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# FUSION OF 2-ELEMENTS IN PERIODIC GROUPS WITH FINITE SYLOW 2-SUBGROUPS

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Abstract. This article contributes to the study of a fusion of subsets in finite Sylow 2-subgroups of periodic groups. We extend well-known theorems on fusion of subsets in Sylow subgroups of finite groups by Burnside and Alperin to periodic groups which contain a finite Sylow 2-subgroup.

**Keywords:** periodic group, Sylow subgroup, fusion.

## 1. Introduction

The goal of this paper is to extend several well-known results on finite groups to periodic groups with a finite Sylow 2-subgroup. More precisely, we prove the following theorems.

**Theorem 1.** *Let $T$ be a finite Sylow 2-subgroup of a periodic group $G$. If $A$ and $B$ are invariant in $T$ and conjugate in $G$ subsets, then $A$ and $B$ are conjugate in $N_G(T)$.*

A similar result for finite groups is due to Burnside [1, Lemma 14.3.1].

The next theorem is an analogue of Alperin's result on a fusion in Sylow subgroups of finite groups [2].

**Theorem 2.** *Suppose that $A$ and $B$ are two subsets in a finite Sylow 2-subgroup $T$ of a periodic group $G$, and $A^x = B$. Then there exist elements $x_i$, Sylow 2-subgroups $Q_i$ of $G$, $1 \leqslant i \leqslant n$, and an element $y \in N_G(T)$ with the following properties*:

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(a) $x = x_1 x_2 \dots x_n y$;

(b) $T \cap Q_i$ *is a tame intersecton, i.e.* $N_T(T \cap Q_i)$ *and* $N_{Q_i}(T \cap Q_i)$ *are Sylow* 2*-subgroups of* $N_G(T \cap Q_i)$;

(c) $x_i \in N_G(T \cap Q_i)$ *and* $x_i$ *is a* 2*-element,* $1 \leqslant i \leqslant n$;

(d) $A \subseteq T \cap Q_1$ *and* $A^{x_1 x_2 \dots x_i} \subseteq T \cap Q_{i+1}$, $1 \leqslant i \leqslant n-1$.

As an easy corollary of Theorem 2, we get the following (again, well-known for finite groups) result on maximal intersections of finite Sylow 2-subgroups in periodic groups.

**Theorem 3.** *Suppose that $T$ is a finite Sylow 2-subgroup of a periodic group $G$, $S$ is a Sylow 2-subgroup of $G$, such that $D = T \cap S$ is a maximal intersection, that is, it is not contained as a proper subset in any larger intersection of distinct Sylow subgroups.*

1. *If $D = 1$, then every two conjugate in $G$ subsets are conjugate in $N_G(T)$.*

2. *If $D \neq 1$, then every subgroup of $T$ which is conjugate to $D$ in $G$ is conjugate to $D$ in $N_G(T)$.*

And, finally, we generalize one more classical result of Burnside (see [1, Theorem 4.2.5].

**Theorem 4.** *Let $R$ be a normal 2-subgroup of a finite Sylow 2-subgroup $T$ of a periodic group $G$. If $R$ is contained in some Sylow 2-subgroup as a noninvariant subgroup, then there is a subgroup $H$ of $T$, such that $H$ contains $R$ and the number $r$ of different subgroups of $N_G(H)$ which are conjugate with $R$ is odd and more than one.*

To give the reader a broad understanding of the background to the problem, next section is devoted to reviewing the most influential of results on the topic.

## 2. Involutions with finite centralizer

The starting point of our study is the celebrated theorem of Shunkov on almost regular involution [3], along with its generalizations and corollaries.

**(2.1)** *If a periodic group contains an involution whose centralizer is finite, then the group is locally finite.*

Later this result was refined by Belyaev and Sesekin [4].

**(2.2.1)** *Suppose that $a$ is an involution in a periodic group $G$, and its centralizer is finite. Then the index of the subgroup $[G, a] = \langle [g, a] \mid g \in G \rangle$ in $G$ is finite and the derived subgroup of $[G, a]$ is finite.*

**(2.2.2)** *Suppose that $a$ and $b$ are involutions in a periodic group $G$. If the centralizers of $a$ and $b$ in $G$ are finite, then the index of the centralizer of $ab$ in $G$ is finite.*

In particular, (2.2.2) implies the following fact.

**(2.3)** *Let $V$ be a noncyclic subgroup of order 4 in a periodic group $G$. If, for every involution $x \in V$, the subgroup $C_G(x)$ is finite, then $G$ is finite.*

Since a finite 2-group with a unique involution—i.e. a 2-group, such that every of its subgroups of order 4 is cyclic—is a cyclic or quaternion group, we have that (2.3) is a natural generalization of the following Shunkov's theorem [5], which he proved several years prior to (2.1).

**(2.4)** *Suppose that a periodic group $G$ contains involutions and all of their centralizers are finite. Then either $G$ is finite, or it contains a normal locally finite subgroup $N$ without involutions and $G/N$ is a finite group with a unique involution.*

We mention one more improvement of (2.1) obtained in [6].

**(2.5)** *There exists a function $f(m)$ such that if $G$ is a finite group containing an involution $i$ such that $|C_G(i)| = m$, then $G$ contains a nilpotent subgroup of class at most two and index at most $f(m)$.*

Note also that in 1987, Belyaev [7] significantly simplified the original proof of Theorem (2.1). Besides, he relaxed the condition of Shunkov's theorem while strengthening its conlucion.

**(2.6)** Recall that an $FC$-radical of a group $G$ is a set $FC(G)$ consisting of those elements whose centralizers are of finite indices in $G$. The intersection of two subgroups of finite index has finite index, therefore, $FC(G)$ is a subgroup.

*Suppose that $a$ is an involution in the group $G$ and $C_G(a)$—the centralizer of $a$ in $G$—is finite. If, for every $g \in G$, the commutator $[a, g]$ has finite order, then the following holds*:

(a) *$G$ is locally finite*;

(b) $[G, a] \leq FC(G)$;

(c) $|G : [G, a]| \leqslant |C_G(a)|$; *in particular,* $|G : FC(G)| \leqslant |C_G(a)|$;

(d) $[FC(G), FC(G)]$ *is a finite subgroup.*

## 3. Normalizer condition for finite subgroups of 2-groups

The following result from [8] is one of important and beautiful corollaries of (2.1).

**(3.1)** *If $F$ is a proper finite subgroup of a 2-group $G$, then $N_G(F) \neq F$.*

*Proof.* We use induction on $|F|$. We may assume that $F \neq 1$. Let $z$ be an involution from the centre of $F$. If $C_G(z) = F$, then, by (2.1), $G$ is locally finite and in this case, by the normalizer condition for nilpotent groups, the statement is true.

Suppose that $C_G(z)$ contains $F$ as a proper subgroup. We may assume that $C_G(z) = G$, and hence $\langle z \rangle \trianglelefteq G$. By induction, $N_{\overline{G}}(\overline{F}) \neq \overline{F}$, for $\overline{F} = F/\langle z \rangle$, $\overline{G} = G/\langle z \rangle$. Therefore, $N_G(F) \neq F$. □

## 4. Orders of intersections of Sylow 2-subgroups

The following result is one more corollary of (2.1). Its proof can be found in [8], but we give it here for the sake of completeness and the reader's convenience.

**(4.1)** *Let $S$ be a Sylow 2-subgroup of a periodic group $G$. Then one of the following statements holds*:

(a) *every Sylow 2-subgroup of $G$ is a conjugate of $S$*;

(b) *For every positive integer $n$ there exists a Sylow 2-subgroup $T$ which is not conjugate to $S$, such that $|T \cap S| > n$.*

*Proof.* Consider the set $M$ whose elements are Sylow 2-subgroups of $G$ which are not conjugate to $S$. Suppose that the statement of the theorem is not true, then the set $M$ is nonempty and there exists $n \in \mathbb{N}$, such that $|S \cap T| \leqslant n$ for every $T \in M$. Choose $T \in M$ such that the order of $D = S \cap T$ is the largest. Clearly $S \neq D \neq T$. By (2.1), $N_S(D) \neq D \neq N_S(T)$.

Let $S_0$ be a subgroup of $N_S(D)$, such that $|S_0 : D| = 2$, and $T_0$ a subgroup of $N_T(D)$, such that $|T_0 : D| = 2$. Then $G_0 = \langle S_0, T_0\rangle$ is a finite subgroup in $N_G(D)$. Suppose that $S_1$, $T_1$ are Sylow subgroups of $\langle S_0, T_0\rangle$ containing $S_0$, $T_0$ respectively. Since $G_0$ is finite, there exists $x \in N_G(D)$, such that $S_1 = T_1^x$. Let $S^*$ be a Sylow 2-subgroup of $G$, containing $S_1$. We have

$$|S \cap S^*| \geqslant |S \cap S_1| \geqslant |S \cap S_0| > D,$$

and hence $S^*$ is conjugate to $S$. Since $|S^* \cap T^x| \geqslant |S_1| > |D|$, we have that $S^*$ is conjugate to $T^x$ and therefore $S$ is conjugate to $T^x$, and we arrived at a contradiction. □

As an immediate corollary of (4.1), we get a well-known result of Shunkov from his unpublished thesis.

**(4.2)** *If a periodic group $G$ possesses a finite Sylow 2-subgroup, then all Sylow 2-subgroups of $G$ are conjugate, and so are finite.*

We remark that not only in a periodic, but even in a locally finite group its Sylow 2-subgroups might not be conjugate or even isomorphic. For example, as it was proved by Hickin [9], the following statement is true.

**(4.3)** *Every infinite countable locally finite $p$-group is isomorphic to some Sylow $p$-subgroup of a simple Hall's universal group $U$ which is a sole countable existentially closed group in a class of all locally finite groups* (on the Hall's universal group see [10]).

## 5. Groups with finite Sylow 2-subgroups

In this section, $M$ will denote the class of all periodic groups which contain a finite Sylow 2-subgroup.

By (4.2), $M$ coincides with the class of periodic groups whose Sylow 2-subgroups are finite and conjugate.

**(5.1)** *If $H \leq G \in M$, then $H \in M$. If $H \trianglelefteq G \in M$, then $G/H \in M$.*

*Proof.* A Sylow 2-subgroup $S_H$ of $H$ is contained in a Sylow 2-subgroup $S_G$ of $G$. By (4.2), $S_G$ is finite, and thus $S_H$ is finite.

Suppose that $H \trianglelefteq G$. By (4.2), every Sylow 2-subgroup of $H$ is conjugate to $S_H$, and therefore,

$$G = HN_G(S_H), \quad G/H \simeq N_G(S_H)/N_H(S_H),$$

i.e. we may assume that $G = N_G(S_H)$.

Suppose first that $H = S_H$, i.e. $H$ is a 2-group. If $S_G/G$ is contained in a 2-group $S_G^*/H$, then $S_G^*$ is a 2-group, and thus $S_G^* = S_G$ and $S_G/G$ is a finite Sylow 2-subgroup of $G/H$. So we may assume that $S_H = 1$ and $H$ does not possess involutions.

Let $S_G^*$ be a Sylow 2-subgroup of $\overline{G} = G/H$ containing $HS_G/H = \overline{S}_G$. By (3.1), $N_{S_G}(\overline{S}_G) \neq \overline{S}_G$. Consider a subgroup $\overline{A}$ of $N_{S^*}(\overline{S}_G)$, such that $|\overline{A} : \overline{S}_G| = 2$, and its full preimage $A$. The subroup $B = NS_G$ is normal in $A$ and, since every Sylow 2-subgroup of $B$ is a conjugate of $S_G$, we have $A = HN_A(S_G)$. Take $x \in N_A(S_G)\setminus S_G$. Then $x = x_1x_2$, where $x_1x_2 = x_2x_1$, $x_2$ is a 2-element, and $x_1$ is an element of odd order. Therefore, $x_2 \in N_A(S_G) \setminus S_G$ and $\langle S_G, x_2\rangle$ is a 2-group, which contains $S_G$ as a proper subgroup, a contradiction. □

## 6. Fusion in a finite Sylow 2-subgroup of a periodic group

Statements (4.2) and (5.1) allow us to extend the proofs of classical theorems on fusion in Sylow subgroups of finite groups, unchanged, to finite Sylow 2-subgroups of periodic groups.

**(6.1)** As an example, we give the *proof of Theorem* 1, simply adapting to periodic groups the proof of the corresponding result on finite groups by Burnside.

Suppose that $A^x = B$ for $x \in G$. Then $B = A^x \leq T^x$, and since $A$ is invariant in $T$, we have that $B$ is invariant in $T^x$, and so $\langle T, T^x \rangle \leq N_G(B)$.

By (5.1) and (4.2), there exists $y \in N_G(B)$, such that $T^{xy} = T$, i.e. $n = xy \in N_G(T)$. In addition, $B = B^y = A^{xy} = A^n$, which means that $A$ and $B$ are conjugates in $N_G(T)$. The theorem is proved.

**(6.2)** Similar reasoning can be applied to the original proof of Alperin's theorem on a fusion in Sylow subgroups of finite groups [2] in order to extend it to the class of periodic groups with finite Sylow 2-subgroups. This is because the mentioned proof only uses the Sylow theorems (not needing the information on the number of Sylow subgroups) which, as it has been shown above, are fully valid for finite Sylow 2-subgroups of periodic groups.

In light of the above, the *proof of Theorem* 2 is obtained on the same lines as that of the original proof by Alperin [2].

**(6.3)** *Proof of Theorem* 3. As it was mentioned in the introduction, this theorem is an easy corollary of Theorem 2. Remind that Theorem 3 concerns maximal intersections of finite Sylow 2-subgroups and is well-known for finite groups.

Suppose that $D = 1$. Consider a nonempty set $A$ of $T$ and $g \in G$, such that $A^g \subseteq T$. If $g \notin N_G(T)$, then $A \subseteq T^{g^{-1}} \neq T$, and thus $A \subseteq T \cap T^{g^{-1}}$, which means that $D = 1$ is not a maximal intersection, a contradiction. So, $g \in N_G(T)$ and item 1 is proved.

Suppose that $D \neq 1$, $D^n \leq T$. Then, by Theorem 2, there are Sylow 2-subgroups $Q_1, \ldots, Q_r$ of $G$ and 2-subgroups $D_1, \ldots, D_r = D^n$, such that $\langle D_{i-1}, D_i \rangle \leq T \cap Q_i$ and $D_i = D_{i-1}^{x_i}$ for suitable $x_i \in N_G(T \cap Q_i)$. Clearly, $D_i$ is a maximal intersection of Sylow 2-subgroups.

If $D_{i-1} = D_i$, then we can take $x_i = 1$. But if $D_{i-1} \neq D_i$, then $T \cap Q_i \geq \langle D_{i-1}, D_i \rangle \neq D_i$, and thus $Q_i = T$ and $x_i \in N_G(T)$. Therefore, $x_i \in N_G(T)$ for every $i$ and $D^n = D^{x_1 \ldots x_r}$ for $x_1 \ldots x_r \in N_G(T)$. □

**(6.4)** *Proof of Theorem* 4. By (4.2), all Sylow 2-subgroups of $G$ are conjugate. Hence, $R \leq T^x$ and $R \ntrianglelefteq T^x$ for some $x$. Thus $R^{x^{-1}} \leq T$ and $R^{x^{-1}} \ntrianglelefteq T$.

By Theorem 2, there exist subgroups $R_0 = R^{x^{-1}}, R_1, \ldots, R_s = R$, such that all of them are conjugate to $R$, $\langle R_{i-1}, R_i \rangle \leq Q_i$ and $R_{i-1}$, $R_i$ are conjugate in $Q_i$ (for some tame intersections $Q_i$ of $T$) with Sylow 2-subgroups of $G$.

We have $R_0 \ntrianglelefteq T$ and $R_i \trianglelefteq T$, thus there exists a pair $(n, n+1)$, $0 \leqslant n < s$, such that $R_n \ntrianglelefteq T$, $R_{n+1} \trianglelefteq T$. By (5.1), $R_{n+1}^y = R$ for some $y \in N_G(T)$ and $R_n$, $R_{n+1}$ are conjugate in $N_G(Q_{n+1})$. Moreover, $N_T(Q_{n+1})$ are Sylow 2-subgroups in $N_G(Q_{n+1})$, and so $|N_G(Q_{n+1}) : (N_G(Q_{n+1}) \cap N(R_{n+1}))|$ is an odd number which is not equal to one. So, $H = Q_{n+1}^y$ is the subgroup sought. □

We should remark that $R \trianglelefteq H$, and hence all subgroups which are conjugate to $R$ in $N_G(H)$ are normal in $H$, too. In particular, if under conditions of (5.4), $R$ is a subgroup from the center of $T$, then for $H$ may be taken a commutative group equal to the product of $r$ subgroups of $T$ which are conjugate to $R$ in $G$.

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