In essentially the same manner as in [7, Section 4], we can formulate a free-variable tableau modification T^{3}Ł∀ of the calculus G^{3}Ł∀ and describe a family of T^{3}Ł∀-proof search algorithms parameterized by a fair tactic. Then Theorem 3 (on constructing G^{3}Ł∀-proofs according to fair tactics) allows us to establish that any algorithm of the family constructs some T^{3}Ł∀-proof for any G^{3}Ł∀-provable sentence. Note that, in [7], we obtained proof search algorithms applicable to prenex G^{2}Ł∀-provable sentences; but now G^{3}Ł∀ provides us with proof search algorithms applicable to the wider class of all (not necessarily prenex) G^{3}Ł∀-provable sentences (cf. also Corollary 1).

6. The mid-hypersequent theorem for G^{3}Ł∀ and its consequences

We say that a G^{3}Ł∀-proof is a *mid-hypersequent* proof if in it all applications of propositional rules are above all applications of quantifier rules.

To transform some G^{3}Ł∀-proofs into mid-hypersequent ones, we will use the following properties (P1–P4), which express permutability of adjacent rule applications and are formulated slightly more generally than we need in the sequel. In each of these properties, the resulting proof is displayed after the initial one. From now on, if a formula (or sequent) occurrence in the conclusion of a rule application is in boldface, then the occurrence is the principal one in the application. Properties P1–P4 can be verified in a straightforward way.

P1: *permutability of two one-premise rule applications, with one of their principal sequent occurrences being an ancestor of the other.* Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be any one-premise inference rules of $\mathrm{G^3Ł}\forall$, except the case where $\mathcal{R}_1 \in \{ (\Rightarrow\forall)^3,\ (\exists\Rightarrow)^3 \}$ and $\mathcal{R}_2 \in \{ (\Rightarrow\exists)^3,\ (\forall\Rightarrow)^3 \}$.

Under this condition, we have cases P1.1–P1.4, depending on which of $\mathcal{R}_1$ and $\mathcal{R}_2$ are antecedent rules, and which are succedent ones.

P1.1. If $\mathcal{R}_1$ is an antecedent rule and $\mathcal{R}_2$ is a succedent rule, then we can perform the following transformation:

$$\dfrac{\dfrac{\dfrac{D}{\mathcal{G}\,|\,\Gamma, F_1 \Rightarrow F_2, \Delta\,|\,\mathcal{H}_1\,|\,\mathcal{H}_2}}{\mathcal{G}\,|\,\Gamma, F_1 \Rightarrow \boldsymbol{A_2}, \Delta\,|\,\mathcal{H}_1}\,\mathcal{R}_2}{\mathcal{G}\,|\,\Gamma, \boldsymbol{A_1} \Rightarrow A_2, \Delta}\,\mathcal{R}_1 \qquad \dfrac{\dfrac{\dfrac{D}{\mathcal{G}\,|\,\Gamma, F_1 \Rightarrow F_2, \Delta\,|\,\mathcal{H}_1\,|\,\mathcal{H}_2}}{\mathcal{G}\,|\,\Gamma, \boldsymbol{A_1} \Rightarrow F_2, \Delta\,|\,\mathcal{H}_2}\,\mathcal{R}_1}{\mathcal{G}\,|\,\Gamma, A_1 \Rightarrow \boldsymbol{A_2}, \Delta}\,\mathcal{R}_2$$

P1.2. If $\mathcal{R}_1$ is a succedent rule and $\mathcal{R}_2$ is an antecedent rule, then the transformation is symmetric to that in P1.1.

P1.3 (resp. P1.4). If both of $\mathcal{R}_1$ and $\mathcal{R}_2$ are antecedent (resp. succedent) rules, then, for a hypersequent that is at the bottom of an appropriate initial proof and has the form $\mathcal{G}\,|\,\Gamma, A_1, A_2 \Rightarrow \Delta$ (resp. $\mathcal{G}\,|\,\Gamma \Rightarrow A_1, A_2, \Delta$), we can carry out a transformation similar to that just given.

E.g., if $\mathcal{R}_1$ is $(\rightarrow\Rightarrow)^3$ and $\mathcal{R}_2$ is $(\Rightarrow\forall)^3$, then the initial and resulting proofs look like:

$$\dfrac{\dfrac{\dfrac{D}{\mathcal{G}\,|\,\Gamma, \mathfrak{p} \Rightarrow [C]^x_a, \Delta\,|\,B \Rightarrow \mathfrak{p}, A}}{\mathcal{G}\,|\,\Gamma, \mathfrak{p} \Rightarrow \forall \boldsymbol{x}\boldsymbol{C}, \Delta\,|\,B \Rightarrow \mathfrak{p}, A}\,(\Rightarrow\forall)^3}{\mathcal{G}\,|\,\Gamma, \boldsymbol{A} \rightarrow \boldsymbol{B} \Rightarrow \forall x C, \Delta}\,(\rightarrow\Rightarrow)^3 \qquad \dfrac{\dfrac{\dfrac{D}{\mathcal{G}\,|\,\Gamma, \mathfrak{p} \Rightarrow [C]^x_a, \Delta\,|\,B \Rightarrow \mathfrak{p}, A}}{\mathcal{G}\,|\,\Gamma, \boldsymbol{A} \rightarrow \boldsymbol{B} \Rightarrow [C]^x_a, \Delta}\,(\rightarrow\Rightarrow)^3}{\mathcal{G}\,|\,\Gamma, A \rightarrow B \Rightarrow \forall \boldsymbol{x}\boldsymbol{C}, \Delta}\,(\Rightarrow\forall)^3$$

P2: *permutability of two one-premise rule applications, with neither of their principal sequent occurrences being an ancestor of the other.* Let rules $\mathcal{R}_1$ and $\mathcal{R}_2$ be as in the first paragraph of P1. Then we can perform this transformation:

$$\dfrac{\dfrac{\dfrac{D}{\mathcal{G}\,|\,\mathcal{H}_1\,|\,\mathcal{H}_2}}{\mathcal{G}\,|\,\mathcal{H}_1\,|\,\boldsymbol{S_2}}\,\mathcal{R}_2}{\mathcal{G}\,|\,\boldsymbol{S_1}\,|\,S_2}\,\mathcal{R}_1 \qquad \dfrac{\dfrac{\dfrac{D}{\mathcal{G}\,|\,\mathcal{H}_1\,|\,\mathcal{H}_2}}{\mathcal{G}\,|\,\boldsymbol{S_1}\,|\,\mathcal{H}_2}\,\mathcal{R}_1}{\mathcal{G}\,|\,S_1\,|\,\boldsymbol{S_2}}\,\mathcal{R}_2$$

E.g., if $\mathcal{R}_1$ is $(\rightarrow\Rightarrow)^3$ and $\mathcal{R}_2$ is $(\Rightarrow\forall)^3$, then the initial and resulting proofs have the forms:

$$\dfrac{\dfrac{\dfrac{D}{\mathcal{G}\,|\,\Gamma_1, \mathfrak{p} \Rightarrow \Delta_1\,|\,B \Rightarrow \mathfrak{p}, A\,|\,\Gamma_2 \Rightarrow [C]^x_a, \Delta_2}}{\mathcal{G}\,|\,\Gamma_1, \mathfrak{p} \Rightarrow \Delta_1\,|\,B \Rightarrow \mathfrak{p}, A\,|\,\Gamma_2 \Rightarrow \forall \boldsymbol{x}\boldsymbol{C}, \Delta_2}\,(\Rightarrow\forall)^3}{\mathcal{G}\,|\,\Gamma_1, \boldsymbol{A} \rightarrow \boldsymbol{B} \Rightarrow \Delta_1\,|\,\Gamma_2 \Rightarrow \forall x C, \Delta_2}\,(\rightarrow\Rightarrow)^3$$

$$\dfrac{\dfrac{\dfrac{D}{\mathcal{G}\,|\,\Gamma_1, \mathfrak{p} \Rightarrow \Delta_1\,|\,B \Rightarrow \mathfrak{p}, A\,|\,\Gamma_2 \Rightarrow [C]^x_a, \Delta_2}}{\mathcal{G}\,|\,\Gamma_1, \boldsymbol{A} \rightarrow \boldsymbol{B} \Rightarrow \Delta_1\,|\,\Gamma_2 \Rightarrow [C]^x_a, \Delta_2}\,(\rightarrow\Rightarrow)^3}{\mathcal{G}\,|\,\Gamma_1, A \rightarrow B \Rightarrow \Delta_1\,|\,\Gamma_2 \Rightarrow \forall \boldsymbol{x}\boldsymbol{C}, \Delta_2}\,(\Rightarrow\forall)^3$$

P3: *permutability of an application of the two-premise rule* $(\Rightarrow\to)^3$ *and two applications* R', R'' *of a one-premise rule, with the principal sequent occurrences of* R', R'' *being ancestors of that of the two-premise rule application.* Let $\mathcal{R}$ be any one-premise inference rule of $\mathrm{G^3Ł\forall}$. Then there are cases P3.1 and P3.2, according as $\mathcal{R}$ is an antecedent or succedent rule.

P3.1. If $\mathcal{R}$ is an antecedent rule, then under conditions (I) and (II) stated below, we can carry out the following transformation:

$$\dfrac{\mathcal{R}\,\dfrac{\overset{D_1}{\mathcal{G}\,|\,\Gamma, F_1 \Rightarrow \Delta\,|\,\mathcal{H}_1}}{\mathcal{G}\,|\,\Gamma, \boldsymbol{C} \Rightarrow \Delta} \qquad \dfrac{\overset{D_2}{\mathcal{G}\,|\,\Gamma, F_2, A \Rightarrow B, \Delta\,|\,\mathcal{H}_2}}{\mathcal{G}\,|\,\Gamma, \boldsymbol{C}, A \Rightarrow B, \Delta}\,\mathcal{R}}{\mathcal{G}\,|\,\Gamma, C \Rightarrow \boldsymbol{A} \to \boldsymbol{B}, \Delta}\,(\Rightarrow\to)^3$$

$$\dfrac{\dfrac{\overset{D_1'}{\mathcal{G}\,|\,\Gamma, F_2 \Rightarrow \Delta\,|\,\mathcal{H}_2} \qquad \overset{D_2}{\mathcal{G}\,|\,\Gamma, F_2, A \Rightarrow B, \Delta\,|\,\mathcal{H}_2}}{\mathcal{G}\,|\,\Gamma, F_2 \Rightarrow \boldsymbol{A} \to \boldsymbol{B}, \Delta\,|\,\mathcal{H}_2}\,(\Rightarrow\to)^3}{\mathcal{G}\,|\,\Gamma, \boldsymbol{C} \Rightarrow A \to B, \Delta}\,\mathcal{R}$$

P3.2. If $\mathcal{R}$ is a succedent rule, then under conditions (I) and (II) stated below, we can perform a similar transformation with a bottom hypersequent of the form $\mathcal{G}\,|\,\Gamma \Rightarrow C, A \to B, \Delta$.

Conditions (I) and (II) mentioned in P3.1 and P3.2 are:

(I) If $\mathcal{R}$ is $(\forall\Rightarrow)^3$ or $(\Rightarrow\exists)^3$, then the proper terms of the three displayed applications of $\mathcal{R}$ are the same.

(II) We construct the proof D_1' thus:

(II.a) Suppose that $\mathcal{R}$ is $(\exists\Rightarrow)^3$ or $(\Rightarrow\forall)^3$, and that a_1 and a_2 are the proper parameters of the two applications of $\mathcal{R}$ displayed in the initial proof on the left and right, respectively; then: $D_1' = D_1$ if $a_1 = a_2$; otherwise, we obtain the proof $\widetilde{D}_1$ (for the root hypersequent of D_1) from D_1 by replacing all occurrences of a_2 with a parameter not occurring in D_1, and next, we get the required proof D_1' from $\widetilde{D}_1$ by replacing all occurrences of a_1 with a_2.

(II.b) If $\mathcal{R}$ is $(\to\Rightarrow)^3$, $(\forall\Rightarrow)^3$, or $(\Rightarrow\exists)^3$, then we obtain D_1' from D_1 as in (a), but instead of parameters, we use semipropositional variables of the type corresponding to the rule $\mathcal{R}$.

E.g., if $\mathcal{R}$ is $(\forall\Rightarrow)^3$, then the initial and resulting proofs look like:

$$\dfrac{(\forall\Rightarrow)^3\,\dfrac{\overset{D_1}{\begin{array}{c}\mathcal{G}\,|\,\Gamma, \mathfrak{p}_1 \Rightarrow \Delta\\ |\,\forall xC \Rightarrow \mathfrak{p}_1\,|\,[C]^x_t \Rightarrow \mathfrak{p}_1\end{array}}}{\mathcal{G}\,|\,\Gamma, \forall \boldsymbol{x}\boldsymbol{C} \Rightarrow \Delta} \qquad \dfrac{\overset{D_2}{\begin{array}{c}\mathcal{G}\,|\,\Gamma, \mathfrak{p}_2, A \Rightarrow B, \Delta\\ |\,\forall xC \Rightarrow \mathfrak{p}_2\,|\,[C]^x_t \Rightarrow \mathfrak{p}_2\end{array}}}{\mathcal{G}\,|\,\Gamma, \forall \boldsymbol{x}\boldsymbol{C}, A \Rightarrow B, \Delta}\,(\forall\Rightarrow)^3}{\mathcal{G}\,|\,\Gamma, \forall xC \Rightarrow \boldsymbol{A} \to \boldsymbol{B}, \Delta}\,(\Rightarrow\to)^3$$

$$\dfrac{\dfrac{\overset{D_1'}{\begin{array}{c}\mathcal{G}\,|\,\Gamma, \mathfrak{p}_2 \Rightarrow \Delta\\ |\,\forall xC \Rightarrow \mathfrak{p}_2\,|\,[C]^x_t \Rightarrow \mathfrak{p}_2\end{array}} \qquad \overset{D_2}{\begin{array}{c}\mathcal{G}\,|\,\Gamma, \mathfrak{p}_2, A \Rightarrow B, \Delta\\ |\,\forall xC \Rightarrow \mathfrak{p}_2\,|\,[C]^x_t \Rightarrow \mathfrak{p}_2\end{array}}}{\mathcal{G}\,|\,\Gamma, \mathfrak{p}_2 \Rightarrow \boldsymbol{A} \to \boldsymbol{B}, \Delta\,|\,\forall xC \Rightarrow \mathfrak{p}_2\,|\,[C]^x_t \Rightarrow \mathfrak{p}_2}\,(\Rightarrow\to)^3}{\mathcal{G}\,|\,\Gamma, \forall \boldsymbol{x}\boldsymbol{C} \Rightarrow A \to B, \Delta}\,(\forall\Rightarrow)^3$$

P4: *permutability of an application of the two-premise rule* $(\Rightarrow\rightarrow)^3$ *and two applications* R', R'' *of a one-premise rule, with neither of the principal sequent occurrences of* R', R'' *being an ancestor of that of the two-premise rule application.* Let $\mathcal{R}$ be any one-premise inference rule of $\mathrm{G}^3\text{Ł}\forall$. Then we can carry out the transformation:

$$\dfrac{\mathcal{R}\,\dfrac{\overset{D_1}{\mathcal{G}\,|\,\mathcal{H}_{1,1}\,|\,\mathcal{H}_{2,0}}}{\mathcal{G}\,|\,\mathcal{H}_{1,1}\,|\,\boldsymbol{S_2}} \qquad \dfrac{\overset{D_2}{\mathcal{G}\,|\,\mathcal{H}_{1,2}\,|\,\mathcal{H}_2}}{\mathcal{G}\,|\,\mathcal{H}_{1,2}\,|\,\boldsymbol{S_2}}\,\mathcal{R}}{\mathcal{G}\,|\,\boldsymbol{S_1}\,|\,S_2}\,(\Rightarrow\rightarrow)^3 \qquad\qquad \dfrac{\dfrac{\overset{D_1'}{\mathcal{G}\,|\,\mathcal{H}_{1,1}\,|\,\mathcal{H}_2} \qquad \overset{D_2}{\mathcal{G}\,|\,\mathcal{H}_{1,2}\,|\,\mathcal{H}_2}}{\mathcal{G}\,|\,\boldsymbol{S_1}\,|\,\mathcal{H}_2}\,(\Rightarrow\rightarrow)^3}{\mathcal{G}\,|\,S_1\,|\,\boldsymbol{S_2}}\,\mathcal{R}$$

Here all the principal formula occurrences in the three displayed applications of $\mathcal{R}$ represent the same formula; and conditions (I) and (II) given in P3 must hold.

E.g., if $\mathcal{R}$ is $(\Rightarrow\forall)^3$, then the initial and resulting proofs have the forms:

$$\dfrac{(\Rightarrow\forall)^3\,\dfrac{\overset{D_1}{\mathcal{G}\,|\,\Gamma_1\Rightarrow\Delta_1\,|\,\Gamma_2\Rightarrow[C]^x_{a_1},\Delta_2}}{\mathcal{G}\,|\,\Gamma_1\Rightarrow\Delta_1\,|\,\Gamma_2\Rightarrow\forall\boldsymbol{xC},\Delta_2} \qquad \dfrac{\overset{D_2}{\mathcal{G}\,|\,\Gamma_1,A\Rightarrow B,\Delta_1\,|\,\Gamma_2\Rightarrow[C]^x_{a_2},\Delta_2}}{\mathcal{G}\,|\,\Gamma_1,A\Rightarrow B,\Delta_1\,|\,\Gamma_2\Rightarrow\forall\boldsymbol{xC},\Delta_2}\,(\Rightarrow\forall)^3}{\mathcal{G}\,|\,\Gamma_1\Rightarrow\boldsymbol{A\rightarrow B},\Delta_1\,|\,\Gamma_2\Rightarrow\forall xC,\Delta_2}\,(\Rightarrow\rightarrow)^3$$

$$\dfrac{\dfrac{\overset{D_1'}{\mathcal{G}\,|\,\Gamma_1\Rightarrow\Delta_1\,|\,\Gamma_2\Rightarrow[C]^x_{a_2},\Delta_2} \qquad \overset{D_2}{\mathcal{G}\,|\,\Gamma_1,A\Rightarrow B,\Delta_1\,|\,\Gamma_2\Rightarrow[C]^x_{a_2},\Delta_2}}{\mathcal{G}\,|\,\Gamma_1\Rightarrow\boldsymbol{A\rightarrow B},\Delta_1\,|\,\Gamma_2\Rightarrow[C]^x_{a_2},\Delta_2}\,(\Rightarrow\rightarrow)^3}{\mathcal{G}\,|\,\Gamma_1\Rightarrow A\rightarrow B,\Delta_1\,|\,\Gamma_2\Rightarrow\forall\boldsymbol{xC},\Delta_2}\,(\Rightarrow\forall)^3$$

Theorem 4 (the mid-hypersequent theorem for $\mathrm{G}^3\text{Ł}\forall$)**.** *Let* $\mathcal{H}$ *be a hypersequent in which each* RPLŁ$\forall$*-formula is prenex. Then any* $\mathrm{G}^3\text{Ł}\forall$*-proof* D *for* $\mathcal{H}$ *can be transformed into a mid-hypersequent* $\mathrm{G}^3\text{Ł}\forall$*-proof* $\widehat{D}$ *for* $\mathcal{H}$*; moreover,* $Q(\widehat{D}) \leqslant Q(D)$*, where* $Q(\mathcal{D})$ *is the number of quantifier rule applications in a* $\mathrm{G}^3\text{Ł}\forall$*-proof* $\mathcal{D}$*.*

Proof.[3] For a $\mathrm{G}^3\text{Ł}\forall$-proof $\mathcal{D}$ and a propositional rule application R in $\mathcal{D}$, let $\mathcal{O}(R)$ be the number of quantifier rule applications above R, and $\mathcal{O}(\mathcal{D})$ be the sum of $\mathcal{O}(R)$ over all propositional rule applications R in $\mathcal{D}$.

We proceed by induction on $\mathcal{O}(D)$, where D is a given proof for $\mathcal{H}$.

1. If $\mathcal{O}(D) = 0$, then D is the desired proof.

2. Otherwise, choose an application R_0 of a (propositional) rule $\mathcal{R}_0$ in D such that $\mathcal{O}(R_0) > 0$ and no application R' with $\mathcal{O}(R') > 0$ is above R_0.

2.1. Suppose $\mathcal{R}_0$ is $(\rightarrow\Rightarrow)^3$. By R_1 denote the (quantifier) rule application that stands immediately above the application R_0. We permute R_0 and R_1 using transformation P1 or P2, and next, by the induction hypothesis, we obtain the desired proof.

2.2. Now suppose $\mathcal{R}_0$ that is $(\Rightarrow\rightarrow)^3$, and the proof for the conclusion $\mathcal{H}_0$ of the application R_0 looks like:

---

[3]It is interesting to note the following. Lemma 7 states that any $\mathrm{G}^3\text{Ł}\forall$-proof D can be transformed into a $\mathrm{G}^3\text{Ł}\forall$-proof $\widehat{D}$ of the same hypersequent $\mathcal{H}$, with any rule applicable to $\mathcal{H}$ backward being applied at the bottom of $\widehat{D}$. However, Theorem 4 cannot be proved only by repeated applications of Lemma 7. Indeed, in the case of the rule $(\forall\Rightarrow)^3$ or $(\Rightarrow\exists)^3$, the resulting proof $\widehat{D}$ obtained using the construction in the proofs of Lemmas 7, 4, and 3 has the same structure above the lowest rule application as the initial $\mathrm{G}^3\text{Ł}\forall$-proof D (i.e., in that part of $\widehat{D}$ the same rules are applied in the same order as in D).

$$\frac{\begin{array}{cc} D_1 & D_2 \\ \mathcal{H}_1 & \mathcal{H}_2 \end{array}}{\mathcal{H}_0}\,\mathcal{R}_0.$$

Then the lowest application in D_1 or D_2, say for definiteness the lowest application R_2 in D_2, is an application of a quantifier rule $\mathcal{R}$.

By the induction hypothesis, we can transform D_1 into a mid-hypersequent proof D_1' for $\mathcal{H}_1$ such that $Q(D_1') \leqslant Q(D_1)$. In the proof D (for $\mathcal{H}$), we replace the subtree D_1 by D_1', thus obtaining a proof D' for $\mathcal{H}$.

Let the principal formula occurrence $\mathcal{F}_2$ in R_2 (which is a formula occurrence in $\mathcal{H}_2$) be an ancestor of an occurrence $\mathcal{F}_0$ in $\mathcal{H}_0$. The formulas A and B in $\mathcal{H}_2$ that originate from the principal occurrence of $A \to B$ in R_0 are quantifier-free. Therefore the occurrence $\mathcal{F}_0$ has an ancestor $\mathcal{F}_1$ in $\mathcal{H}_1$, and all $\mathcal{F}_i$ $(i = 0, 1, 2)$ represent the same formula.

Using the construction in the proof of the hp-invertibility of the rule $\mathcal{R}$ (see Lemma 4), from the proof D_1' for $\mathcal{H}_1$, we construct a proof D_1'' for the premise of an application R_1 of $\mathcal{R}$ with $\mathcal{H}_1$ as the conclusion and $\mathcal{F}_1$ as the principal formula occurrence. Here if $\mathcal{R}$ is $(\forall \Rightarrow)^3$ or $(\Rightarrow \exists)^3$, then the proper term of the application R_1 (of $\mathcal{R}$) is taken to be the proper term of the application R_2 (of $\mathcal{R}$). Let D_1''' be the proof (for $\mathcal{H}_1$) obtained from the proof D_1'' by the application R_1.

Given the mid-hypersequent proof D_1', it is not hard to see that D_1'' is also a mid-hypersequent proof $\big($i.e., $\mathcal{O}(D_1'') = 0\big)$ and $Q(D_1'') \leqslant Q(D_1')$. Hence, $\mathcal{O}(D_1''') = 0$ and $Q(D_1''') \leqslant Q(D_1') + 1$.

Next, in the proof D' (for $\mathcal{H}$), we replace the subtree D_1' by D_1''' and get a proof D'' for $\mathcal{H}$. Using transformation P3 or P4, in D'' we permute the application R_0 (of the two-premise rule $\mathcal{R}_0$) and the applications R_1 and R_2 (of the quantifier rule $\mathcal{R}$), which stand immediately above R_0; and we have a proof D''' for $\mathcal{H}$ as a result.

From $\mathcal{O}(D_1''') = 0$, $Q(D_1''') \leqslant Q(D_1') + 1 \leqslant Q(D_1) + 1$, and the forms of transformations P3 and P4, it follows that $\mathcal{O}(D''') < \mathcal{O}(D)$ and $Q(D''') \leqslant Q(D)$. Now by the induction hypothesis, we can construct the desired proof from D'''. □

Remark 5. In contrast to Theorem 4 above, Theorems 10 and 18 in [7] (i.e., the mid-hypersequent theorems for $\mathrm{G^1Ł\forall}$ and $\mathrm{G^2Ł\forall}$) require an initial hypersequent to be only of the form $\Rightarrow A$, where A is a prenex RPL∀-formula.

Further, our proof of Theorem 4 differs much from well-known proofs of the mid-sequent theorem for (variants of) classical sequent calculi with structural rules; see [17] and, e.g., [15, §§ 54–55]. Those structural rules are useful for transforming formal proofs and include thinning, or weakening, in succedent, whose hypersequent version is not sound and thus is not admissible for the calculus $\mathrm{G^3Ł\forall}$.

Our proof is also considerably different from the proof of Theorem 8.29 in [5], i.e., of the mid-hypersequent theorem for some hypersequent calculi for fuzzy logics not including Ł∀ or RPL∀. This is mainly because the two-premise rule of $\mathrm{G^3Ł\forall}$ is context-sharing, but each of the two-premise logical rules in [5, Theorem 8.29] is or is easily made context-independent (cf., e.g., [16, item 3.1.5] and note also that context-independent rules facilitate permuting rule applications, but make bottom-up proof search too nondeterministic).

Theorem 5. *Suppose that A is a prenex* RPL∀*-formula. Then the following are equivalent:* (1) $\vdash_{\mathrm{G^1Ł\forall}} A$, (2) $\vdash_{\mathrm{G^2Ł\forall}} A$, (3) $\vdash_{\mathrm{G^3Ł\forall}} A$.

$$
\begin{array}{cl}
\Rightarrow \mathfrak{q}_1 \,|\, \mathfrak{q}_1 \Rightarrow \mathfrak{q}_3 \,|\, \mathfrak{q}_3 \Rightarrow \exists x\forall y\exists z B(x,y,z) \,|\, \mathfrak{q}_3 \Rightarrow \mathfrak{q}_4 & \\
\,|\, \mathfrak{q}_4 \Rightarrow \mathfrak{q}_5 \,|\, \mathfrak{q}_5 \Rightarrow \exists z B(t_3,a_2,z) \,|\, \mathfrak{q}_5 \Rightarrow B(t_3,a_2,t_5) & \\
\,|\, \mathfrak{q}_4 \Rightarrow B(t_3,a_2,t_4) \,|\, \mathfrak{q}_1 \Rightarrow \mathfrak{q}_2 \,|\, \mathfrak{q}_2 \Rightarrow \exists z B(t_1,a_1,z) \,|\, \mathfrak{q}_2 \Rightarrow B(t_1,a_1,t_2) & \\
\hline
\Rightarrow \mathfrak{q}_1 \,|\, \mathfrak{q}_1 \Rightarrow \mathfrak{q}_3 \,|\, \mathfrak{q}_3 \Rightarrow \exists x\forall y\exists z B(x,y,z) \,|\, \mathfrak{q}_3 \Rightarrow \mathfrak{q}_4 \,|\, \mathfrak{q}_4 \Rightarrow \boldsymbol{\exists z B(t_3,a_2,z)} & (\Rightarrow \exists)^3 \\
\,|\, \mathfrak{q}_4 \Rightarrow B(t_3,a_2,t_4) \,|\, \mathfrak{q}_1 \Rightarrow \mathfrak{q}_2 \,|\, \mathfrak{q}_2 \Rightarrow \exists z B(t_1,a_1,z) \,|\, \mathfrak{q}_2 \Rightarrow B(t_1,a_1,t_2) & \\
\hline
\Rightarrow \mathfrak{q}_1 \,|\, \mathfrak{q}_1 \Rightarrow \mathfrak{q}_3 \,|\, \mathfrak{q}_3 \Rightarrow \exists x\forall y\exists z B(x,y,z) \,|\, \mathfrak{q}_3 \Rightarrow \boldsymbol{\exists z B(t_3,a_2,z)} & (\Rightarrow \exists)^3 \\
\,|\, \mathfrak{q}_1 \Rightarrow \mathfrak{q}_2 \,|\, \mathfrak{q}_2 \Rightarrow \exists z B(t_1,a_1,z) \,|\, \mathfrak{q}_2 \Rightarrow B(t_1,a_1,t_2) & \\
\hline
\Rightarrow \mathfrak{q}_1 \,|\, \mathfrak{q}_1 \Rightarrow \mathfrak{q}_3 \,|\, \mathfrak{q}_3 \Rightarrow \exists x\forall y\exists z B(x,y,z) \,|\, \mathfrak{q}_3 \Rightarrow \boldsymbol{\forall y\exists z B(t_3,y,z)} & (\Rightarrow \forall)^3 \\
\,|\, \mathfrak{q}_1 \Rightarrow \mathfrak{q}_2 \,|\, \mathfrak{q}_2 \Rightarrow \exists z B(t_1,a_1,z) \,|\, \mathfrak{q}_2 \Rightarrow B(t_1,a_1,t_2) & \\
\hline
\Rightarrow \mathfrak{q}_1 \,|\, \mathfrak{q}_1 \Rightarrow \boldsymbol{\exists x\forall y\exists z B(x,y,z)} & (\Rightarrow \exists)^3 \\
\,|\, \mathfrak{q}_1 \Rightarrow \mathfrak{q}_2 \,|\, \mathfrak{q}_2 \Rightarrow \exists z B(t_1,a_1,z) \,|\, \mathfrak{q}_2 \Rightarrow B(t_1,a_1,t_2) & \\
\hline
\Rightarrow \mathfrak{q}_1 \,|\, \mathfrak{q}_1 \Rightarrow \exists x\forall y\exists z B(x,y,z) \,|\, \mathfrak{q}_1 \Rightarrow \boldsymbol{\exists z B(t_1,a_1,z)} & (\Rightarrow \exists)^3 \\
\hline
\Rightarrow \mathfrak{q}_1 \,|\, \mathfrak{q}_1 \Rightarrow \exists x\forall y\exists z B(x,y,z) \,|\, \mathfrak{q}_1 \Rightarrow \boldsymbol{\forall y\exists z B(t_1,y,z)} & (\Rightarrow \forall)^3 \\
\hline
\Rightarrow \boldsymbol{\exists x\forall y\exists z B(x,y,z)} & (\Rightarrow \exists)^3
\end{array}
$$

FIGURE 1. The $G^3Ł\forall$-proof search tree $\widetilde{D}_3$

$$
\begin{array}{cl}
\Rightarrow \exists x\forall y\exists z B(x,y,z) \,|\, \Rightarrow \exists z B(t_3,a_2,z) \,|\, \Rightarrow B(t_3,a_2,t_5) \,|\, \Rightarrow B(t_3,a_2,t_4) & \\
\,|\, \Rightarrow \exists z B(t_1,a_1,z) \,|\, \Rightarrow B(t_1,a_1,t_2) & \\
\hline
\Rightarrow \exists x\forall y\exists z B(x,y,z) \,|\, \Rightarrow \boldsymbol{\exists z B(t_3,a_2,z)} \,|\, \Rightarrow B(t_3,a_2,t_4) & (\Rightarrow \exists)^2 \\
\,|\, \Rightarrow \exists z B(t_1,a_1,z) \,|\, \Rightarrow B(t_1,a_1,t_2) & \\
\hline
\Rightarrow \exists x\forall y\exists z B(x,y,z) \,|\, \Rightarrow \boldsymbol{\exists z B(t_3,a_2,z)} \,|\, \Rightarrow \exists z B(t_1,a_1,z) \,|\, \Rightarrow B(t_1,a_1,t_2) & (\Rightarrow \exists)^2 \\
\hline
\Rightarrow \exists x\forall y\exists z B(x,y,z) \,|\, \Rightarrow \boldsymbol{\forall y\exists z B(t_3,y,z)} \,|\, \Rightarrow \exists z B(t_1,a_1,z) \,|\, \Rightarrow B(t_1,a_1,t_2) & (\Rightarrow \forall)^2 \\
\hline
\Rightarrow \boldsymbol{\exists x\forall y\exists z B(x,y,z)} \,|\, \Rightarrow \exists z B(t_1,a_1,z) \,|\, \Rightarrow B(t_1,a_1,t_2) & (\Rightarrow \exists)^2 \\
\hline
\Rightarrow \exists x\forall y\exists z B(x,y,z) \,|\, \Rightarrow \boldsymbol{\exists z B(t_1,a_1,z)} & (\Rightarrow \exists)^2 \\
\hline
\Rightarrow \exists x\forall y\exists z B(x,y,z) \,|\, \Rightarrow \boldsymbol{\forall y\exists z B(t_1,y,z)} & (\Rightarrow \forall)^2 \\
\hline
\Rightarrow \boldsymbol{\exists x\forall y\exists z B(x,y,z)} & (\Rightarrow \exists)^2
\end{array}
$$

FIGURE 2. The $G^2Ł\forall$-proof search tree $\widetilde{D}_2$

Proof. (1) and (2) are equivalent by Theorem 15 in [7]. (3) follows from (1) by Theorem 2. We will show that (3) implies (2).

In view of Theorem 4, it is enough to transform any mid-hypersequent $G^3Ł\forall$-proof D_3 for A into some $G^2Ł\forall$-proof for A. Let $\widetilde{D}_3$ be the $G^3Ł\forall$-proof search tree for A consisting of all quantifier rule applications in D_3; $\mathcal{H}_3$ be the only top hypersequent in $\widetilde{D}_3$; and $\widetilde{\mathcal{H}}_3$ be the hypersequent that is obtained from $\mathcal{H}_3$ by removing all sequents containing quantifiers. Since, in the $G^3Ł\forall$-proof D_3, the hypersequent $\mathcal{H}_3$ is proved using propositional rules only, and axioms of $G^3Ł\forall$ are defined in terms of atomic sequents only, we see that $\vdash_{G^3Ł\forall} \widetilde{\mathcal{H}}_3$.

To avoid cumbersome notation, first we will perform the transformation in the case when A has the form $\exists x\forall y\exists z B(x,y,z)$ (where $B(x,y,z)$ is a quantifier-free RPL$\forall$-formula, and x, y, z are distinct variables), and $\widetilde{D}_3$ has the form given in Figure 1; then we will explain why a similar transformation can be carried out in the general case. The result of simultaneously replacing all occurrences of x, y, z in $B(x,y,z)$ with terms s_1, s_2, s_3, respectively, is denoted by $B(s_1, s_2, s_3)$.

From $\widetilde{D}_3$ we can construct the G^{2}Ł∀-proof search tree $\widetilde{D}_2$ given in Figure 2 by starting with the hypersequent $\Rightarrow A$ and applying the rules $(\Rightarrow \exists)^2$ and $(\Rightarrow \forall)^2$ backward, according to how the rules $(\Rightarrow \exists)^3$ and $(\Rightarrow \forall)^3$ are applied backward in $\widetilde{D}_3$ (such a correspondence between rule applications is natural and is not described for brevity). Let $\mathcal{H}_2$ be the top hypersequent in $\widetilde{D}_2$; and $\widetilde{\mathcal{H}}_2$ be the hypersequent consisting of all quantifier-free sequents of $\mathcal{H}_2$.

To complete our proof in the case being considered, it remains to show that $\vdash_{\mathrm{G^2Ł\forall}} \mathcal{H}_2$. For this, it is sufficient to establish that $\vDash \widetilde{\mathcal{H}}_3$ implies $\vDash \widetilde{\mathcal{H}}_2$. Indeed, $\vdash_{\mathrm{G^3Ł\forall}} \widetilde{\mathcal{H}}_3$ and the soundness of G^{3}Ł∀ (see Theorem 1) guarantee that $\vDash \widetilde{\mathcal{H}}_3$. If we prove that the latter implies $\vDash \widetilde{\mathcal{H}}_2$, then first we will obtain $\vdash_{\mathrm{G^2Ł\forall}} \widetilde{\mathcal{H}}_2$ by the completeness of G^{2}Ł∀ for quantifier-free hypersequents (see Proposition 14 in [7]), and next we will get $\vdash_{\mathrm{G^2Ł\forall}} \mathcal{H}_2$ because the external weakening rule similar to the rule $(\mathrm{ew})^3$ in Lemma 3 is admissible for G^{2}Ł∀.

The hypersequent $\widetilde{\mathcal{H}}_2$ has the form:

$$\Rightarrow B(t_3, a_2, t_5) \mid\, \Rightarrow B(t_3, a_2, t_4) \mid\, \Rightarrow B(t_1, a_1, t_2)\,;$$

and the hypersequent $\widetilde{\mathcal{H}}_3$ has the form:

$$\begin{gathered}\Rightarrow \mathfrak{q}_1 \mid \mathfrak{q}_1 \Rightarrow \mathfrak{q}_3 \mid \mathfrak{q}_3 \Rightarrow \mathfrak{q}_4 \mid \mathfrak{q}_4 \Rightarrow \mathfrak{q}_5 \mid \mathfrak{q}_5 \Rightarrow B(t_3, a_2, t_5) \\ \mid \mathfrak{q}_4 \Rightarrow B(t_3, a_2, t_4) \mid \mathfrak{q}_1 \Rightarrow \mathfrak{q}_2 \mid \mathfrak{q}_2 \Rightarrow B(t_1, a_1, t_2)\,.\end{gathered}$$

For a hypersequent $\mathcal{H}$, we write $\nvDash \mathcal{H}$ to denote that $\mathcal{H}$ is not valid.

The condition $\nvDash \widetilde{\mathcal{H}}_2$ is equivalent to the existence of an hs-interpretation M_2 and a valuation ν_2 such that these three inequalities hold:

$$1 > |B(t_3, a_2, t_5)|_{M_2,\nu_2}, \quad 1 > |B(t_3, a_2, t_4)|_{M_2,\nu_2}, \quad 1 > |B(t_1, a_1, t_2)|_{M_2,\nu_2}.$$

The condition $\nvDash \widetilde{\mathcal{H}}_3$ is satisfied iff there exist an hs-interpretation M_3 and a valuation ν_3 for which all these inequalities hold:

$$\begin{aligned} 1 > |\mathfrak{q}_1|_{M_3,\nu_3} > |\mathfrak{q}_3|_{M_3,\nu_3} > |\mathfrak{q}_4|_{M_3,\nu_3} &> |\mathfrak{q}_5|_{M_3,\nu_3} > |B(t_3, a_2, t_5)|_{M_3,\nu_3}, \\ |\mathfrak{q}_4|_{M_3,\nu_3} &> |B(t_3, a_2, t_4)|_{M_3,\nu_3}, \\ |\mathfrak{q}_1|_{M_3,\nu_3} > |\mathfrak{q}_2|_{M_3,\nu_3} &> |B(t_1, a_1, t_2)|_{M_3,\nu_3}. \end{aligned}$$

Clearly, $\nvDash \widetilde{\mathcal{H}}_2$ implies $\nvDash \widetilde{\mathcal{H}}_3$, as required in the given case.

In the general case, it is obvious that from $\widetilde{D}_3$ we can similarly construct a G^{2}Ł∀-proof search tree $\widetilde{D}_2$. Then the assertion "$\nvDash \widetilde{\mathcal{H}}_2$ implies $\nvDash \widetilde{\mathcal{H}}_3$" follows from the next observation, which is easily justified using induction on the height of the tree $\widetilde{D}_3$.

We can represent the hypersequent $\widetilde{\mathcal{H}}_3$ as a directed acyclic graph by associating, to each sequent member in $\widetilde{\mathcal{H}}_3$, a unique vertex and, to each sequent of the form $F_1 \Rightarrow F_2$, an edge from F_1 to F_2. In this graph, there is exactly one source, and all vertices corresponding to RPL∀-formulas are sinks. The condition $\nvDash \widetilde{\mathcal{H}}_3$ is equivalent to the existence of an hs-interpretation M_3 and a valuation ν_3 such that, for each edge $F_1 \Rightarrow F_2$, the inequality $|F_1|_{M_3,\nu_3} > |F_2|_{M_3,\nu_3}$ holds and so does the inequality $1 > |F|_{M_3,\nu_3}$ for the source F of the graph. □

Corollary 3. *Suppose A is a prenex Ł∀-formula. Then for each $i = 1, 2, 3$ we have: $\vdash_{\mathrm{G^iŁ\forall}} A$ iff $\vdash_{\mathrm{GŁ\forall}} A$.*

Proof. By Theorem 17 given in [7], $\vdash_{\mathrm{G^1Ł\forall}} A$ iff $\vdash_{\mathrm{GŁ\forall}} A$. It remains to apply Theorem 5 stated above. □

Corollary 4. *Let $\mathfrak{S}$ be a signature such that the validity problem for existential sentences of classical logic over $\mathfrak{S}$ is undecidable. Then the* G^{3}Ł∀*-provability problem for existential* Ł∀*-sentences over $\mathfrak{S}$ is undecidable.*

Proof. By Theorem 21 in [7], the corresponding problem for G^{2}Ł∀ is undecidable; and the result follows by Theorem 5 above. □

7. Infinitary hypersequent calculi for Ł∀ and RPL∀

We are to consider infinitary calculi based on GŁ∀ and G^{3}Ł∀. But before this, we give some preliminaries.

A (possibly empty) tuple of terms $t_1, \ldots, t_m$ is also written $\vec{t}$. An RPL∀-formula A is written $A(\vec{x})$ or $A(x_1, \ldots, x_m)$ to mean that all free variables of A are among $x_1, \ldots, x_m$. The result of simultaneously replacing all free occurrences of $x_1, \ldots, x_m$ in an RPL∀-formula $A(\vec{x})$ with terms $t_1, \ldots, t_m$, respectively, is denoted by $A(\vec{t})$. The string $\exists x_1 \ldots \exists x_m$ is abbreviated to $\exists \vec{x}$. We write $\mathsf{Q}\vec{x}$ to mean $\mathsf{Q}_1 x_1 \ldots \mathsf{Q}_m x_m$, where $\mathsf{Q}_i \in \{\forall, \exists\}$ for $i = 1, \ldots, m$.

For a positive integer n, the set $\{1, 2, \ldots, n\}$ is denoted by $1..n$.

By $(A \vee B)$ denote $((A \to B) \to B)$; then

$$|A \vee B|_{M,\nu} = \max(|A|_{M,\nu},\, |B|_{M,\nu}) \quad \text{and} \quad \Big|\bigvee_{i=1}^{n} A_i\Big|_{M,\nu} = \max_{i \in 1..n} |A_i|_{M,\nu}$$

for every interpretation M and every valuation ν (cf., e.g., [5, Section 6.2]).

The Herbrand universe of A is denoted by $\mathcal{U}(A)$ and is, as usual, the set of all terms that are built from the function symbols of A, and a new nullary function symbol if there are no nullary function symbols in A.

A prenex RPL∀-formula $\mathsf{Q}\vec{x}B$, where B is quantifier-free, is called *rectified* if all the variables in $\vec{x}$ are distinct. The (existential) *Skolem form* of a rectified prenex RPL∀-sentence is defined in the usual manner by removing universal quantifiers (cf., e.g., [5, Definition 8.32]).

For a real number r, define $\vDash^{>r} A$ to hold exactly if $|A|_{M,\nu} > r$ for every interpretation M and every valuation ν.

Theorem 6 (the approximate Herbrand theorem for Ł∀; [4, Theorem 3], [5, Theorem 8.38])**.** *Suppose that P is a quantifier-free* Ł∀*-formula, and $\exists\vec{x}P(\vec{x})$ is rectified and closed. Then $\vDash \exists\vec{x}P(\vec{x})$ iff, for all real numbers $r < 1$, there exist tuples $\vec{t}_1, \ldots, \vec{t}_n$ of terms from $\mathcal{U}(P)$ such that*

$$\vDash^{>r} \bigvee_{i=1}^{n} P(\vec{t}_i).$$

The multiset containing exactly n copies of an element E is denoted by $[E]^n$; and the multiset sum of n copies of a multiset $\mathcal{M}$, by $\mathcal{M}^n$. We also write $[E_i]_{i \in 1..n}$ for the multiset sum of $[E_1]^1, \ldots, [E_n]^1$.

Now we state the following theorem claiming that an infinitary calculus based on GŁ∀ is complete for prenex Ł∀-sentences.

Theorem 7 ([5, Theorem 8.48], [13, Theorem 6.1.11])**.** *Let A be a prenex* Ł∀*-sentence. Then* $\vDash A$ *iff A is provable in* GŁ∀ *extended with the rule:*

$$\frac{\bar{0} \Rightarrow [A]^k \quad \text{for all positive integers } k}{\Rightarrow A}.$$

The proof of this theorem in [5, p. 226] (and a similar proof in [13, p. 268]) takes the Skolem form $\exists \vec{x} P^F(\vec{x})$ of a valid prenex Ł∀-sentence A, so $\vDash \exists \vec{x} P^F(\vec{x})$. Then, for each positive integer k, the proof uses Theorem 6 to obtain some tuples $\vec{t}_1, \dots, \vec{t}_n$ of terms from $\mathcal{U}(P^F)$ such that $\vdash_{\text{GŁ}\forall} (\bar{0} \Rightarrow [B]^k)$, where $B = \bigvee_{i=1}^n P^F(\vec{t}_i)$. Finally, the proof employs [5, Lemma 8.34] to derive $(\bar{0} \Rightarrow [A]^k)$ from $(\bar{0} \Rightarrow [B]^k)$ in GŁ∀.

[5, Lemma 8.34] reads in our notation as follows: *Let $\exists \vec{x} P^F(\vec{x})$ be the Skolem form of $\mathsf{Q}\vec{y}P(\vec{y})$, and $\vec{t}_1, \dots, \vec{t}_n$ be tuples of terms from $\mathcal{U}(P^F)$. Then $(\Rightarrow \mathsf{Q}\vec{y}P(\vec{y}))$ is derivable from the hypersequent $(\Rightarrow P^F(\vec{t}_1) \mid \dots \mid \Rightarrow P^F(\vec{t}_n))$ using* (ew), (ec), $(\Rightarrow \forall)$, *and* $(\Rightarrow \exists)$, *where in* $(\Rightarrow \forall)$, *any variable-free term not occurring in the conclusion may be used in the premise.*

Let us point out two errors in the above proof.

First, the bottom hypersequent of any GŁ∀-derivation from $\mathcal{H} = (\bar{0} \Rightarrow [B]^k)$ that does use $\mathcal{H}$ must contain the disjunctions $\bigvee$ from B (or more precisely, the implications $\to$ simulating the disjunctions in B). But A does not contain such disjunctions. So, in general, there is no GŁ∀-derivation of $(\bar{0} \Rightarrow [A]^k)$ from $\mathcal{H}$ in which $\mathcal{H}$ is used.

Second, [5, Lemma 8.34] is not applicable to multiple-conclusion hypersequents, such as $(\bar{0} \Rightarrow [B]^k)$ for $k > 1$. We can try to extend Lemma 8.34 so that the extended lemma is applicable to the hypersequent $(\bar{0} \Rightarrow [B]^k)$. However, a straightforward extension fails even for hypersequents without $\vee$ and $\to$. Indeed, given the Ł∀-sentence $\forall x R(x)$ and its Skolem form $R(a)$ with R being a unary predicate symbol, it is clear that $(\bar{0} \Rightarrow \forall x R(x), \forall x R(x))$ is not derivable from $(\bar{0} \Rightarrow R(a), R(a))$ in GŁ∀, even if in $(\Rightarrow \forall)$, any closed term not occurring in the conclusion may be used in the premise.

We give a correct and more detailed proof of the last theorem.

Proof of Theorem 7. This theorem follows from Proposition 3 and Theorem 8, which are given below. □

Proposition 3. *Suppose that A is an* Ł∀*-formula, and* $\vdash_{\text{GŁ}\forall} (\bar{0} \Rightarrow [A]^k)$ *for all positive integers k. Then* $\vDash A$.

Proof. By the soundness of GŁ∀ (see [5, Theorem 8.46]), $\vDash (\bar{0} \Rightarrow [A]^k)$ for all positive integers k. Then for every interpretation M, every valuation ν, and each positive integer k, we have $-1 \leqslant k \cdot (|A|_{M,\nu} - 1)$, i.e., $1 - 1/k \leqslant |A|_{M,\nu}$. So $\vDash A$. □

Theorem 8. *Suppose A is a prenex* Ł∀*-sentence and* $\vDash A$*. Then* $\vdash_{\text{GŁ}\forall} (\bar{0} \Rightarrow [A]^k)$ *for all positive integers k.*

Proof. **I.** Suppose first that A is rectified.

Consider the Skolem form $\exists \vec{x} B(\vec{x})$ of A, where B is quantifier-free.

Let us show that $\vDash \exists \vec{x} B(\vec{x})$. Since $\vDash A$, it suffices to prove that, in constructing the Skolem form of A, one removal of $\forall$ preserves validity. To this end, suppose that $\vDash \exists \vec{u}\, \forall v A_1(\vec{u}, v)$, where $\vec{u} = \langle u_1, \dots, u_l \rangle$ and $v \notin \{u_1, \dots, u_l\}$, and that the term

$f(\vec{u})$ is free for v in $A_1(\vec{u}, v)$. We want to get $\vDash \exists\vec{u}A_1(\vec{u}, f(\vec{u}))$. For every interpretation M with a domain $\mathcal{D}$ and every valuation ν, we have

$$1 = |\exists\vec{u}\forall v A_1(\vec{u}, v)|_{M,\nu} = \sup_{d_1\in\mathcal{D}} \ldots \sup_{d_l\in\mathcal{D}} \inf_{d\in\mathcal{D}} |A_1(\vec{u}, v)|_{M,\,\widehat{\nu}[v\mapsto d]} \leqslant$$
$$\leqslant \sup_{d_1\in\mathcal{D}} \ldots \sup_{d_l\in\mathcal{D}} |A_1(\vec{u}, v)|_{M,\,\widehat{\nu}[v\mapsto |f(\vec{u})|_{M,\widehat{\nu}}]} = \sup_{d_1\in\mathcal{D}} \ldots \sup_{d_l\in\mathcal{D}} |A_1(\vec{u}, f(\vec{u}))|_{M,\,\widehat{\nu}} =$$
$$= |\exists\vec{u}A_1(\vec{u}, f(\vec{u}))|_{M,\nu},$$

where $\widehat{\nu} = \nu[u_1 \mapsto d_1] \ldots [u_l \mapsto d_l]$. Thus $\vDash \exists\vec{u}A_1(\vec{u}, f(\vec{u}))$; and so $\vDash \exists\vec{x}B(\vec{x})$.

Fix any positive integer k. By Theorem 6, there exist tuples $\vec{t_1}, \ldots, \vec{t_n}$ of terms from $\mathcal{U}(B)$ such that

$$\vDash^{>(1-1/k)} \bigvee_{i=1}^{n} B(\vec{t_i}).$$

Hence the hypersequent

$$\mathcal{H} = \bigl(\bar{0} \Rightarrow [B(\vec{t_1})]^k \bigm| \ldots \bigm| \bar{0} \Rightarrow [B(\vec{t_n})]^k\bigr)$$

is valid.

Next we show that $\vDash \mathcal{H}$ implies $\vdash_{\mathrm{GL}\forall} (\bar{0} \Rightarrow [A]^k)$, extending the technique used in [15, § 55] to prove a version of Herbrand's theorem for classical logic; our main addition to that technique is renamings of several occurrences of the same parameter in a hypersequent to new distinct parameters by means of Lemma 8 (see below).

We assume harmlessly that $\vec{t_i} \neq \vec{t_j}$ for $i \neq j$. Further, to simplify the notation, let

$$A = \exists w\forall x\exists y\forall z C(w, x, y, z),$$

where C is quantifier-free, and $n = 2$; it will be clear how our reasoning extends to the general case. The Skolem form of A is

$$\exists\vec{x}B(\vec{x}) = \exists w\exists y C(w,\, f(w),\, y,\, g(w, y)),$$

where f and g are Skolem function symbols not occurring in A. Then the valid hypersequent $\mathcal{H}$ has the form:

$$\mathcal{H} = \bigl(\bar{0} \Rightarrow [C(t_{11},\, f(t_{11}),\, t_{12},\, g(t_{11}, t_{12}))]^k \bigm| \bar{0} \Rightarrow [C(t_{21},\, f(t_{21}),\, t_{22},\, g(t_{21}, t_{22}))]^k\bigr),$$

where $t_{11}, t_{12}, t_{21}, t_{22} \in \mathcal{U}(B)$.

To each term t that begins with f or g and occurs in the terms shown explicitly in the above expression for $\mathcal{H}$, we associate a unique parameter a_t not occurring in $\mathcal{H}$. Suppose that s is a term occurring in $\mathcal{H}$ or in the terms shown explicitly in the above expression for $\mathcal{H}$;[4] then by $\widetilde{s}$ we denote the result of substituting a_t for each maximal occurrence of each subterm t in s such that t begins with f or g (an occurrence of a subterm starting with f or g is *maximal* if it is not within another such occurrence). E.g.:

if $s = g(f(f(b)), f(b))$, then $\widetilde{s} = a_s$;

if $s = e(f(b), f(f(b)), f(b))$, then $\widetilde{s} = e(a_{f(b)}, a_{f(f(b))}, a_{f(b)})$.

Let $\widetilde{\mathcal{H}}$ come from $\mathcal{H}$ by replacing each formula $R(s_1, \ldots, s_l)$, where R is a predicate symbol and $s_1, \ldots, s_l$ are terms, by $R(\widetilde{s_1}, \ldots, \widetilde{s_l})$.

Given that $\vDash \mathcal{H}$, we claim that $\vDash \widetilde{\mathcal{H}}$. Indeed, $\widetilde{\mathcal{H}}$ and $\mathcal{H}$ are quantifier-free; therefore, $\widetilde{\mathcal{H}}$ (resp. $\mathcal{H}$) is valid iff so is the propositional hypersequent $\widetilde{\mathcal{H}}_{prop}$ (resp. $\mathcal{H}_{prop}$)

[4] Note that, in particular, we are to cover the case when some of w, x, y, z do not occur in $C(w, x, y, z)$, hence the disjunction of the two conditions on s.

obtained from $\widetilde{\mathcal{H}}$ (resp. $\mathcal{H}$) by regarding each non-constant atomic RPŁ∀-formula E as a new propositional variable p_E. In constructing $\widetilde{\mathcal{H}}$ from $\mathcal{H}$, each non-constant atomic RPŁ∀-formula $R(s_1, \dots, s_l)$ is replaced by $R(\widetilde{s_1}, \dots, \widetilde{s_l})$, and nothing else changes. Hence $\widetilde{\mathcal{H}}_{prop}$ results from substituting propositional variables for ones in $\mathcal{H}_{prop}$, which is valid. Thus $\widetilde{\mathcal{H}}_{prop}$ is valid too; and so is $\widetilde{\mathcal{H}}$.

But the calculus GŁ∀ is complete for quantifier-free GŁ∀-hypersequents (see [5, Theorem 6.24]), so $\vdash_{\mathrm{G}Ł\forall} \widetilde{\mathcal{H}}$.

Since f and g are new to A, we have

$$\widetilde{\mathcal{H}} = \big(\bar{0} \Rightarrow [C(\widetilde{t_{11}},\, a_{f(t_{11})},\, \widetilde{t_{12}},\, a_{g(t_{11},t_{12})})]^k \,\big|\, \bar{0} \Rightarrow [C(\widetilde{t_{21}},\, a_{f(t_{21})},\, \widetilde{t_{22}},\, a_{g(t_{21},t_{22})})]^k\big).$$

Recall that $\langle t_{11}, t_{12}\rangle \neq \langle t_{21}, t_{22}\rangle$. Choose the longest term from $g(t_{11}, t_{12})$ and $g(t_{21}, t_{22})$; say it is $g(t_{11}, t_{12})$. (From terms of the same length, any one of them can be taken as the longest.) It is not hard to check that the parameter $a_{g(t_{11},t_{12})}$ occurs in $\widetilde{\mathcal{H}}$ only as shown explicitly in the above expression for $\widetilde{\mathcal{H}}$. By Lemma 8 (which is given below),

$$\vdash_{\mathrm{G}Ł\forall} \big(\bar{0} \Rightarrow [C(\widetilde{t_{11}},\, a_{f(t_{11})},\, \widetilde{t_{12}},\, b_i)]_{i\in 1..k} \,\big|\, \bar{0} \Rightarrow [C(\widetilde{t_{21}},\, a_{f(t_{21})},\, \widetilde{t_{22}},\, a_{g(t_{21},t_{22})})]^k\big),$$

where $b_1, \dots, b_k$ are new distinct parameters. Hence, by k applications of the rule $(\Rightarrow \forall)$ and then by k applications of the rule $(\Rightarrow \exists)$, we get the GŁ∀-provable hypersequent

$$\widetilde{\mathcal{H}}_1 = \big(\bar{0} \Rightarrow [\exists y \forall z C(\widetilde{t_{11}},\, a_{f(t_{11})},\, y,\, z)]^k \,\big|\, \bar{0} \Rightarrow [C(\widetilde{t_{21}},\, a_{f(t_{21})},\, \widetilde{t_{22}},\, a_{g(t_{21},t_{22})})]^k\big).$$

Now choose the longest term from $f(t_{11})$ and $g(t_{21}, t_{22})$; for definiteness it is $g(t_{21}, t_{22})$. So the parameter $a_{g(t_{21},t_{22})}$ occurs in $\widetilde{\mathcal{H}}_1$ only as shown. Using Lemma 8, applying $(\Rightarrow \forall)$ k times, and next applying $(\Rightarrow \exists)$ k times, from $\widetilde{\mathcal{H}}_1$ we obtain the GŁ∀-provable hypersequent

$$\widetilde{\mathcal{H}}_2 = \big(\bar{0} \Rightarrow [\exists y \forall z C(\widetilde{t_{11}},\, a_{f(t_{11})},\, y,\, z)]^k \,\big|\, \bar{0} \Rightarrow [\exists y \forall z C(\widetilde{t_{21}},\, a_{f(t_{21})},\, y,\, z)]^k\big).$$

If $t_{11} = t_{21}$, then the two sequents in $\widetilde{\mathcal{H}}_2$ are the same. In this case, we apply the rule (ec) to $\widetilde{\mathcal{H}}_2$ and get $\vdash_{\mathrm{G}Ł\forall} \big(\bar{0} \Rightarrow [\exists y \forall z C(\widetilde{t_{11}},\, a_{f(t_{11})},\, y,\, z)]^k\big)$; next, k applications of $(\Rightarrow \forall)$ and k applications of $(\Rightarrow \exists)$ give $\vdash_{\mathrm{G}Ł\forall} \big(\bar{0} \Rightarrow [\exists w \forall x \exists y \forall z C(w, x, y, z)]^k\big)$ as required.

Suppose that $t_{11} \neq t_{21}$. As before, from $f(t_{11})$ and $f(t_{21})$ we pick the longest term, say $f(t_{11})$; so $a_{f(t_{11})}$ occurs in $\widetilde{\mathcal{H}}_2$ only as shown. Applying Lemma 8 to $\widetilde{\mathcal{H}}_2$ and then the rules $(\Rightarrow \forall)$ and $(\Rightarrow \exists)$, we obtain the GŁ∀-provable hypersequent

$$\widetilde{\mathcal{H}}_3 = \big(\bar{0} \Rightarrow [\exists w \forall x \exists y \forall z C(w, x, y, z)]^k \,\big|\, \bar{0} \Rightarrow [\exists y \forall z C(\widetilde{t_{21}},\, a_{f(t_{21})},\, y,\, z)]^k\big).$$

Finally, by Lemma 8, $(\Rightarrow \forall)$, and $(\Rightarrow \exists)$, from $\vdash_{\mathrm{G}Ł\forall} \widetilde{\mathcal{H}}_3$ we obtain

$$\vdash_{\mathrm{G}Ł\forall} \big(\bar{0} \Rightarrow [\exists w \forall x \exists y \forall z C(w, x, y, z)]^k \,\big|\, \bar{0} \Rightarrow [\exists w \forall x \exists y \forall z C(w, x, y, z)]^k\big);$$

whence by (ec), we get $\vdash_{\mathrm{G}Ł\forall} \big(\bar{0} \Rightarrow [\exists w \forall x \exists y \forall z C(w, x, y, z)]^k\big)$ as required.

II. Now suppose that A is not rectified.

Let A be of the form $\mathsf{Q}_1 x_1 \dots \mathsf{Q}_m x_m A_0$, where $\mathsf{Q}_i \in \{\forall, \exists\}$ for $i = 1, \dots, m$ and A_0 is quantifier-free. To get a rectified prenex Ł∀-sentence A', we eliminate from A each quantifier occurrence $\mathsf{Q}_i x_i$ such that $x_i = x_j$ and $i < j$ for some j. Clearly, $\vDash A'$. We apply the argument of item **I** to A' and thus have $\vdash_{\mathrm{G}Ł\forall} (\bar{0} \Rightarrow [A']^k)$ for all positive integers k.

Given a GŁ∀-proof of $(\bar{0} \Rightarrow [A']^k)$, we obtain the required GŁ∀-proof of $(\bar{0} \Rightarrow [A]^k)$ by restoring the eliminated quantifier occurrences and inserting applications of the rules $(\Rightarrow \forall)$ and $(\Rightarrow \exists)$ for them. Indeed, such insertions can be easily performed, because in such an application of $(\Rightarrow \mathsf{Q})$ (where Q is a quantifer), the premise has the form $\mathcal{G} \,|\, \Gamma \Rightarrow B, \Delta$, and the conclusion $\mathcal{G} \,|\, \Gamma \Rightarrow \mathsf{Q}xB, \Delta$. □

Lemma 8. *Suppose that k is a positive integer; b is a parameter; $A(b)$ is an Ł∀-formula;*

$$\mathcal{H}(b) = \big(\mathcal{G} \,\big|\, \Gamma \Rightarrow [A(b)]^k, \Delta\big);$$

$\vdash_{\mathrm{G}Ł\forall} \mathcal{H}(b)$*; b does not occur in $\mathcal{G}, \Gamma, \Delta$; $b_1, \ldots, b_k$ are distinct parameters none of which occurs in $\mathcal{H}(b)$; and $A(b_i)$ comes from $A(b)$ by substituting b_i for each occurrence of b. Then*

$$\vdash_{\mathrm{G}Ł\forall} \big(\mathcal{G} \,\big|\, \Gamma \Rightarrow [A(b_i)]_{i\in 1..k}, \Delta\big).$$

Proof. It is readily seen that $\vdash_{\mathrm{G}Ł\forall} \mathcal{H}(b_i)$ for each $i = 1, \ldots, k$. Using the rule (mix) $k-1$ times, from $\mathcal{H}(b_1), \ldots, \mathcal{H}(b_k)$ we obtain

$$\mathcal{G} \,\big|\, \Gamma^k \Rightarrow [A(b_1)]^k, \ldots, [A(b_k)]^k, \Delta^k.$$

Then applying (split) $k-1$ times gives

$$\mathcal{G} \,\big|\, [\Gamma \Rightarrow A(b_1), \ldots, A(b_k), \Delta]^k.$$

Finally, by (ec) $k-1$ times, we get

$$\mathcal{G} \,\big|\, \Gamma \Rightarrow A(b_1), \ldots, A(b_k), \Delta, \quad \text{or briefly} \quad \mathcal{G} \,\big|\, \Gamma \Rightarrow [A(b_i)]_{i\in 1..k}, \Delta.$$

□

Now we extend our proof of Theorem 7 to establish the same theorem for a prenex RPL∀-sentence and the calculi G^1Ł∀ and G^3Ł∀.

Theorem 9. *Let A be a prenex* RPL∀*-sentence and $\mathfrak{C}$ be either* G^1Ł∀ *or* G^3Ł∀. *Then $\vDash A$ iff A is provable in $\mathfrak{C}$ extended with the rule:*

$$\frac{\bar{0} \Rightarrow [A]^k \quad \textit{for all positive integers } k}{\Rightarrow A}.$$

Proof. The above proof of Theorem 7, which consists of the proofs of Proposition 3 and Theorem 8, carries over to Theorem 9 in the following way.

1. The proof of Proposition 3 with an RPL∀-formula A in place of an Ł∀-formula A and with $\mathfrak{C}$ in place of GŁ∀ is essentially the same, because the calculi G^1Ł∀ and G^3Ł∀ are sound (see [7, Theorem 3] and Theorem 1).

2. To employ the proof of Theorem 8 for the case of a valid prenex RPL∀-sentence A and the calculus G^1Ł∀, the following items 2.1–2.3 are sufficient.

2.1. The approximate Herbrand theorem for RPL∀, i.e., Theorem 6 for a rectified existential RPL∀-sentence $\exists\vec{x}P(\vec{x})$, holds. Indeed, the proof of Theorem 8.38 in [5] (i.e., of the approximate Herbrand theorem for Ł∀) does for RPL∀: truth constants $\bar{r}$ are treated just as $\bar{0}$ ($\bar{0}$ is denoted by $\perp$ in [5]).

2.2. The calculus G^1Ł∀ is complete for quantifier-free G^1Ł∀-hypersequents $\mathcal{H}$, i.e., $\vDash \mathcal{H}$ implies $\vdash_{\mathrm{G}^1Ł\forall} \mathcal{H}$ (see Propositions 14 and 11 in [7]). Here a G^1Ł∀-*hypersequent* is a hypersequent containing no semipropositional variables of type 0.

2.3. The rules that look like the rules[5] (mix), (split), (ec), ($\Rightarrow \forall$), and ($\Rightarrow \exists$) of GŁ∀ but have G^1Ł∀-hypersequents as their premises and conclusions are admissible for G^1Ł∀. Let us justify the claim. Denote these rules by $\mathcal{R}_{\text{mix}}$, $\mathcal{R}_{\text{split}}$, $\mathcal{R}_{\text{ec}}$, $\mathcal{R}_{\Rightarrow\forall}$, and $\mathcal{R}_{\Rightarrow\exists}$, respectively. The rule $\mathcal{R}_{\text{mix}}$ is admissible for G^1Ł∀ by [7, Lemma 8]; $\mathcal{R}_{\text{split}}$, by [7, Lemma 7]; and $\mathcal{R}_{\text{ec}}$, by [7, Lemma 5]. The rules $\mathcal{R}_{\Rightarrow\forall}$ and $\mathcal{R}_{\Rightarrow\exists}$ are obviously admissible for G^1Ł∀ (cf. the proof of Theorem 4 in [7]).

3. Finally, Theorem 8 for a valid prenex RPL∀-sentence A and the calculus G^3Ł∀ follows from the same theorem for G^1Ł∀ (see item 2 of the current proof) and Theorem 2. □

8. Conclusion

For the logics Ł∀ and RPL∀, we presented the analytic hypersequent calculus G^3Ł∀, whose rules are repetition-free; established its key proof-theoretic properties pertinent to proof search; and thus provided foundations for various bottom-up proof search algorithms.

Recall that the previously known (finitary) analytic calculi for Ł∀ or RPL∀ are the hypersequent calculi GŁ∀, G^1Ł∀, and G^2Ł∀. They are not repetition-free, and GŁ∀ has structural rules. But the calculus G^3Ł∀ is repetition-free and does not have structural rules; moreover, all of its inference rules are height-preserving invertible. So, for bottom-up proof search, it has a clear advantage over the other mentioned calculi. We also showed that G^3Ł∀ proves any sentence provable in GŁ∀, or G^1Ł∀, or G^2Ł∀.

Now we are especially interested in whether every G^3Ł∀-provable Ł∀-sentence is GŁ∀-provable, and in relationships between G^3Ł∀ and Hájek's Hilbert-type calculus for RPL∀ from [1] (some preliminary results are in [18]).

Another interesting problem is to describe in syntactic terms nontrivial classes of hypersequent calculi that have some or all of the proof-theoretic properties established for G^3Ł∀. Our way of proving the properties of the calculus G^3Ł∀ relies on its being repetition-free and thus provides a clue to a generalization. Cf., e.g., [19] and [20], which give sufficient conditions for several properties of some sequent calculi, in particular, for the invertibility of inference rules.

In the present paper, we also pointed out errors in the earlier proofs of the completeness of the GŁ∀-based infinitary calculus for prenex Ł∀-sentences; we gave the new correct proof of the claim, and extended the proof to show the completeness of the G^3Ł∀-based infinitary calculus for prenex RPL∀-sentences.

The paper [4] gives a proof that the GŁ∀-based infinitary calculus is complete for arbitrary Ł∀-sentences. However, we found significant gaps and an error in the proof; G. Metcalfe, an author of [4], confirmed our findings. Thus it is an open question whether the GŁ∀-based (resp. G^3Ł∀-based) infinitary calculus is complete for arbitrary Ł∀-sentences (resp. RPL∀-sentences).

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