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EFFICIENT COMPUTATION OF FAVARD CONSTANTS AND THEIR CONNECTION TO EULER POLYNOMIALS AND NUMBERS

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Dedicated to the memory of my teacher V. L. Miroshnichenko

Abstract. We discuss problems of calculating the Favard constants, which are often used in approximation theory and their connection to Euler numbers and polynomials. Simple effective recurrence formulas for computation of the Favard constants are found. The application of the results to one problem of extremal functional interpolation allowing the solution to be expressed in an explicit form is demonstrated.

Keywords: Favard constants, Euler numbers, Euler polynomials, recurrence formulas, approximation theory.

1. Introduction

In approximation theory, when solving extremal problems, obtaining error estimates for approximation, and in a number of other problems, we often have to deal [1–3] with comparison functions of the form

$$\varphi_n(x) = \frac{4}{\pi^{n+1}} \sum_{\nu=0}^{\infty} \frac{\sin[(2\nu+1)\pi x - \pi n/2]}{(2\nu+1)^{n+1}}, \quad n = 0, 1, \ldots, \tag{1}$$

called *Euler perfect splines* as they are explicitly defined by means of Euler polynomials. Apparently, Schoenberg [4, 5] was the first person to call the function $\varphi_n(x)$

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an Euler spline, however his spline $\mathcal{E}_n(x)$ differed in its normalization and shift by $1/2$ for even n. As Cavaretta [6] reports, the definition of *perfect splines*, that is, such that the values of their higher derivatives on every interval of the mesh are equal in their absolute value but differ in signs on adjacent intervals, was introduced by Glaeser [7] in 1967. But the functions (1) were first used in 1937 by Favard [8], and also by Akhiezer and Krein [9] to construct a linear method of approximation in the space of trigonometric polynomials. Due to that fact, they are sometimes referred to as the Akhiezer–Krein–Favard, or just Favard, functions or sums (see, e.g., [2, 10, 11]).

The final results of research and the estimates are generally written using the values of these functions at some points, or their norms, expressed via the Favard constants

$$\mathcal{K}_n = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu(n+1)}}{(2\nu+1)^{n+1}} = \pi^n\|\varphi_n\|_\infty. \tag{2}$$

In 1939 when solving an important expremal problem by Kolmogorov [12], it turned out that the functions $\varphi_n(x)$ were extremal in the inequality between uniform norms on the axis of derivatives of differentiable functions, and also in the comparison theorem. Error estimates for approximation by interpolation splines and derivatives with accurate constants are expressed via the Favard constants (see [13]).

The Favard constants can be explicitly represented by means of values of Euler polynomials, Bernoulli and Euler numbers. The main aim of this article is to put together such kind of formulas, and also to provide simple recurrent formulas with a finite number of summands for the Favard constants. A few of results presented here has been announced in paper [14].

2. Euler perfect splines

It is well-known (see, e.g., [15, 1.14], [16, 5.4.6]) that given $0 \le x \le 1$, function (1) coincides up to a factor with the Euler polynomial $E_n(x)$ of power n, in particular,

$$\varphi_n(x) = \frac{1}{n!}E_n(x). \tag{3}$$

Recall [15, 1.14] that for Euler polynomials, $2e^{xt}/(e^t+1)$ is a generating function. Its expansion into power series in a variable t has the form

$$\frac{2e^{xt}}{e^t+1} = \sum_{k=0}^{\infty}E_k(x)\frac{t^k}{k!}.$$

According to [17, 24.6.8], Euler polynomials can also be defined using explicit formulas:

$$E_n(x) = \frac{1}{2^n}\sum_{k=1}^{n+1}\sum_{j=0}^{k-1}(-1)^j\binom{n+1}{k}(x+j)^n. \tag{4}$$

Function (1) is periodic of period 2, and with $1 \le x \le 2$, the equality

$$\varphi_n(x) = -\varphi_n(x-1)$$

holds, so the function is extended in a periodic way beyond the segment $[0,2]$. It is well-known [18] that $E_0(x) \equiv 1$, and with $n \ge 1$, the equality

$$E_n(0) = -E_n(1) \tag{5}$$

is valid, which with the property

$$E_n'(x) = nE_{n-1}(x) \tag{6}$$

allows us to state that function (1) at the gluing points $x = 0$ and $x = 1$ (and at other ones with respect to periodicity) has a C^{n-1} smoothness, and the higher derivative $\varphi_n^{(n)}(x)$ equals 1 on the interval $(0,1)$ and -1 on the adjacent interval $(1,2)$. Therefore, it is a cardinal spline of degree n of minimal defect 1 with knots at integer points of a number line.

Hence, function (1) which we are interested in, being an Euler perfect spline, can be explicitly expressed using Euler polynomial of a corresponding degree. The polynomials $E_n(x)$ are sufficiently well studied, and with even value $n = 2m$ they obtain zero values at the extreme points, that is,

$$E_{2m}(0) = E_{2m}(1) = 0, \quad m \geq 1,$$

and are strongly convex, reaching their maximum and minimum at the midpoint. With odd $n = 2m-1$, the Euler polynomial $E_{2m-1}(x)$ reaches its largest and smallest values at its boundaries, being a monotonic function on the segment $[0,1]$.

3. Favard constants

According to (2) and (3), the Favard constants are the highest values of Euler polynomials on the segment [0,1], multiplied by $\pi^n/n!$, that is,

$$\mathcal{K}_n = \frac{\pi^n}{n!}\|E_n\|_\infty,$$

then from the mentioned properties of Euler polynomials we infer that

$$\mathcal{K}_{2m-1} = (-1)^m \pi^{2m-1} \varphi_{2m-1}(0) = \frac{(-1)^m \pi^{2m-1}}{(2m-1)!} E_{2m-1}(0), \tag{7}$$

$$\mathcal{K}_{2m} = (-1)^m \pi^{2m} \varphi_{2m}\left(\frac{1}{2}\right) = \frac{(-1)^m \pi^{2m}}{(2m)!} E_{2m}\left(\frac{1}{2}\right). \tag{8}$$

According to [18], we can write

$$\mathcal{K}_{2m-1} = \frac{(-1)^{m-1}\pi^{2m-1} 2(2^{2m}-1)}{(2m)!} B_{2m},$$

$$\mathcal{K}_{2m} = \frac{(-1)^m \pi^{2m}}{2^{2m}(2m)!} E_{2m},$$

where B_k are Bernoulli numbers, the values of Bernoulli polynomials $B_k(x)$ as $x = 0$, E_k are Euler numbers defined by the values of the polynomials $E_k(x)$ at the point $x = 1/2$, in particular,

$$E_k = 2^k E_k\left(\frac{1}{2}\right). \tag{9}$$

It is well-known [15, 18] that all Euler numbers with odd indices equal 0, and the even-indexed numbers are integers and strongly alternating:

$$E_0 = 1, \;\; E_2 = -1, \;\; E_4 = 5, \;\; E_6 = -61, \;\; \ldots$$

Bernoulli numbers with odd indices also equal zero (except for B_1), and the ones with even indices also have alternating signs, but are rational fractions:

$$B_0 = 1,\ \ B_1 = -\frac{1}{2},\ \ B_2 = \frac{1}{6},\ \ B_4 = -\frac{1}{30},\ \ \ldots$$

This circumstance allows us to write the several first Favards constants, that is,

$$\mathcal{K}_0 = 1,\ \ \mathcal{K}_1 = \frac{\pi}{2},\ \ \mathcal{K}_2 = \frac{\pi^2}{8},\ \ \mathcal{K}_3 = \frac{\pi^3}{24},\ \ \mathcal{K}_4 = \frac{5\pi^4}{384},$$

moreover [1, p.103], the inequalities

$$1 = \mathcal{K}_0 < \mathcal{K}_2 < \ldots < \frac{4}{\pi} < \ldots < \mathcal{K}_3 < \mathcal{K}_1 = \frac{\pi}{2} \tag{10}$$

are valid.

4. Calculation of Favard constants via Euler numbers

Our motivation for writing this article includes the fact that the majority of papers concerning in some way the Favard constants do not mention their connection to Bernoulli and Euler numbers. Even the authors of monographs on approximation theory limit themselves to formula (2) which defines the Favard constants by means of infinite series and, possibly, with the chain of inequalities (10) (see, for example, [1, 3, 19]). As a result, some researchers do not understand whether values of the function $\varphi_n(x)$ can be found at particular points, or whether the Favard constants can be obtained without summation of the series.

We have already mentioned that the Favard constants can be expressed via Bernoulli and Euler numbers, and for these numbers there are explicit (finite) and recurrent formulas. Hence, the explicit or recurrent formulas can be written for the Favard constants as well. In 2001, during the conference «Methods of spline functions», dedicated to the memory of Yu. S. Zavialov, V. L. Miroshnichenko in his report informed [20] that he had obtained simple recurrent formulas for defining the Favard constants.

Theorem 1 ([20]). *The following equalities hold:*

$$\mathcal{K}_n = \frac{\pi^n}{2^n n!}|T_n|, \qquad n = 0, 1, \ldots,$$

where T_n are integers defined via the recurrent formula

$$T_n = 1 - \sum_{k=1}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k} T_{n-2k}, \quad n = 3, 4, \ldots; \qquad T_0 = T_1 = T_2 = 1$$

Here and after, the square brackets in the upper limit of summation mark the integer part of a number.

Unfortunately, the proof of this theorem was not published until the passing of V. L. Miroshnichenko.

Lemma 1. *The following equalities hold:*

$$E_n(0) = \frac{(-1)^n}{2^n} \sum_{k=0}^{n} \binom{n}{k} E_k, \qquad n = 0, 1, \ldots \tag{11}$$

Proof directly follows due to the well-known [15, 18] representation of the Euler polynomial:

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}.$$

Theorem 2. *The following equalities hold:*

$$\mathcal{K}_n = \frac{(-1)^{\left[\frac{n-1}{2}\right]}}{n!} \left(\frac{\pi}{2}\right)^n \sum_{k=0}^{n-1} \binom{n}{k} E_k = \frac{1}{n!} \left(\frac{\pi}{2}\right)^n \left| \sum_{k=0}^{n-1} \binom{n}{k} E_k \right|, \quad n = 1, 2, \ldots \tag{12}$$

Proof. Since given even $n = 2m$, $n > 0$, the value of the polynomial $E_n(0)$ equals 0, Lemma 1 provides us with a recurrent formula for calculating even-indexed Euler numbers (the odd-indexed ones equal 0)

$$E_{2m} = -\sum_{k=0}^{m-1} \binom{2m}{2k} E_{2k}, \qquad m = 1, 2, \ldots \tag{13}$$

Taking into account (9) from (8), we obtain

$$\mathcal{K}_{2m} = \frac{(-1)^{m-1}}{(2m)!} \left(\frac{\pi}{2}\right)^{2m} \sum_{k=0}^{m-1} \binom{2m}{2k} E_{2k}, \qquad m = 1, 2, \ldots$$

In the case when $n = 2m - 1$, again from formula (11) we have that

$$E_{2m-1}(0) = -\frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m-1}{2k} E_{2k}, \qquad m = 1, 2, \ldots, \tag{14}$$

so we insert this expression into (7), and as a result we obtain

$$\mathcal{K}_{2m-1} = \frac{(-1)^{m-1}}{(2m-1)!} \left(\frac{\pi}{2}\right)^{2m-1} \sum_{k=0}^{m-1} \binom{2m-1}{2k} E_{2k}, \qquad m = 1, 2, \ldots$$

Combining the expressions for even- and odd-indexed Favard constants, we ascertain that the theorem is valid. □

Note that in formula (12) from Theorem 2 under the summation sign there are summands with Euler numbers with odd indices which equal 0, hence, this formula can be written in the form which does not include these zero summands by only keeping the terms with even Euler numbers.

Corollary 1. *The following equalities hold:*

$$\mathcal{K}_n = \frac{(-1)^{\left[\frac{n-1}{2}\right]}}{n!} \left(\frac{\pi}{2}\right)^n \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k} E_{2k}, \qquad n = 1, 2, \ldots$$

Theorem 2 provides us with a simple formula (12) for direct calculation of Favard constants by means of Euler numbers without summation of infinite series. And the Euler numbers themselves can be calculated using recurrent formula (13).

Compare formula (12) for Favard constants via Euler numbers to the formula proposed by V. L. Miroshnichenko in Theorem 1. A recurrent formula for the numbers T_n always contain only terms of similar parity. Then for even indices it can be written in the form

$$T_{2m} = 1 - \sum_{k=1}^{m-1} \binom{2m}{2k} T_{2m-2k}, \quad m = 2, \ldots; \qquad T_0 = T_2 = 1.$$

Note that the number T_0 is not used in the recurrent formula, but is only needed for calculation of K_0, where only the absolute value is employed, hence, the initial sign of T_0 does not affect anything, and we may assume that $T_0 = -1$. Such modification allows us to rewrite the last recurrent formula in the form

$$T_{2m} = -T_0 - \sum_{k=1}^{m-1} \binom{2m}{2k} T_{2m-2k}, \quad m = 2, \ldots; \qquad T_0 = -1.$$

Now the obtained formula coincides with formula (13), however, the calculations start with numbers of different signs. Therefore, we have that $T_{2m} = -E_{2m}$ (taking into account the change in the sign of T_0).

The Favard constants $\mathcal{K}_n$ actually constitute the values of the maxima (with a coefficient) of the Euler perfect splines $\varphi_n(x)$, that is, the extremal values of the Euler polynomials $E_n(x)$ on the segment $[0, 1]$. Moreover, with even value of n, the Favard constant is expressed using the Euler number E_n, and with an odd one both maximum and minimum get displaced to the borders of the segment (and differ in their signs), hence, it seems natural to consider an analogue to the Euler numbers, shifted to the point $x = 0$.

In a way similar to (9), we introduce the numbers

$$E_n^* = 2^n E_n(0), \tag{15}$$

to which we will refer to as *shifted Euler numbers*. These numbers are quite similar in their properties to the regular Euler numbers E_n.

$$E_0^* = 1, \ \ E_1^* = -1, \ \ E_3^* = 2, \ \ E_5^* = -16, \ \ E_7^* = 272, \ \ \ldots$$

These numbers only differ from Bernoulli numbers in their factor [17, 24.4.26]

$$E_n^* = \frac{2^{n+1}}{n+1}(1 - 2^{n+1}) B_{n+1},$$

so a number of researchers conduct studies of their problems in terms of Bernoulli numbers. For us it is more convenient to introduce dissimilar numbers which in their properties are close to regular Euler numbers. Such numbers along with regular Euler numbers are sometimes considered in classic literature as well [21, p.27]. As can be noticed in the following description of the properties, they also include only integers, and all the even-indexed ones (except for E_0^*) turn into 0.

Lemma 2. *With $n = 0, 1, \ldots$ the following equalities hold:*

$$E_n^* = (-1)^n \sum_{k=0}^{n} \binom{n}{k} E_k, \tag{16}$$

$$E_n = \sum_{k=0}^{n} \binom{n}{k} E_k^*. \tag{17}$$

Proof. The first equality is proved in Lemma 1, and we obtain the second one directly from the expansion formula [18, 23.1.7]

$$E_n(x+h) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) h^{n-k}$$

given $x = 0$ and $h = 1/2$. □

Note that under the summation sign in equality (17) all the summands with even indices except for $k = 0$ equal zero, hence, this equality can be rewritten in the form

$$E_n = 1 + \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} E^*_{2k-1}. \tag{18}$$

Taking into account that $E_{2m-1} = 0$, we arrive at a recurrent formula

$$E^*_{2m-1} = -1 - \sum_{k=1}^{m-1} \binom{2m-1}{2k-1} E^*_{2k-1}. \tag{19}$$

Comparing this formula to the formula from Theorem 1 for the odd T_n, we notice that they coincide for $T_{2m-1} = -E^*_{2m-1}$.

Thus, we obtain the proof of Miroshnichenko's [20] theorem. Actually, it provides formulas (7) and (8), and recurrent formulas (13) and (19) are added for the regular and shifted Euler numbers E_n and E^*_n (more precisely, his values T_n differ in their sign from these numbers). We can restate this theorem in terms of Euler numbers.

Theorem 3. *The following equalities hold:*

$$\mathcal{K}_{2m-1} = \frac{(-1)^m}{(2m-1)!} \left(\frac{\pi}{2}\right)^{2m-1} E^*_{2m-1}, \qquad m = 1, 2, \ldots,$$

$$\mathcal{K}_{2m} = \frac{(-1)^m}{(2m)!} \left(\frac{\pi}{2}\right)^{2m} E_{2m}, \qquad m = 0, 1, \ldots$$

Note that Theorem 3 demonstrates that Favard constants up to a factor are just Euler numbers (regular or shifted, depending on their parity). And Lemma 2 provides us with simple recurrent formulas for the numbers E_n and E^*_n, both via the regular Euler numbers E_n and the shifted ones E^*_n.

Note that equalities (17) and (19) allow us to express extrema of Euler polynomials of any degree using shifted Euler numbers. Therefore, we can infer the next

Theorem 4. *The following equalities hold:*

$$\mathcal{K}_n = \frac{(-1)^{\left[\frac{n}{2}\right]}}{n!} \left(\frac{\pi}{2}\right)^n \sum_{k=0}^{n-1} \binom{n}{k} E^*_k = \frac{1}{n!} \left(\frac{\pi}{2}\right)^n \left| \sum_{k=0}^{n-1} \binom{n}{k} E^*_k \right|, \quad n = 1, 2, \ldots \tag{20}$$

Under the summation sign in equality (20) from Theorem 4, the summands with even indices (except for $k = 0$) turn into 0, hence, the equality can be written only using nonzero summands.

Corollary 2. *The following equalities hold:*

$$\mathcal{K}_n = \frac{(-1)^{\left[\frac{n}{2}\right]}}{n!}\left(\frac{\pi}{2}\right)^n \left[1 + \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{n}{2k-1} E^*_{2k-1}\right], \qquad n = 0, 1, \ldots$$

5. Favard constants via Bernoulli and Genocchi numbers

Earlier we mentioned the formula for odd-indexed Favard constants which used Bernoulli numbers. Apparently, there are also uniform formulas which employ Bernoulli numbers for all Favard constants.

Theorem 5. *The following equalities hold*

$$\mathcal{K}_n = \frac{(-1)^{\left[\frac{n}{2}\right]}}{(n+1)!}\left(\frac{\pi}{2}\right)^n \sum_{k=1}^{n} \binom{n+1}{k} 2^k(1-2^k)B_k, \quad n = 1, 2, \ldots \tag{21}$$

Proof directly follows from (15), from a well-known formula [18, 23.1.20], connecting the shifted Euler numbers (or the values of Euler polynomials given $x = 0$) to Bernoulli numbers

$$E^*_{n-1} = 2^{n-1}E_{n-1}(0) = \frac{2^n}{n}(1-2^n)B_n,$$

and from Theorem 4.

Similarly to what we did before, we can keep only nonzero summands in formula (21).

Corollary 3. *The following equalities hold:*

$$\mathcal{K}_n = \frac{1}{n!}\left(\frac{\pi}{2}\right)^n \left|1 + \frac{1}{n+1} \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{n+1}{2k} 2^{2k}(1-2^{2k})B_{2k}\right|, \qquad n = 0, 1, \ldots$$

For Bernoulli numbers, equalities (see [18]), similar to the equalities from Lemma 2 hold, namely,

$$(-1)^n B_n = \sum_{k=0}^{n} \binom{n}{k} B_k,$$

which provides us with a possibility to calculate them recurrently. Recall that all odd-indexed Bernoulli numbers (except for B_1) equal 0.

In the literature (see [22]), well-known are Genocchi polynomials $G_n(x)$, which are actually the same Euler polynomials, but with a different normalization, that is,

$$G_n(x) = nE_{n-1}(x),$$

and Genocchi numbers G_n are defined in a way similar to Bernoulli numbers,

$$G_n = G_n(0), \qquad G_0 = 0,$$

therefore,

$$G_n = 2(1-2^n)B_n = nE_{n-1}(0) = n2^{1-n}E^*_{n-1}.$$

These equalities and Theorem 5 allow us to directly write the formulas for Favard constants using Genocchi numbers.

Theorem 6. *The following equalities hold:*

$$\mathcal{K}_n = \frac{(-1)^{\left[\frac{n}{2}\right]}}{(n+1)!}\left(\frac{\pi}{2}\right)^n \sum_{k=1}^{n} \binom{n+1}{k} 2^{k-1} G_k, \quad n = 1, 2, \ldots$$

For Genocchi polynomials, the identity

$$G_n(x+h) = \sum_{k=0}^{n} \binom{n}{k} G_k(x) h^{n-k},$$

is valid [22], which infers the identity for Genocchi numbers

$$(-1)^n G_n = \sum_{k=0}^{n} \binom{n}{k} G_k,$$

allowing us to calculate them recurrently. Other relations for Genocchi numbers can be found in paper [23]. Here we provide several first nonzero Genocchi numbers:

$$G_1 = 1, \quad G_2 = -1, \quad G_4 = 1, \quad G_6 = -3, \quad G_8 = 17, \ldots$$

6. Recurrent formulas for Favard constants

Above we have provided the formulas for calculation of Favard constants by means of any Euler, Bernoulli, or Genocchi numbers. These special numbers are widespread and can be found in many subfields of mathematics and other disciplines. They are studied in depth, and a lot of formulas exist for their calculation. For example, the explicit representation of Euler polynomial (4) together with formulas (7) and (8) directly provide explicit formulas for Favard constants:

$$\mathcal{K}_n = \frac{1}{2^n}\left(\frac{\pi}{2}\right)^n \sum_{k=1}^{n+1}\sum_{j=0}^{k-1} (-1)^{j+\left[\frac{n+1}{2}\right]} \binom{n+1}{k} \left(j + \frac{n+1}{2} - \left[\frac{n+1}{2}\right]\right)^n.$$

However, pure recurrent formulas which can allow us to calculate Favard constants without engaging some special numbers can apparently be of not less interest. Representations (7) and (8) allow us to reword Theorems 2 and 4 (more precisely, their corollaries) by substituting in them Euler and shifted Euler numbers by even- and odd-indexed Favard constants.

Theorem 7. *The equalities*

$$\mathcal{K}_n = \left| \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^k}{(n-2k)!}\left(\frac{\pi}{2}\right)^{n-2k} \mathcal{K}_{2k} \right|, \quad n = 1, 2, \ldots, \tag{22}$$

hold, where $\mathcal{K}_0 = 1$.

Theorem 8. *The equalities*

$$\mathcal{K}_n = \left| \frac{1}{n!}\left(\frac{\pi}{2}\right)^n + \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{(-1)^k}{(n-2k+1)!}\left(\frac{\pi}{2}\right)^{n-2k+1} \mathcal{K}_{2k-1} \right|, \quad n = 1, 2, \ldots, \tag{23}$$

hold, where $\mathcal{K}_0 = 1$.

Formulas (22), (23) from Theorems 7 and 8 have been partially obtained in paper [24], however, in that study, Favard constants with even indices are calculated using different constants also only with even indices, and vise versa — the ones with odd indices are only calculated by means of constants with odd indices. Our formulas are of a general form and they are uniform for even and odd indices.

Note that the formulas in Theorems 7 and 8 can be combined in one formula.

Corollary 4. *The following equalities hold:*

$$\mathcal{K}_0 = 1, \qquad \mathcal{K}_n = \frac{1}{2}\left|\frac{1}{n!}\left(\frac{\pi}{2}\right)^n - \sum_{k=0}^{n-1}\frac{(-1)^{n(k+1)+\left[\frac{k}{2}\right]}}{(n-k)!}\left(\frac{\pi}{2}\right)^{n-k}\mathcal{K}_k\right|, \quad n = 1, 2, \dots$$

All the formulas for Favard constants stated earlier are not very complicated, and after several calculations they provide us with explicit values for the required constants (a computer program can be created). However, it turns out that we can obtain even more simple recurrent formulas.

Theorem 9. *The following equalities hold*

$$\mathcal{K}_0 = 1,\ \mathcal{K}_1 = \frac{\pi}{2}, \qquad \mathcal{K}_n = \frac{\pi}{2n}\sum_{k=1}^{\left[\frac{n}{2}\right]}\mathcal{K}_{2k-1}\mathcal{K}_{n-2k}, \quad n = 2, 3, \dots, \tag{24}$$

$$\mathcal{K}_0 = 1, \qquad \mathcal{K}_n = \frac{\pi}{2n}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\mathcal{K}_{2k}\mathcal{K}_{n-1-2k}, \quad n = 1, 2, \dots \tag{25}$$

Proof. We will use a quadratic recurrent relation [17, 24.14.3] for Euler polynomials:

$$\sum_{k=0}^{n}\binom{n}{k}E_k(h)E_{n-k}(x) = 2E_{n+1}(x+h) + 2(1-x-h)E_n(x+h). \tag{26}$$

Suppose that $x = h = 0$, then we obtain the identity

$$\sum_{k=0}^{n}\binom{n}{k}E_k^*E_{n-k}^* = E_{n+1}^* + 2E_n^*,$$

which will be meaningful only given even $n = 2m$, in particular,

$$E_{2m+1}^* = \sum_{k=1}^{m}\binom{2m}{2k-1}E_{2k-1}^*E_{2m-2k+1}^*, \quad m \geq 1. \tag{27}$$

Now let $x = 0$ and $h = 1/2$ in (26). We obtain the equality

$$\sum_{k=0}^{n}\binom{n}{k}E_k^*E_{n-k} = E_{n+1} + E_n,$$

which only interests us given odd $n = 2m - 1$. We have that

$$E_{2m} = \sum_{k=1}^{m}\binom{2m-1}{2k-1}E_{2k-1}^*E_{2m-2k}, \quad m \geq 1. \tag{28}$$

According to the equalities from Theorem 3, shifted and regular Euler numbers can be expressed via Favard constants:

$$E^*_{2m-1} = (-1)^m (2m-1)! \left(\frac{2}{\pi}\right)^{2m-1} \mathcal{K}_{2m-1}, \qquad m = 1, 2, \ldots,$$

$$E_{2m} = (-1)^m (2m)! \left(\frac{2}{\pi}\right)^{2m} \mathcal{K}_{2m}, \qquad m = 0, 1, \ldots$$

Placing them into identity (27) and (28), we arrive at the relations

$$\mathcal{K}_{2m-1} = \frac{\pi}{2(2m-1)} \sum_{k=1}^{m-1} \mathcal{K}_{2k-1}\mathcal{K}_{2m-2k-1}, \qquad m = 2, 3, \ldots,$$

$$\mathcal{K}_{2m} = \frac{\pi}{4m} \sum_{k=1}^{m} \mathcal{K}_{2k-1}\mathcal{K}_{2m-2k}, \qquad m = 1, 2, \ldots,$$

which provide us with formula (24).

Now we return to identity (26), suppose that $x = h = 1/2$, and due to (5), we obtain

$$\sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k} = -E^*_{n+1},$$

and given $n = 2m$, we have the equality

$$E^*_{2m+1} = -\sum_{k=0}^{m} \binom{2m}{2k} E_{2k} E_{2m-2k}, \quad m \geq 0, \tag{29}$$

which in terms of Favard constants takes the form

$$\mathcal{K}_{2m-1} = \frac{\pi}{2(2m-1)} \sum_{k=0}^{m-1} \mathcal{K}_{2k}\mathcal{K}_{2m-2k-2}, \qquad m = 1, 2, \ldots$$

Now formula (25) is also proved. □

Take into account that if n is even, then formulas (24) and (25) coincide. For the odd ones they differ, but can be combined into one.

Corollary 5. *The following equalities hold:*

$$\mathcal{K}_0 = 1, \qquad \mathcal{K}_1 = \frac{\pi}{2}, \qquad \mathcal{K}_n = \frac{\pi}{4n} \sum_{k=0}^{n-1} \mathcal{K}_k \mathcal{K}_{n-1-k}, \quad n = 2, 3, \ldots$$

7. Tangent and secant numbers

All special numbers (Euler, Bernoulli, Genocchi) which we consider here possess a property that half of them are zero ones. And of course only nonzero numbers are of our interest. Therefore, in academic literature some researchers only itemize nonzero numbers, moreover, they keep the original names of these numbers. This fact, in turn, leads to some confusion. The second issue which adds even more confusion is that the numbers which we take into consideration are strongly alternating, hence, we need to look carefully at their signs. However, in some problems where the signs are not needed, they are dropped, while their names still remain the same.

As we mentioned already, the same numbers can be encountered in absolutely different problems. Therefore, there are different ways to define them. For example, for Euler numbers we use the most classical definition — the value of Euler polynomials at the particular point $x = 1/2$. But they can also be defined as the coefficients of expansion into Taylor series of a *hyperbolic secant* function, that is,

$$\operatorname{sech} x = \frac{1}{\cosh x} = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \frac{61}{720}x^6 + \ldots = \sum_{m=0}^{\infty} \frac{E_{2m}}{(2m)!}x^{2m},$$

and if we expand a *trigonometric secant* into a series, then all the coefficients are positive:

$$\sec x = \frac{1}{\cos x} = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \ldots = \sum_{m=0}^{\infty} \frac{|E_{2m}|}{(2m)!}x^{2m}.$$

We can see that the expansion coefficients for these series only differ in their signs, however, in literature, in both cases they can be called Euler numbers. Proper terminology refers to the coefficients in the second case as *secant numbers*.

The shifted Euler numbers E_n^* which we introduced have actually been studied before. They are encountered when expanding a *hyperbolic tangent* into a series

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \ldots = -\sum_{m=1}^{\infty} \frac{E_{2m-1}^*}{(2m-1)!}x^{2m-1},$$

and when expanding a *trigonometric tangent*, all coefficients are again of a single sign:

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \ldots = \sum_{m=1}^{\infty} \frac{|E_{2m-1}^*|}{(2m-1)!}x^{2m-1}.$$

As for the expansion coefficients for a hyperbolic tangent, there is no standard term, and they are usually expressed using Bernoulli numbers even though that is not always convenient. And the positive coefficients became known as *tangent numbers*.

In a number of problems, for example, when studying the quantity of alternating permutations [25, 26], there arises a necessity to deal with secant and tangent numbers at the same time. Expansion of the function

$$\tan x + \sec x = \sum_{n=0}^{\infty} \frac{A_n}{n!}x^n \tag{30}$$

into series leads to emergence of a number sequence A_n, which contains at the same time tangent $A_{2m-1} = |E_{2m-1}^*| = (-1)^m E_{2m-1}^*$ and secant $A_{2m} = |E_{2m}| = (-1)^m E_{2m}$ numbers (or Euler and Bernoulli numbers without taking into account their signs, although they differ from Bernoulli numbers in their factor).

Here again there is no common term for referring to the sequence of numbers A_n. One can encounter such terms as Bernoulli–Euler numbers, or *up/down* numbers [27]; in OEIS (The Online Encyclopedia of Integer Sequences) list [28], this sequence is called *Euler zigzag numbers*.

It is worth noting that tangent numbers are related to *Eulerian numbers* and *Eulerian polynomials* [29], which emerged in L. Euler's studies of the alternating sums

$$\sum_{k=1}^{n}(-1)^k k^m.$$

Take into account that Euler and Eulerian numbers are different objects. However, there is a large confusion about them. Eulerian polynomial are sometimes referred to as *Euler–Frobenius polynomials* [30], but also a term Euler polynomials can be encountered (see, for example, a collection of articles by S. L. Sobolev on roots of such polynomials [31]). For a rigorous definition of Euler–Frobenius polynomials and a description of their properties we refer the reader to [30], and here we will only briefly mention the way to obtain them using B-splines.

It is well-known (see, for example, [1,30,32]) that a B-spline $M_{i,n}(x)$ of order n (of degree $n-1$) with knots $t_i,\dots,t_{i+n}$ is a divided difference of order n with respect to the variable t from the truncated power function $n(t-x)_+^{n-1}$ by the points $t_i,\dots,t_{i+n}$. Here we are only interested in B-splines possessing knots uniformly located in integers, which can be defined using the formula

$$M_n(x)=M_{i,n}(x)=\frac{1}{(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)_+^{n-1}.$$

The support of the B-spline $M_n(x)$ is the segment $[0,n]$, and the knots are the points $0,1,\dots,n$. The values of the B-spline in the knots, multiplied by $n!$, are the Eulerian numbers $E(n,k)$, in particular,

$$E(n,k)=n!\,M_{n+1}(k+1),\qquad k=0,1,\dots,n-1.$$

And the polynomial $\Pi_n(x)$ of degree $n-1$ with the coefficients from the Eulerian numbers $E(n,k)$ (or from the values of the B-spline $M_{n+1}(x)$ in the knots of the mesh, multiplied by $n!$) is the Euler–Frobenius polynomial

$$\Pi_n(x)=\sum_{k=0}^{n-1}x^kE(n,k)=n!\sum_{k=0}^{n-1}x^kM_{n+1}(k+1). \tag{31}$$

We now give the expressions for the polynomials $\Pi_n(x)$ for small n:

$$\begin{aligned}\Pi_0(x)&=1, &\qquad \Pi_3(x)&=x^2+4x+1,\\ \Pi_1(x)&=1, & \Pi_4(x)&=x^3+11x^2+11x+1,\\ \Pi_2(x)&=x+1, & \Pi_5(x)&=x^4+26x^3+66x^2+26x+1.\end{aligned}$$

It is worth noting that given $x=-1$, from the values of Euler–Frobenius polynomial, we infer the shifted Euler numbers

$$\Pi_n(-1)=-E_n^*. \tag{32}$$

If the polynomials are defined with coefficients which equal the values of B-splines in half-integers, we arrive at the polynomials

$$\Pi_n^*(x)=n!\sum_{k=0}^{n}x^kM_{n+1}\Big(k+\frac{1}{2}\Big), \tag{33}$$

which we call *shifted Euler–Frobenius polynomials*. We give the expressions for the polynomials $\Pi_n^*(x)$ for small n:

$$\begin{aligned}\Pi_1^*(x)&=x+1, &\qquad \Pi_3^*(x)&=\frac{1}{8}x^3+\frac{23}{8}x^2+\frac{23}{8}x+\frac{1}{8},\\ \Pi_2^*(x)&=\frac{1}{4}x^2+\frac{3}{2}x+\frac{1}{4}, & \Pi_4^*(x)&=\frac{1}{16}x^4+\frac{19}{4}x^3+\frac{115}{8}x^2+\frac{19}{4}x+\frac{1}{16}.\end{aligned}$$

And now we have that, when $x = -1$, the values of the shifted Euler–Frobenius polynomials give us the Euler numbers

$$\Pi_n^*(-1) = E_n. \tag{34}$$

Because we know that Favard constants can be expressed using the numbers E_n and E_n^* (Theorem 3), then formulas (32) and (34) provide us with another means for calculating Favard constants via the values of B-splines. Note that the polynomials $\Pi_n(x)$ and $\Pi_n^*(x)$ (with some factor) were studied in depth in [30] and [32, Ch.III].

In general, combinatorial problems only require the absolute values of the Euler numbers E_{2n} and E_{2n-1}^*, therefore, they consider positive numbers A_n. As for us, when expanding into a series (30), we substitute the functions with the hyperbolic ones, that is,

$$\tanh x + \operatorname{sech} x = \sum_{n=0}^{\infty} \frac{\mathbb{E}_n}{n!} x^n,$$

so the new number sequence $\mathbb{E}_n$ contains nonzero Euler numbers and shifted Euler numbers taking into account their signs, in particular, $\mathbb{E}_{2m-1} = E_{2m-1}^*$, $\mathbb{E}_{2m} = E_{2m}$, while $A_n = |\mathbb{E}_n|$ for every n. We will call the numbers $\mathbb{E}_n$ *united Euler numbers.*

Earlier we have shown that the numbers T_n in V. L. Miroshnichenko's theorem (Theorem 1) are nonzero Euler and shifted Euler numbers, which differ in their signs and are written in one sequence, that is, $T_n = -\mathbb{E}_n$ and $|T_n| = A_n$.

Now we can reword the theorems from Section 4 in terms of united Euler numbers. But we will start from the recurrent formulas for numbers $\mathbb{E}_n$. Equalities (16), (17) from Lemma 2 provide us with recurrent formulas for calculating the numbers E_{2m-1}^* and E_{2m}. Although our new sequence $\mathbb{E}_n$ entirely consists of these numbers, we can not turn them directly into equalities for $\mathbb{E}_n$, because the mentioned equalities include zero Euler numbers which we did not include into the united sequence. However, equalities (14) and (19) for calculating E_{2m-1}^* via E_{2m} and E_{2m-1}^*, respectively, can be rewritten in the form

$$\mathbb{E}_{2m-1} = -\sum_{k=0}^{m-1} \binom{2m-1}{2k} \mathbb{E}_{2k}, \qquad \mathbb{E}_{2m-1} = -1 - \sum_{k=1}^{m-1} \binom{2m-1}{2k-1} \mathbb{E}_{2k-1}, \tag{35}$$

and also equalities (13) and (18) for calculating E_{2m} get the form

$$\mathbb{E}_{2m} = -\sum_{k=0}^{m-1} \binom{2m}{2k} \mathbb{E}_{2k}, \qquad \mathbb{E}_{2m} = 1 + \sum_{k=1}^{m} \binom{2m}{2k-1} \mathbb{E}_{2k-1}. \tag{36}$$

Combining them pairwise, we obtain two recurrent formulas:

$$\mathbb{E}_n = -\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k} \mathbb{E}_{2k}, \qquad n = 1, 2, \ldots,$$

$$\mathbb{E}_n = (-1)^n \left(1 + \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{n}{2k-1} \mathbb{E}_{2k-1} \right), \qquad n = 0, 1, \ldots$$

But we can also combine formulas (35) for odd numbers,

$$\mathbb{E}_{2m-1} = -\frac{1}{2}\left(1 + \sum_{k=0}^{2m-2}\binom{2m-1}{k}\mathbb{E}_k\right), \qquad m = 1, 2, \dots,$$

and, in a similar way, formulas (36) for even numbers:

$$\mathbb{E}_{2m} = \frac{1}{2}\sum_{k=1}^{2m-1}\binom{2m}{k}(-1)^{k+1}\mathbb{E}_k = \frac{1}{2}\left(1 - \sum_{k=0}^{2m-1}\binom{2m}{k}(-1)^k\mathbb{E}_k\right), \quad m = 1, 2, \dots.$$

And finally, we can combine the last formulas into one, which will have the form

$$\mathbb{E}_n = (-1)^n\frac{1}{2}\left(1 + \sum_{k=0}^{n-1}\binom{n}{k}(-1)^{(k+1)(n+1)}\mathbb{E}_k\right), \qquad n = 1, 2, \dots.$$

Apart from that, we also have quadratic recurrent formulas (27), (28), and (29), which can be written in terms of the united Euler numbers almost without any modifications,

$$\mathbb{E}_{2m+1} = \sum_{k=1}^{m}\binom{2m}{2k-1}\mathbb{E}_{2k-1}\mathbb{E}_{2m-2k+1}, \quad m \geq 1,$$

$$\mathbb{E}_{2m+1} = -\sum_{k=0}^{m}\binom{2m}{2k}\mathbb{E}_{2k}\mathbb{E}_{2m-2k}, \quad m \geq 0,$$

$$\mathbb{E}_{2m} = \sum_{k=1}^{m}\binom{2m-1}{2k-1}\mathbb{E}_{2k-1}\mathbb{E}_{2m-2k}, \quad m \geq 1,$$

or in the following form:

(37)
$$\mathbb{E}_n = \sum_{k=1}^{\left[\frac{n}{2}\right]}\binom{n-1}{2k-1}\mathbb{E}_{2k-1}\mathbb{E}_{n-2k}, \quad n = 2, 3, \dots,$$

(38)
$$\mathbb{E}_n = (-1)^n\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-1}{2k}\mathbb{E}_{2k}\mathbb{E}_{n-2k-1}, \quad n = 1, 2, \dots$$

Also the formula combining (37) and (38) will have the form

$$\mathbb{E}_{n+1} = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}(-1)^{(n+1)(k+1)}\mathbb{E}_k\mathbb{E}_{n-k}, \qquad n = 1, 2, \dots$$

Thus, the recurrent formulas for $\mathbb{E}_n$ are written (we even have several different variants). Now we will rewrite the formulas for Favard constants using these numbers. Theorems 2 and 4 will be very close to Corollaries 1 and 2, hence, we will only rewrite Theorem 3. It will have a very simple form.

Theorem 10. *The following equalities hold:*

$$\mathcal{K}_n = \frac{1}{n!}\left(\frac{\pi}{2}\right)^n|\mathbb{E}_n| = (-1)^{\left[\frac{n+1}{2}\right]}\frac{1}{n!}\left(\frac{\pi}{2}\right)^n\mathbb{E}_n, \qquad n = 0, 1, \dots$$

Note that the last theorem and the recurrent relations for the united Euler numbers $\mathbb{E}_n$ can be rewritten in terms of the Euler zigzag numbers A_n, although some of the formulas can become more complex, as for them the alternation of signs of numbers is essential.

8. Favard interpolation problem

We have already noted that in a number of papers on approximation theory dealing with Favard constants and extremal functions, sometimes their connection to Euler polynomials and numbers is not mentioned. In our opinion, if this connection was used, for example, in the following studies, it could have simplified the research and even help to notice new regularities (see [33–37]). Using paper [33] as an example, we will show that we can come closer to obtaining a new explicit kind of solution.

The authors of paper [38] present a review of results of a research on problems of extremal functional interpolation. One of the problems is as follows.

Suppose that $Y_{n,p} = \{\boldsymbol{y} : \boldsymbol{y} = \{y_k\}_{k\in\mathbb{Z}}, \|\Delta^n \boldsymbol{y}\|_{l_p} \le 1\}$ is a class of interpolated sequences $\boldsymbol{y} = \{y_k\}_{k\in\mathbb{Z}}$ satisfying the conditions

$$\sum_{k\in\mathbb{Z}} |\Delta^n y_k|^p \le 1, \quad 1 \le p < \infty; \qquad \sup_{k\in\mathbb{Z}} |\Delta^n y_k| \le 1, \quad p = \infty,$$

and $\Delta^n \boldsymbol{y}$ is a sequence whose members are the values of an operator of an ordinary finite difference of order n on the sequence $\boldsymbol{y}$, that is,

$$\Delta^n y_k = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} y_{k+m}, \quad k \in \mathbb{Z},$$

and AC is a set of local absolutely continuous functions. We denote the class of functions interpolating the sequence $\boldsymbol{y}$ in integer points of the number line by

$$\Phi_p(\boldsymbol{y}) = \{f : f^{(n-1)} \in AC, f^{(n)} \in L_p(\mathbb{R}), f(k) = y_k \ \forall k \in \mathbb{Z}\}$$

The problem of extremal interpolation requires finding a *constant of extremal functional interpolation*

$$A_{n,p} = \sup_{\boldsymbol{y}\in Y_{n,p}} \inf_{f\in\Phi_p(\boldsymbol{y})} \|f^{(n)}\|_{L_p}. \tag{39}$$

This problem for $p = \infty$ was brilliantly solved in 1965 by Subbotin [39], and later in 1967 he obtained a solution for the rest of the cases $1 \le p < \infty$ [33].

As Yu. N. Subbotin recalled, the origin of this problem lies in the theory of difference schemes. It is well-known that difference schemes are used for the numerical solution of differential equations. Derivatives of functions are replaced by difference approximations, in particular, finite differences (divided differences for nonuniform meshes). As a PhD student, Yu. N. Subbotin got this problem from his scientific advisor S. B. Stechkin, and the problem itself had emerged in conversations with N. N. Yanenko about a posteriori analysis of approximated solutions and studies on convergence of difference schemes.

A similar problem of minimizing $f^{(n)}$ for an arbitrary mesh on a finite segment was solved in 1940 by Favard [40], who tried to obtain an estimation for the n-th derivative of a function through the largest of the given divided differences of that function. He analyzed the cases for $n = 1$ and $n = 2$ in detail, and for an arbitrary n he showed that the emerging constant does not depend on both functions, the mesh, or the number of knots, and proposed a constructive algorithm for finding a function interpolating mesh data with the n-th derivative, which "does not greatly exceed" the given divided differences.

In review [38], the authors report that for nonuniform meshes, the best constant has not been obtained, and even asymptomatics is unknown. In relation to that, we will mention the papers of C. de Boor [41–43], in which he approached a study and estimation of the constant in the Favard problem and his interest was sparked by a question about these constants from H.-O. Kreiss, who apparently was looking for a shortcut in computing error bounds for a given finite difference approximation to the solution of an ordinary differential equation [44]. De Boor improved Favard's algorithm and established some estimations for the constant of interest.

Later de Boor [45] obtained a connection of the constant from the Favard problem to the condition number of the B-splines basis, and based on a large number of calculations, suggested that this constant is of order 2^n. In 1999, Scherer and Shadrin [46] obtained an upper estimation of condition number for a B-spline basis of $n2^n$ form (and, respectively, the constant in the Favard problem), close to the hypothesis suggested by de Boor. And earlier in 1978, Lyche [47] showed that the order of the condition number can not be less than 2^n.

It relation to that, note a recent work by Novikov and Shevaldin [48], in which the problem for an arbitrary nonuniform mesh and for a geometric one is considered for $n = 2$. The authors calculated the optimal constant for the geometrical mesh, and obtained a corridor for an constant in the case of an arbitrary mesh. Note that simple calculations show that the extremal function will be the interpolation parabolic spline due to Subbotin, whose knots are located strictly in the middle between the points of interpolation. Such kind of splines was studied in detail in monograph [32]. The constant $A_{2,\infty} = 2$, obtained by Yu. N. Subbotin for the case of a uniform mesh, constitutes an optimal constant for the general case as well.

We now return to Stechkin–Yanenko problem (39) for the uniform mesh. The constant for $p = \infty$, found by Subbotin [39], was expressed in terms of a series from expression (2), and in terms of Favard constants this expression is equal to

$$A_{n,\infty} = \left(\frac{\pi}{2}\right)^n \mathcal{K}_n^{-1} = \frac{n!}{|\mathbb{E}_n|}. \tag{40}$$

The extremal function which realized the lower boundary in (39) was exactly the Euler perfect spline.

The lower estimation for the value $A_{n,\infty}$ was written using the L_1-norm of the polynomial $Q_{n-1}(x)$ on the segment $[0, 1]$, which is expressed via Bernoulli polynomials as follows:

$$Q_{n-1}(x) = \frac{2^{2n-1}}{n}\left[(-1)^n B_n\left(\frac{1-x}{4}\right) - B_n\left(\frac{1+x}{4}\right)\right].$$

As for the upper estimation, Euler–Frobenius polynomials (31) and shifted Euler–Frobenius polynomials (33) were used, in particular, their values given $x = -1$, which infer, according to (32) and (34), Euler numbers.

The solution proposed by Subbotin [33] for the rest of the cases $1 \le p < \infty$ was obtained in the form

$$A_{n,p} = (n-1)!\, \|Q_{n-1}\|^{-1}_{L_q[0,1]}, \tag{41}$$

where $1/p+1/q=1$. Since given $p=\infty$, the same formula (41) was used to obtain the required constant [39], we can assume that with all $1\le p\le\infty$ for calculation of the constant $A_{n,p}$, formula (41) is valid.

Review [38] reports that instead of calculating the norm of the polynomial $Q_{n-1}(x)$ given $p<\infty$, only the asymptotics for the constants $A_{n,p}$ given $n\to\infty$ has been obtained.

Actually, for some values of p the norm of the polynomial $Q_{n-1}(x)$ can be calculated if it is considered as an Euler polynomial. Indeed, we will sequentially use the properties of Bernoulli and Euler polynomials [17, 24.4.3] and [17, 24.4.23], then we have that

$$Q_{n-1}(x)=\frac{2^{2n-1}}{n}\left[B_n\left(\frac{x+1}{4}+\frac{1}{2}\right)-B_n\left(\frac{1+x}{4}\right)\right]=2^{n-1}E_{n-1}\left(\frac{1+x}{2}\right).$$

With $0\le x\le 1$, the argument of the Euler polynomial will lie on the segment $[1/2,1]$, hence, formula (41) can be rewritten in the equivalent form

$$A_{n,p}=\frac{(n-1)!}{2^{n-1}2^{1/q}}\|E_{n-1}\|^{-1}_{L_q[1/2,1]}.$$

Therefore,

$$\max_{0\le x\le 1}|Q_{n-1}(x)|=2^{n-1}\max_{1/2\le x\le 1}|E_{n-1}(x)|=|\mathbb{E}_{n-1}|=(n-1)!\left(\frac{2}{\pi}\right)^{n-1}\mathcal{K}_{n-1}.$$

Then according to (41), for $p=1$ we obtain

$$A_{n,1}=\frac{(n-1)!}{|\mathbb{E}_{n-1}|}=\left(\frac{\pi}{2}\right)^{n-1}\mathcal{K}^{-1}_{n-1}. \tag{42}$$

The explicit value of the constant $A_{n,p}$ can be obtained given $p=2$ as well.

Lemma 3. *If $n+m$ is even, then the following equality holds:*

$$\int_{1/2}^{1}E_n(t)E_m(t)dt=-\frac{n}{m+1}\int_{1/2}^{1}E_{n-1}(t)E_{m+1}(t)dt.$$

Proof. It suffices to apply differentiation rule (6), the integration by parts formula, and the property that even-indexed Euler polynomials turn into 0 on the borders of the segment $[0,1]$, and the odd-indexed ones are zero in the midpoint of the segment. □

Lemma 3 allows us to reduce the calculation of the L_2-norm of the polynomial $E_{n-1}(x)$ to the calculation of the L_1-norm of the polynomial $E_{2n-2}(x)$, that is, the following formula is valid:

$$\int_{1/2}^{1}E^2_{n-1}(t)dt=(-1)^{n-1}\frac{((n-1)!)^2}{(2n-2)!}\int_{1/2}^{1}E_{2n-2}(t)dt.$$

Therefore,

$$\|Q_{n-1}\|^2_{L_2[0,1]}=2^{2n-1}\|E_{n-1}\|^2_{L_2[0.5,1]}=2^{2n-1}\frac{((n-1)!)^2}{(2n-2)!}\|E_{2n-2}\|_{L_1[1/2,1]}$$

$$=\frac{((n-1)!)^2}{(2n-1)!}|\mathbb{E}_{2n-1}|=((n-1)!)^2\left(\frac{2}{\pi}\right)^{2n-1}\mathcal{K}_{2n-1}.$$

As a result, we obtain

$$A_{n,2} = \sqrt{\frac{(2n-1)!}{|\mathbb{E}_{2n-1}|}} = \left(\frac{\pi}{2}\right)^{n-1/2} \mathcal{K}_{2n-1}^{-1/2}.$$

The values of the constants $A_{n,p}$ can be also expressed via Euler numbers or Favard constants for the values of p, given which q will be an integer. To do that, it is necessary to calculate an integral of the corresponding power of Euler polynomial on the segment $[1/2, 1]$. The formulas for calculating definite integrals of the product of Euler polynomials can be found in paper [49]. With respect to our case, the values for the required integrals are expressed in terms of regular and shifted Euler numbers.

Theorem 11 ([49]). *For an integer $q \geq 1$, the equality*

$$\int_{1/2}^{1} E_n^q(t)dt = \frac{(n!)^q}{2^{qn+1}} \left(\sum_{k=0}^{qn} (-1)^k \sum_{\substack{j_1+\ldots+j_{q-1}=k \\ 0\leq j_1,\ldots,j_{q-1}\leq k}} \frac{k!}{j_1!\cdots j_{q-1}!} \times \frac{\widetilde{E}_{j_1,\ldots,j_{q-1},k}}{(n-j_1)!\cdots(n-j_{q-1})!(n+k+1)!} \right),$$

holds, where

$$\widetilde{E}_{j_1,\ldots,j_{q-1},k} = (-1)^{qn+1} E^*_{n-j_1} \cdots E^*_{n-j_{q-1}} E^*_{n+k+1} - E_{n-j_1} \cdots E_{n-j_{q-1}} E_{n+k+1},$$

moreover, given negative indices, both kinds of Euler numbers are considered zero ones.

Note that in the expression for $\widetilde{E}_{j_1,\ldots,j_{q-1},k}$ at least one of $2qn$ Euler numbers equals 0, and if their indices are of different parity, then the entire value equals 0. To write the expression for the q-norm of Euler polynomial using Favard constants, we need to keep only nonzero summands. For an arbitrary q that is hard to achieve, so we will limit ourselves to the case when $q = 3$. Consider separately the cases of even and odd degree of the polynomial. Then these expressions will have the form

$$\|E_{2m-1}\|^3_{L_3[1/2,1]} = \frac{((2m-1)!)^3}{2^{6m-2}} \left| \sum_{k=m}^{2m-1} \binom{2k-1}{2m-1} \frac{2E^*_{2m+2k-1}E^*_{4m-2k-1}}{(2m+2k-1)!(4m-2k-1)!} + \sum_{k=1}^{2m-1} \sum_{\substack{j=1 \\ k-m+1\leq j\leq m}}^{2k} \binom{2k}{2j-1} \frac{E_{2m+2k}E_{2m-2j}E_{2m-2k+2j-2}}{(2m+2k)!(2m-2j)!(2m-2k+2j-2)!} \right|,$$

$$\|E_{2m}\|^3_{L_3[1/2,1]} = \frac{((2m)!)^3}{2^{6m+1}} \left| \binom{4m}{2m} \frac{E^*_{6m+1}}{(6m+1)!} + \sum_{k=1}^{2m} \sum_{\substack{j=1 \\ k-m+1\leq j\leq m}}^{2k} \binom{2k}{2j-1} \frac{E^*_{2m+2k+1}E^*_{2m-2j+1}E^*_{2m-2k+2j-1}}{(2m+2k+1)!(2m-2j+1)!(2m-2k+2j-1)!} \right|.$$

Since

$$A_{n,3/2} = \frac{(n-1)!}{2^{n-2/3}} \|E_{n-1}\|^3_{L_3[1/2,1]},$$

then taking into account the equalities from Theorem 10, we obtain

$$A_{2m,3/2} = \Big(\frac{\pi}{2}\Big)^{2m-2/3} \Bigg| \sum_{k=m}^{2m-1} 2\binom{2k-1}{2m-1} \mathcal{K}_{2m+2k-1}\mathcal{K}_{4m-2k-1}$$
$$+ \sum_{k=1}^{2m-1} \sum_{\substack{j=1 \\ k-m+1\le j\le m}}^{2k} \binom{2k}{2j-1} \mathcal{K}_{2m+2k}\mathcal{K}_{2m-2j}\mathcal{K}_{2m-2k+2j-2} \Bigg|^{-1/3},$$

$$A_{2m+1,3/2} = \Big(\frac{\pi}{2}\Big)^{2m+1/3} \Bigg| \binom{4m}{2m} \mathcal{K}_{6m+1}$$
$$+ \sum_{k=1}^{2m} \sum_{\substack{j=1 \\ k-m+1\le j\le m}}^{2k} \binom{2k}{2j-1} \mathcal{K}_{2m+2k+1}\mathcal{K}_{2m-2j+1}\mathcal{K}_{2m-2k+2j-1} \Bigg|^{-1/3}.$$

After the work by Subbotin [33] in 1967, Schoenberg [4] in 1969 repeated the results for $p = 2, \infty$ and wrote the values of the constants using series. Apart from that, he calculated a constant for some space $L_{[1]}$, in which he defined the norm of the function $f(x)$ as

$$\|f\|_{L_{[1]}} = \sum_{j\in\mathbb{Z}} \operatorname*{ess\,sup}_{x\in I_j^n} |f(x)|,$$

where

$$I_j^n = \Big(j + \frac{n-1}{2}, j + \frac{n+1}{2}\Big), \qquad j \in \mathbb{Z}.$$

Obviously, $L_{[1]} \subset L_1$, hence, the constant $A_{n,[1]}$ for such space can not be less than $A_{n,1}$. It turned out to coincide with $A_{n,\infty}$, that is, it equals

$$A_{n,[1]} = \Big(\frac{\pi}{2}\Big)^n \mathcal{K}_n^{-1} = \frac{n!}{|\mathbb{E}_n|}.$$

As Subbotin [33] notes, the constants have to be monotone with respect to p, therefore, as a consequence, we obtain inequalities connecting adjacent Favard constants or Euler numbers.

Theorem 12. *For every $n \ge 1$, the following inequalities hold:*

$$\frac{2}{\pi}\mathcal{K}_n^2 \le \mathcal{K}_{2n-1} \le \frac{\pi}{2}\mathcal{K}_{n-1}^2, \qquad \frac{|\mathbb{E}_n|}{n!} \le \sqrt{\frac{|\mathbb{E}_{2n-1}|}{(2n-1)!}} \le \frac{|\mathbb{E}_{n-1}|}{(n-1)!},$$
$$|E_{2n}| \le 2n|E_{2n-1}^*|, \qquad |E_{2n+1}^*| \le (2n+1)|E_{2n}|.$$

We decided not to provide inequalities for adjacent Bernoulli numbers, which can be obtained from comparing the numbers $A_{n,p}$ for $p = 1$ and $p = \infty$, since in the literature stronger inequalities can be found [50]. In handbooks [17, 18], the reader can find inequalities between adjacent (nonzero) Euler numbers E_{2n}, which are also stronger compared to the ones that we obtain. However, we have not succeeded in finding in the literature any inequalities connecting the maxima of adjacent Euler polynomials (that is, E_{2n} numbers with E_{2n-1}^* numbers, or E_{2n+1}^*, in other words — adjacent $\mathbb{E}_n$ numbers), and inequalities for Favard constants.

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References

[1] N. Korneichuk, *Exact constants in approximation theory*, Cambridge University Press, Cambridge, 1991. Zbl 0722.41001

[2] V.M. Tikhomirov, *Approximation theory*, in *Analysis. II. Convex analysis and approximation theory, Encycl. Math. Sci.*, **14**, Springer, Berlin-Heidelberg, 1990, 93–255. Zbl 0780.41001

[3] R.A. DeVore, G.G. Lorentz, *Constructive approximation*, Springer, Berlin, 1993. Zbl 0797.41016

[4] I.J. Schoenberg, *Cardinal interpolation and spline functions*, J. Approx. Theory, **2**:2 (1969), 167–206. Zbl 0202.34803

[5] I.J. Schoenberg, *Cardinal interpolation and spline functions: II interpolation of data of power growth*, J. Approx. Theory, **6**:4 (1972), 404–420. Zbl 0268.41004

[6] A.S. Cavaretta, Jr., *On cardinal perfect splines of least sup-norm on the real axis*, J. Approx. Theory, **8**:4 (1973), 285–303. Zbl 0264.41004

[7] G. Glaeser, *Prolongement extremal de fonctions différentiables*, J. Approx. Theory, **8**:3 (1973), 249–261. Zbl 0259.41011

[8] J. Favard, *Sur les meilleurs procédés d'approximation de certaines classes des fonctions par des polynomes trigonométriques*, Bull. Sci. Math., II. Ser., **61** (1937), 209–224, 243–256. Zbl 0017.25101

[9] N.I. Akhiezer, M.G. Krein, *On the best approximation of periodic functions*, Dokl. Akad. Nauk SSSR, **15**:3 (1937), 107–111. JFM 63.0227.01

[10] Yu.N. Subbotin, *Norms of interpolational splines of odd degree in the spaces* W_2^k, Math. Notes, **44**:6 (1988), 958–962. Zbl 0708.41010

[11] V.V. Zhuk, S.Yu. Pimenov, *On norms of Akhiezer-Krein-Favard sums*, Vestnik St. Peterburg Univ., Prikl. Math. Inf. Pros. Upravl., **2006**:4 (2006), 37–47.

[12] A.N. Kolmogorov, *On inequalities between the upper bounds of the successive derivatives of an arbitrary function on an infinite interval*, Amer. Math. Soc. Transl., **1949**:2 (1949). MR0031009

[13] Yu.S. Volkov, Yu.N. Subbotin, *Fifty years of Schoenberg's problem on the convergence of spline interpolation*, Proc. Steklov Inst. Math., **288**, suppl. 1 (2015), S222–S237. Zbl 1323.41011

[14] Yu.S. Volkov, *On one problem of extremal functional interpolation and Favard constants*, Dokl. Math., **102**:3 (2020).

[15] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher transcendental functions*, Vol.1, McGraw-Hill, New York, 1953. Zbl 0051.30303

[16] A.P. Prudnikov, A.Yu. Brychkov, O.L. Marichev, *Integrals and series. Vol. 1: Elementary functions*, Gordon & Breach Science Publishers, New York etc., 1986. Zbl 0733.00004

[17] K. Dilcher, *Bernoulli and Euler polynomials*, NIST handbook of mathematical functions, U.S. Dept. Commerce, Washington, 2010, 587–599. MR2655364

[18] M. Abramovitz, I.A. Stegun (eds.), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards, John Wiley & Sons, New York etc., 1972. Zbl 0543.33001

[19] A.I. Stepanets, *Methods of Approximation Theory*, VSP, Utrecht, 2005. Zbl 1082.42001

[20] V.L. Miroshnichenko, *On exact estimates of approximation by interpolation splines*, Siberian conf. «Methods of spline functions» / Abstracts, Sobolev Inst. Math., 2001, 55–56.

[21] N.E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer, Berlin, 1924. JFM 50.0315.02

[22] A.F. Horadam, *Genocchi polynomials*, Applications of Fibonacci numbers. Vol. 4, Kluwer, Dordrecht, 1990, 145–166. Zbl 0749.11019

[23] M.K. Kerimov, *The Rayleigh function: Theory and methods for its calculation*, Comput. Math. Math. Phys., **39**:12 (1999), 1883–1925. Zbl 0979.33013

[24] S.G. Gocheva-Ilieva, I.H. Feschiev, *New recursive representations for the Favard constants with application to multiple singular integrals and summation of series*, Abstr. Appl. Anal., (2013), Article ID 523618. Zbl 1366.42021

[25] A.G. Kuznetsov, I.M. Pak, A.E. Postnikov, *Increasing trees and alternating permutations*, Russ. Math. Surv., **49**:6 (1994), 79–114. Zbl 0842.05025
[26] R. Adin, Y. Roichman, *Standard Young tableaux*, Handbook of Enumerative Combinatorics, ed. M. Bóna, CRC Press, Boca Raton, 2015, 895–974. Zbl 1330.05162
[27] V.I. Arnold, *Bernoulli-Euler updown numbers associated with function singularities, their combinatorics and arithmetics*, Duke Math. J., **63**:2 (1991), 537–555. Zbl 0755.58015
[28] https://oeis.org/
[29] L. Comtet, *Advanced combinatorics: The art of finite and infinite expansions*, D. Reidel Publishing Company, Dordrecht, 1974. Zbl 0283.05001
[30] I.J. Schoenberg, *Cardinal spline interpolation*, SIAM, Philadelphia, 1973. Zbl 0264.41003
[31] G.V. Demidenko, V.L. Vaskevich (eds.), *Selected Works of S.L. Sobolev. Volume I: Equations of mathematical physics, computational mathematics, and cubature formulas*, Springer, New York, 2006. Zbl 1117.01022
[32] S.B. Stechkin, Yu.N. Subbotin, *Splines in numerical mathematics*, Nauka, Moscow, 1976. Zbl 0462.65007
[33] Yu.N. Subbotin, *Interpolation by functions with nth derivative of minimum norm*, Proc. Steklov Inst. Math., **88** (1967), 31–63. Zbl 0214.31602
[34] Yu.N. Subbotin, *Approximation by spline functions and estimates of diameters*, Proc. Steklov Inst. Math., **109** (1971), 39–67. Zbl 0285.41010
[35] Yu.N. Subbotin, *Extremal problems of functional interpolation and interpolation-in-the-mean splines*, Proc. Steklov Inst. Math., **138** (1977), 127–185. Zbl 0377.41011
[36] E.V. Mishchenko, *Determination of Riesz bounds for the spline basis with the help of trigonometric polynomials*, Sib. Math. J., **51**:4 (2010), 660–666. Zbl 1214.65006
[37] E.V. Mishchenko, *Construction of polynomials of a special kind and examples of their application in the theory of power series and in the wavelet theory*, Russian Math. **55**:1 (2011), 19–32. Zbl 1241.65003
[38] Yu.N. Subbotin, S.I. Novikov, V.T. Shevaldin, *Extremal function interpolation and splines*, Tr. Inst. Mat. Mekh. UrO RAN, **24**:3 (2018), 200–225. MR3865503
[39] Yu.N. Subbotin, *On the relations between finite differences and the corresponding derivatives*, Proc. Steklov Inst. Math., **78** (1965), 24–42. Zbl 0154.05601
[40] J. Favard, *Sur l'interpolation*, J. Math. Pures Appl., **19**:3 (1940), 281–306. Zbl 0026.01201
[41] C. de Boor, *A smooth and local interpolant with small k-th derivative*, Numeri. Solut. Bound. Value Probl. ordin. differ. Equat., Proc. Symp. Baltimore 1974, Academic Press, New York, 1975, 177–197. Zbl 0317.65003
[42] C. de Boor, *How small can one make the derivatives of an interpolating function?*, J. Approx. Theory, **13**:2 (1975), 105–116. Zbl 0293.41007
[43] C. de Boor, *On 'best' interpolation*, J. Approx. Theory, **16**:1 (1976), 28–42. Zbl 0314.41001
[44] H.-O. Kreiss, *Difference approximations for singular perturbation problems*, Numer. Solut. Bound. Value Probl. ordin. differ. Equat., Proc. Symp. Baltimore 1974, Academic Press, New York, 1975, 199–211. Zbl 0313.65072
[45] C. de Boor, *The exact condition of the B-spline basis may be hard to determine*, J. Approx. Theory, **60**:3 (1990), 344–359. Zbl 0687.41011
[46] K. Scherer, A.Yu. Shadrin, *New upper bound for the B-spline basis condition number II. A proof of de Boor's 2^k-conjecture*, J. Approx. Theory, **99**:2 (1999), 217–229. Zbl 0955.41013
[47] T. Lyche, *A note on the condition numbers of the B-spline bases*, J. Approx. Theory, **22**:3 (1978), 202–205. Zbl 0377.41013
[48] S.I. Novikov, V.T. Shevaldin, *On the connection between the second divided difference and the second derivative*, Tr. Inst. Mat. Mekh. UrO RAN, **26**:2 (2020), 216–224. MR4131105
[49] J. Liu, H. Pan, Y. Zhang, *On the integral of the product of the Appell polynomials*, Integral Transforms Spec. Funct., **25**:9 (2014), 680–685. Zbl 1327.11019
[50] F. Qi, *A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers*, J. Comput. Appl. Math., **351** (2019), 1–5. Zbl 1425.11043

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