S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 17, стр. 2055–2067 (2020) УДК 517.956.25
DOI 10.33048/semi.2020.17.137 MSC 35J20, 35J25, 35J62

EXISTENCE OF A SOLUTION TO A NONLINEAR ELLIPTIC EQUATION IN A MUSIELAK–ORLICZ–SOBOLEV SPACE FOR AN UNBOUNDED DOMAIN

L.M. KOZHEVNIKOVA, A.P. KASHNIKOVA

Abstract. We consider a class of second-order elliptic equations with nonlinearities defined by generalized N-functions. The existence of a weak solution to the Dirichlet problem in a reflexive Musielak–Orlicz–Sobolev space is proved for an arbitrary unbounded domain.

Keywords: Musielak–Orlicz-Sobolev space, Δ_2-condition, Dirichlet problem, existence of a solution, pseudomonotone operator, unbounded domain.

1. Introduction

Let Ω be an arbitrary unbounded domain in $\mathbb{R}^n = \{\mathrm{x} = (x_1, x_2, \dots, x_n)\}$, $n \geq 2$. We consider the Dirichlet problem for a second-order quasilinear elliptic equation of the form

$$-\mathrm{div}\, \mathrm{a}(\mathrm{x}, u, \nabla u) + a_0(\mathrm{x}, u, \nabla u) = F(\mathrm{x}), \quad \mathrm{x} \in \Omega; \tag{1}$$

with homogeneous boundary condition

$$u\Big|_{\partial\Omega} = 0. \tag{2}$$

The general boundary value problem of variational type for a high-order elliptic equation in divergent form with nonlinearities of polynomial form was considered by F. Browder [1] in an arbitrary domain without the conditions of boundedness or smoothness for the boundary of the domain. It is proved that the corresponding operator from a reflexive Banach space into its dual is pesudomonotone, and this fact implies the existence of a solution to the problem under consideration.

Kozhevnikova, L.M., Kashnikova A.P., Existence of a solution to a nonlinear elliptic equation in a Musielak–Orlicz–Sobolev space for an unbounded domain.

Supported by RFBR (Grant 18-01-00428).

Received April 9, 2020, published December, 11, 2020.

Following [1], L.M.Kozevnikova and A.Sh.Kamaletdinov (see [2]) established the existence of a weak solution to the Dirichlet problem in an arbitrary domain Ω for an anisotropic equation (1) with variable nonlinearity exponents. Earlier L.M. Kozhevnikova and A.A.Hadzhi proved in [3] that there exists a weak solution to the Dirichlet problem in an arbitrary unbounded domain Ω for an anisotropic elliptic equation (1) with nonlinearilities defined by N-functions.

The following existence results are known for quasilinear elliptic equations in Musielak–Orlicz–Sobolev spaces. In [4], [5], the existence of solutions is proved under some assumptions like the Δ_2–condition and also the uniform convexity of the generalized N-function M, which guarantee that the Musielak–Orlicz–Sobolev space is reflexive. The study of the problems of the existence of solutions to variational boundary value problems for quasilinear elliptic equations in nonreflexive spaces (provided that the complementary function $\overline{M}$ satisfies the Δ_2-condition) is the contents of [6], [7]. The existence of weak solutions for second-order differential equations with the Dirichlet or Neumann boundary condition in a separable nonreflexive space was established in [8] by constructing super- and subsolutions.

It should be noted that the authors do not know any results on the existence of solutions to nonlinear equations in Musielak–Orlicz–Sobolev spaces for unbounded domains. In the present article, we prove an existence theorem for a solution to problem (1), (2) for an arbitrary unbounded domain Ω in a reflexive Musielak–Orlicz–Sobolev space.

2. Musielak–Orlicz–Sobolev Spaces

In this section, we give necessary information from the theory of generalized N-functions and Musielak–Orlicz spaces (see [9], [10]).

Suppose that a Carathéodory function $M(\mathrm{x}, z) : \Omega\times\mathbb{R} \to \mathbb{R}_+$ satisfies the following conditions:

(1) $M(\mathrm{x}, \cdot)$ is an N-function with respect to $z \in \mathbb{R}$, i.e., it is convex downwards, nondecreasing, even, continuous, $M(\mathrm{x}, 0) = 0$ for a.e. $\mathrm{x} \in \Omega$, and

$$\inf_{\mathrm{x}\in\Omega} M(\mathrm{x}, z) > 0 \quad \text{for all} \quad z \neq 0,$$

$$\lim_{z\to 0} \sup_{\mathrm{x}\in\Omega} \frac{M(\mathrm{x}, z)}{z} = 0,$$

$$\lim_{z\to \infty} \inf_{\mathrm{x}\in\Omega} \frac{M(\mathrm{x}, z)}{z} = \infty;$$

(2) $M(\cdot, z)$ is a measurable function with respect to $\mathrm{x} \in \Omega$ for any $z \in \mathbb{R}$. Such a function $M(\mathrm{x}, z)$ is called a Musielak–Orlicz function or a generalized N-function.

The complementary function $\overline{M}(\mathrm{x}, \cdot)$ to the Musielak–Orlicz function $M(\mathrm{x}, \cdot)$ in the sense of Young is defined by the following equality for a.e. $\mathrm{x} \in \Omega$ and any $z \geq 0$:

$$\overline{M}(\mathrm{x}, z) = \sup_{y\geq 0} \left(yz - M(\mathrm{x}, y)\right).$$

This implies Young's inequality:

$$|zy| \leq M(\mathrm{x}, z) + \overline{M}(\mathrm{x}, y), \quad z, y \in \mathbb{R}, \quad \mathrm{x} \in \Omega. \tag{3}$$

If, for every positive constant l, we have

$$\lim_{z\to \infty} \sup_{\mathrm{x}\in\Omega} \frac{M(\mathrm{x}, lz)}{P(\mathrm{x}, z)} = 0 \tag{4}$$

then this is designated as $M \prec\prec P$ and M is said to grow slower than P at ∞.

A Musielak–Orlicz function M satisfies the Δ_2-condition if there exist $c > 0$ and $z_0 \geq 0$ such that the inequality

$$M(\mathrm{x}, 2z) \leq cM(\mathrm{x}, z)$$

holds for a.e. $\mathrm{x} \in \Omega$ and any $z \geq z_0$.

The Δ_2-condition is equivalent to the fulfillment for a.e. $\mathrm{x} \in \Omega$ and any $z \geq z_0$ of the inequality

$$M(\mathrm{x}, lz) \leq c(l)M(\mathrm{x}, z), \tag{5}$$

where l is an arbitrary number greater than one, $c(l) > 0$.

Henceforth, in the article, we assume that the Δ_2-condition for the generalized N-functions holds for all values $z \geq 0$ (i.e., $z_0 = 0$). For the complementary N-function $\overline{M}(\mathrm{x}, z)$, the Δ_2-condition takes the form:

$$\overline{M}(\mathrm{x}, lz) \leq c(l)\overline{M}(\mathrm{x}, z), \quad z \geq 0. \tag{6}$$

If a generalized N-function $M(\mathrm{x}, z)$ satisfies the Δ_2-condition then, due to convexity and inequality (5), the inequality

$$M(\mathrm{x}, y + z) \leq cM(\mathrm{x}, z) + cM(\mathrm{x}, y), \quad \mathrm{x} \in \Omega, \quad z, y \in \mathbb{R} \tag{7}$$

holds.

There exist three Musielak–Orlicz classes.

$\mathcal{L}_M(\Omega)$ is the generalized Musielak–Orlicz class of measurable functions $v : \Omega \to \mathbb{R}$ such that

$$\varrho_{M,\Omega}(v) = \int_\Omega M(\mathrm{x}, v(\mathrm{x}))d\mathrm{x} < \infty.$$

$L_M(\Omega)$ is the generalized Musielak–Orlicz space, which is the least space containing the class $\mathcal{L}(\Omega)$ with the Luxembourg norm

$$\|v\|_{M,\Omega} = \inf \left\{ \lambda > 0 \;\middle|\; \varrho_{M,\Omega}\left(\frac{v(\mathrm{x})}{\lambda}\right) \leq 1 \right\}.$$

Below, in the notations $\|\cdot\|_{M,Q}$, $\varrho_{M,Q}(\cdot)$, we omit the index $Q = \Omega$.

$E_M(\Omega)$ is the closure in the norm $\|u\|_{M,\Omega}$ of bounded measurable functions with compact support in $\overline{\Omega}$. The embeddings $E_M(\Omega) \subset \mathcal{L}_M(\Omega) \subset L_M(\Omega)$ hold.

A Musielak–Orlicz function $M(\mathrm{x}, z)$ is called locally integrable if

$$\varrho_{M,Q}(z) = \int_Q M(\mathrm{x}, z)d\mathrm{x} < \infty$$

for every $z \in \mathbb{R}$ and every measurable set $Q \subset \Omega$ such that $\operatorname{meas}(Q) < \infty$. We will assume that M and $\overline{M}$ are locally integrable generalized N-functions.

The space $E_M(\Omega)$ is separable and $(E_M(\Omega))^* = L_{\overline{M}}(\Omega)$. If M satisfies the Δ_2-condition then $E_M(\Omega) = \mathcal{L}_M(\Omega) = L_M(\Omega)$ and $L_M(\Omega)$ is separable. The space $L_M(\Omega)$ is reflexive if and only if the Musielak–Orlicz functions M and $\overline{M}$ satisfy the Δ_2-condition.

Define the Musielak–Orlicz–Sobolev space

$$W^1_M(\Omega) = \{v \in L_M(\Omega) \mid |\nabla v| \in L_M(\Omega)\}$$

with the norm

$$\|v\|_{W^1_M(\Omega)} = \|v\|_M + \|\nabla v\|_M.$$

Define the space $\mathring{W}^1_M(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the $*$-weak topology over $W^1_M(\Omega)$. The spaces $W^1_M(\Omega)$, $\mathring{W}^1_M(\Omega)$ are Banach (see [10, Theorem 10.2]).

The space dual to $\mathring{W}^1_M(\Omega)$ is defined as

$$W^{-1}_{\overline{M}}(\Omega) = \{F = f_0 - \operatorname{div} \mathrm{f},\ f_0 \in L_{\overline{M}}(\Omega),\ \mathrm{f} = (f_1, \dots, f_n) \in (L_{\overline{M}}(\Omega))^n\}.$$

A sequence of functions $v^j \in L_M(\Omega)$ converges modularly to $v \in L_M(\Omega)$ if there exists a positive constant $k > 0$ such that

$$\lim_{j\to\infty} \varrho_M \left(\frac{v^j - v}{k} \right) = 0.$$

If M satisfies the Δ_2-condition then the modular and norm topologies coincide.

If $v \in L_M(\Omega)$ then we have the inequalities

$$\|v\|_M \le \varrho_M(v) + 1; \tag{8}$$

if $\|v\|_M \le 1$ then $\varrho_M(v) \le \|v\|_M$;
if $\|v\|_M > 1$ then $\|v\|_M \le \varrho_M(v)$.

The following inequality holds (see [11, Chapter II, Section 9, inequality 9.21])

$$\varrho_M \left(\frac{v}{\|v\|_M} \right) \le 1, \tag{9}$$

in which equality holds if the N-function $M(\mathrm{x}, z)$ satisfies the Δ_2-condition.

Moreover, if the N-function $M(\mathrm{x}, z)$ satisfies the Δ_2-condition then for $v \in L_M(\Omega)$ we have the inequalities

$$\varrho_M(v) \le \varrho_M \left(l \frac{v}{\|v\|_M} \right) \le c(l) \varrho_M \left(\frac{v}{\|v\|_M} \right) = c(l), \quad l = \max\left(\|v\|_M, 1\right). \tag{10}$$

Also, if M and $\overline{M}$ are complementary functions, $u \in L_M(\Omega)$, and $v \in L_{\overline{M}}(\Omega)$, then Hölder's inequality holds:

$$\left| \int_\Omega u(\mathrm{x}) v(\mathrm{x}) d\mathrm{x} \right| \le 2\|u\|_M \|v\|_{\overline{M}}. \tag{11}$$

The following embedding theorem is valid (see [6, Theorem 4]):

Lemma 1. *Suppose that a Musielak–Orlicz function $M(\mathrm{x}, z)$ satisfies the conditions*

$$\int_1^\infty \frac{M^{-1}(\mathrm{x}, z)}{z^{\frac{n+1}{n}}} dz = \infty, \quad \int_0^1 \frac{M^{-1}(\mathrm{x}, z)}{z^{\frac{n+1}{n}}} dz < \infty \tag{12}$$

and

$$M_*^{-1}(\mathrm{x}, z) = \int_0^z \frac{M^{-1}(\mathrm{x}, \tau)}{\tau^{\frac{n+1}{n}}} d\tau, \quad \mathrm{x} \in \Omega,\ z \ge 0.$$

Then $M_*(\mathrm{x}, z)$ is a generalized N-function and $\mathring{W}^1_M(Q) \hookrightarrow L_{M_*}(Q)$ for any bounded subdomain $Q \subset \Omega$. Moreover, the embedding $\mathring{W}^1_M(Q) \hookrightarrow L_P(Q)$ exists and is compact for any Musielak–Orlicz function $P \prec\prec M_*$ integrable over Q.

3. Statement of the Results

We assume that the functions

$$\mathrm{a}(\mathrm{x}, s_0, \mathrm{s}) = (a_1(\mathrm{x}, s_0, \mathrm{s}), \cdots, a_n(\mathrm{x}, s_0, \mathrm{s})), \quad a_0(\mathrm{x}, s_0, \mathrm{s})$$

are measurable with respect to $\mathrm{x} \in \Omega$ for $\mathbf{s} = (s_0, \mathrm{s}) = (s_0, s_1, \ldots, s_n) \in \mathbb{R}^{n+1}$ and continuous in $\mathbf{s} \in \mathbb{R}^{n+1}$ for a.e. $\mathrm{x} \in \Omega$.

Suppose there exist nonnegative functions $\Psi, \Psi_0, \ \phi \in L_1(\Omega)$ and positive constants $\widehat{A}, \overline{a}$ such that the following inequalities hold for a.e. $\mathrm{x} \in \Omega, \ \ s_0 \in \mathbb{R}, \ \mathrm{s}, \mathrm{t} \in \mathbb{R}^n$, $\mathrm{s} \neq \mathrm{t}$:

$$\mathrm{a}(\mathrm{x}, s_0, \mathrm{s}) \cdot \mathrm{s} + a_0(\mathrm{x}, s_0, \mathrm{s}) s_0 \geq \overline{a} M(\mathrm{x}, s_0) + \overline{a} M(\mathrm{x}, |\mathrm{s}|) - \phi(\mathrm{x}); \tag{13}$$

$$\overline{M}(\mathrm{x}, |\mathrm{a}(\mathrm{x}, s_0, \mathrm{s})|) \leq \Psi(\mathrm{x}) + \widehat{A} M(\mathrm{x}, |\mathrm{s}|) + \widehat{A} M(\mathrm{x}, s_0); \tag{14}$$

$$\overline{M}(\mathrm{x}, |a_0(x, s_0, \mathrm{s})|) \leq \Psi_0(\mathrm{x}) + \widehat{A}_0 M(\mathrm{x}, |\mathrm{s}|) + \widehat{A}_0 M(\mathrm{x}, s_0); \tag{15}$$

$$(\mathrm{a}(\mathrm{x}, s_0, \mathrm{s}) - \mathrm{a}(\mathrm{x}, s_0, \mathrm{t})) \cdot (\mathrm{s} - \mathrm{t}) > 0. \tag{16}$$

Here the Musielak–Orlicz functions $M(\mathrm{x}, z), \overline{M}(\mathrm{x}, z)$ satisfy the Δ_2-condition. As an example, one can consider, for example, the function $M(\mathrm{x}, z) = |z|^{p(\mathrm{x})}(|\ln |z|| + 1), \quad 1 < p(\cdot) < \infty$.

From (14), (15), using (8), for every $u \in \mathring{W}^1_M(\Omega)$ we deduce the estimate

$$\begin{aligned} \|\mathbf{a}(\mathrm{x}, u, \nabla u)\|_{\overline{M}} &= \|\mathrm{a}(\mathrm{x}, u, \nabla u)\|_{\overline{M}} + \|a_0(\mathrm{x}, u, \nabla u)\|_{\overline{M}} \\ &\leq \varrho_{\overline{M}}(|a_0(\mathrm{x}, u, \nabla u)|) + \varrho_{\overline{M}}(|\mathrm{a}(\mathrm{x}, u, \nabla u)|) + 2 \\ &\leq C_1 \varrho_M(|\nabla u|) + C_1 \varrho_M(|u|) + C_2 < \infty. \end{aligned} \tag{17}$$

Further, from an element

$$\mathbf{a}(\mathrm{x}, u, \nabla u) \in (L_{\overline{M}}(\Omega))^{n+1},$$

for $v \in \mathring{W}^1_M(\Omega)$ define some functional $\mathbf{A}(u)$ by the equality

$$\langle \mathbf{A}(u), v \rangle = \int\limits_{\Omega} (\mathrm{a}(\mathrm{x}, u, \nabla u) \cdot \nabla v + a_0(\mathrm{x}, u, \nabla u) v) \, d\mathrm{x}. \tag{18}$$

Using Hölder's inequality (11), we infer the following inequalities for functions $u, v \in \mathring{W}^1_M(\Omega)$:

$$|\langle \mathbf{A}(u), v \rangle| \leq 2\|a_0\|_{\overline{M}} \|v\|_M + 2\|\mathrm{a}\|_{\overline{M}} \|\nabla v\|_M \leq 2\|\mathbf{a}(\mathrm{x}, u, \nabla u)\|_{\overline{M}} \|v\|_{W^1_M(\Omega)}. \tag{19}$$

It follows from (17), (19) that the functional

$$\mathbf{A}(u), \ u(\mathrm{x}) \in \mathring{W}^1_M(\Omega)$$

defined by equality (18) in $\mathring{W}^1_M(\Omega)$ is bounded.

We will assume that $F = f_0 - \operatorname{div} \mathrm{f} \in W^{-1} L_{\overline{M}}(\Omega)$, then we can define the functional $\mathbf{F}$:

$$\langle \mathbf{F}, v \rangle = \int\limits_{\Omega} (\mathrm{f} \cdot \nabla v + f_0 v) \, d\mathrm{x}.$$

Definition 1. *Refer as a weak solution to problem* (1), (2) *with*

$$F = f_0 - \operatorname{div} \mathrm{f} \in W^{-1}_{\overline{M}}(\Omega)$$

to a function $u \in \mathring{W}^1_M(\Omega)$ *satisfying the integral identity*

$$\langle \mathbf{A}(u), v\rangle = \langle \mathbf{F}, v\rangle \tag{20}$$

for any function $v \in \mathring{W}^1_M(\Omega)$.

Theorem 1. *Suppose the fulfillment of conditions* (13)–(16), (5), (6), (12). *Then for any* $F \in W^{-1}L_{\overline{M}}(\Omega)$ *there exists at least one solution to problem* (1), (2).

4. Auxiliary Assertions

Lemma 2. *Suppose that condition* (6) *is fulfilled and* $\{v^j\}_{j\in\mathbb{N}}$, v *are functions in* $L_M(\Omega)$ *such that*

$$\|v^j\|_M \le C, \quad j \in \mathbb{N},$$

$$v^j \to v, \quad j \to \infty, \quad \text{a.e. in} \quad \Omega.$$

Then $v^j \rightharpoonup v$ *weakly in* $L_M(\Omega)$, $j \to \infty$.

Lemma 3. *Let* $0 < \varepsilon < 1$ *and let* M_0 *be a Musielak–Orlicz function such that*

$$\int_0^1 \frac{M_0^{-1}(\mathrm{x}, z)}{z^{1+\varepsilon}} dz < \infty, \quad \int_1^\infty \frac{M_0^{-1}(\mathrm{x}, z)}{z^{1+\varepsilon}} dz = \infty.$$

Then the generalized N-function M defined as

$$M^{-1}(\mathrm{x}, z) = \int_0^z \frac{M_0^{-1}(\mathrm{x}, \tau)}{\tau^{1+\varepsilon}} d\tau,$$

satisfies the condition $M_0 \prec\prec M$.

The proof for an N-function can be found in [12, Lemma 4.14]; it is carried out similarly for a Musielak–Orlicz function.

Below we will use Vitali's theorem in the following form (see [13, Chapter III, Section 6, Theorem 15]).

Lemma 4. *Let* v^j, $j \in \mathbb{N}$, v *be measurable functions in a bounded domain* Q *such that*

$$v^j \to v \quad \text{a.e. in} \quad Q, \quad j \to \infty,$$

and the integrals

$$\int\limits_Q |v^j(\mathrm{x})| d\mathrm{x}, \quad j \in \mathbb{N},$$

are uniformly absolutely continuous. Then

$$v^j \to v \quad \text{strongly in} \quad L_1(Q), \quad j \to \infty.$$

Proposition 1. *Suppose the fulfillment of conditions* (13) – (16), (5), (6), $\{u^j\}_{j\in\mathbb{N}}$, $u \in W^1_M(\Omega)$, *and*

$$\|u^j\|_{W^1_M(\Omega)} \le C, \quad j \in \mathbb{N}, \tag{21}$$

$$u^j \to u \quad \text{a.e. in} \quad \Omega, \quad j \to \infty. \tag{22}$$

Assume that the function sequence

(23)
$$q^j(\mathrm{x}) = \big(\mathrm{a}(\mathrm{x}, u^j, \nabla u^j) - \mathrm{a}(\mathrm{x}, u^j, \nabla u)\big) \cdot \nabla(u^j - u) \to 0 \quad a.e.\ in \quad \Omega, \quad j \to \infty.$$

Then, for some subsequence,

(24)
$$\mathbf{a}(\mathrm{x}, u^j, \nabla u^j) \rightharpoonup \mathbf{a}(\mathrm{x}, u, \nabla u) \quad in \quad (L_{\overline{M}}(\Omega))^{n+1}, \quad j \to \infty.$$

Proof. Denote by $\Omega' \subset \Omega$ the subset of full measure of the points for which convergences (22), (23) hold, conditions (13)–(15), (5), (6) are fulfilled, and the functions Ψ, Ψ_0, ϕ take finite values. Prove the convergence

(25)
$$\nabla u^j \to \nabla u \quad \text{everywhere in} \quad \Omega', \quad j \to \infty.$$

To the contrary, suppose that, at some point $\mathrm{x}^* \in \Omega'$, there is no convergence. Put $s_0^j = u^j(\mathrm{x}^*)$, $s_0 = u(\mathrm{x}^*)$, $\mathrm{s}^j = \nabla u^j(\mathrm{x}^*)$, $\mathrm{s} = \nabla u(\mathrm{x}^*)$.

Suppose that the sequence $\{M(\mathrm{x}^*, |\mathrm{s}^j|)\}_{j\in\mathbb{N}}$ is unbounded. Using inequalities (3), (5), (6) and conditions (13)–(15), for $\varepsilon \in (0,1)$ we infer

$$\begin{aligned}
q^j(\mathrm{x}^*) &= \Big(\mathrm{a}(\mathrm{x}^*, s_0^j, \mathrm{s}^j) - \mathrm{a}(\mathrm{x}^*, s_0^j, \mathrm{s})\Big) \cdot (\mathrm{s}^j - \mathrm{s}) \\
&\geq \mathrm{a}(\mathrm{x}^*, s_0^j, \mathrm{s}^j) \cdot \mathrm{s}^j - \varepsilon \overline{M}(\mathrm{x}^*, |\mathrm{a}(\mathrm{x}^*, s_0^j, \mathrm{s}^j)|) - \varepsilon M(\mathrm{x}^*, |\mathrm{s}^j|) \\
&\quad - C_1(\varepsilon)\overline{M}(\mathrm{x}^*, |\mathrm{a}(\mathrm{x}^*, s_0^j, \mathrm{s})|) - C_2(\varepsilon) M(\mathrm{x}^*, |\mathrm{s}|) \\
&\geq \overline{a} M(\mathrm{x}^*, s_0^j) + \overline{a} M(\mathrm{x}^*, |\mathrm{s}^j|) - \phi(\mathrm{x}^*) - \varepsilon \overline{M}(\mathrm{x}^*, |\mathrm{a}(\mathrm{x}^*, s_0^j, \mathrm{s}^j)|) - \varepsilon \overline{M}(\mathrm{x}^*, |a_0(\mathrm{x}^*, s_0^j, \mathrm{s}^j)|) \\
&\quad - \varepsilon M(\mathrm{x}^*, |\mathrm{s}^j|) - C_1(\varepsilon)\overline{M}(\mathrm{x}^*, |\mathrm{a}(\mathrm{x}^*, s_0^j, \mathrm{s})|) - C_2(\varepsilon) M(\mathrm{x}^*, |\mathrm{s}|) - C_3(\varepsilon) M(\mathrm{x}^*, s_0^j) \\
&\geq (\overline{a} - \varepsilon C_4) M(\mathrm{x}^*, |\mathrm{s}^j|) - C_6(\varepsilon) M(\mathrm{x}^*, |\mathrm{s}|) - C_7(\varepsilon) M(\mathrm{x}^*, s_0^j) - C_5.
\end{aligned}$$

Choosing $\varepsilon < \overline{a}/C_4$, we obtain the estimate

$$q^j(\mathrm{x}^*) \geq C_8 M(\mathrm{x}^*, |\mathrm{s}^j|) - C_6 M(\mathrm{x}^*, |\mathrm{s}|) - C_7 M(\mathrm{x}^*, s_0^j) - C_5.$$

In view of the assumption, we conclude that $\{q^j(\mathrm{x}^*)\}_{j\in\mathbb{N}}$ is unbounded, which contradicts (23). Therefore, the sequence $\{\mathrm{s}^j\}_{j\in\mathbb{N}}$ is bounded.

Let $\mathrm{s}^* = (s_1^*, s_2^*, \dots, s_n^*)$ be one of the partial limits of the sequence $\mathrm{s}^j = (s_1^j, s_2^j, \dots, s_n^j)$ for $j \to \infty$; then, with account taken of (22), we have

(26)
$$s_0^j \to s_0, \quad \mathrm{s}^j \to \mathrm{s}^*, \quad j \to \infty.$$

Therefore, (23), (26) and the continuity of $\mathrm{a}(\mathrm{x}^*, s_0, \mathrm{s})$ in $\mathbf{s} = (s_0, \mathrm{s})$ imply that

$$(\mathrm{a}(\mathrm{x}^*, s_0, \mathrm{s}^*) - \mathrm{a}(\mathrm{x}^*, s_0, \mathrm{s})) \cdot (\mathrm{s}^* - \mathrm{s}) = 0;$$

consequently, $\mathrm{s} = \mathrm{s}^*$ by (16). This contradicts the fact that there is no convergence at the point x^*.

Thus, (22), (25) and the continuity of $\mathbf{a}(\mathrm{x}, s_0, \mathrm{s})$ in $\mathbf{s} = (s_0, \mathrm{s})$ imply that

$$\mathbf{a}(\mathrm{x}, u^j, \nabla u^j) \to \mathbf{a}(\mathrm{x}, u, \nabla u) \quad \text{a.e. in} \quad \Omega, \quad j \to \infty.$$

Moreover, (10), (17), (21) imply the boundedness of $\{\mathbf{a}(\mathrm{x}, u^j, \nabla u^j)\}_{j\in\mathbb{N}}$ in $(L_{\overline{M}}(\Omega))^{n+1}$. Due to Lemma 2, we get the weak convergences (24). □

5. Proof of the Existence Theorem

The proof of Theorem 1 is based on an assertion about the pseudomontonicity of the operator $\mathbf{A}$.

Definition 2. *An operator $A : V \to V'$ is called pseudomonotone if*

(i) A is a bounded operator;

(ii) the conditions $u^j \rightharpoonup u$ weakly in V and $\limsup\limits_{j\to\infty}\langle A(u^j), u^j - u\rangle \le 0$ imply that

$$\liminf_{j\to\infty}\langle A(u^j), u^j - v\rangle \ge \langle A(u), u - v\rangle \tag{27}$$

for every $v \in V$.

Lemma 5. *Suppose that V is a reflexive separable Banach space and an operator $A : V \to V'$ possesses the following properties: A is pseudomonotone and coercive, i.e.,*

$$\frac{\langle A(u), u\rangle}{\|u\|} \to \infty \tag{28}$$

as $\|u\| \to \infty$. Then the mapping $A : V \to V'$ is surjective, i.e., for every $F \in V'$ there exists $u \in V$ such that $A(u) = F$ [14, Chapter II, Section 2, Theorem 2.7].

Proposition 2. *Suppose the fulfillment of conditions* (13)–(16), (5), (6), (12). *Then the operator*

$$\mathbf{A} : \mathring{W}^1_M(\Omega) \to W^{-1}_{\overline{M}}(\Omega)$$

defined by (18) *is pseudomonotone.*

Proof. The boundedness of $\mathbf{A}$ follows from (17), (19). Consider a sequence $\{u^j\}_{j=1}^{\infty}$ in $\mathring{W}^1_M(\Omega)$ such that

$$u^j \rightharpoonup u \quad \text{weakly in} \quad \mathring{W}^1_M(\Omega), \quad j \to \infty; \tag{29}$$

$$\limsup_{j\to\infty}\langle \mathbf{A}(u^j), u^j - u\rangle \le 0. \tag{30}$$

Show that

$$\mathbf{A}(u^j) \rightharpoonup \mathbf{A}(u) \quad \text{weakly in} \quad W^{-1}_{\overline{M}}(\Omega), \quad j \to \infty; \tag{31}$$

$$\langle \mathbf{A}(u^j), u^j - u\rangle \to 0, \quad j \to \infty. \tag{32}$$

Obviously, (31), (32) imply (27).

First of all, convergence (29) and inequality (10) yield the estimates:

$$\|u^j\|_{W^1_M(\Omega)} \le C_1, \quad j = 1, 2, \ldots; \tag{33}$$

$$\varrho_M(u^j) + \varrho_M(|\nabla u^j|) \le C_2, \quad j = 1, 2, \ldots. \tag{34}$$

Moreover, combining (17) and (34), we infer

$$\|\mathbf{a}(\mathrm{x}, u^j, \nabla u^j)\|_{\overline{M}} \le C_3, \quad j = 1, 2, \ldots. \tag{35}$$

Fix arbitrary $R > 0$. By Lemma 1, the space $\mathring{W}^1_M(\Omega(R+1))$ is compactly embedded in $L_P(\Omega(R+1))$ for any Musielak–Orlicz function $P(\mathrm{x}, z)$ with $P \prec\prec M_*$. By Lemma 3, $M \prec\prec M_*$ and $\mathring{W}^1_M(\Omega(R+1))$ is compactly embedded in $L_M(\Omega(R+1))$.

Let $\eta_R(r) = \min(1, \max(0, R+1-r))$. Using (7), (34), we deduce the inequalities

$$\varrho_M(|\nabla(u^j\eta_R)|) \le \varrho_M(|\nabla u^j| + |u^j|) \le C_4\varrho_M(|\nabla u^j|) + C_4\varrho_M(|u^j|) \le C_5, \quad j = 1, 2, \dots.$$

Therefore, the function sequence $\{u^j\eta_R\}_{j=1}^{\infty}$ is bounded in $\mathring{W}_M^1(\Omega(R+1))$. Since the embedding

$$\mathring{W}_M^1(\Omega(R+1)) \subset L_M(\Omega(R+1)),$$

is compact, we have the strong convergences on a subsequence

$$u^j\eta_R \to u\eta_R \quad \text{in} \quad L_M(\Omega(R+1)), \quad j \to \infty,$$

which imply the strong convergences

$$u^j \to u \quad \text{in} \quad L_M(\Omega(R)), \quad j \to \infty \tag{36}$$

and the convergence on a subsequence $u^j \to u$ a.e. in $\Omega(R)$. Using the diagonal process, we establish the convergence

$$u^j \to u \quad \text{a.e.} \quad \text{in} \quad \Omega, \quad j \to \infty. \tag{37}$$

Put

$$\Delta^j(\mathrm{x}) = \big(\mathrm{a}(\mathrm{x}, u^j, \nabla u^j) - \mathrm{a}(\mathrm{x}, u, \nabla u)\big) \cdot \nabla(u^j - u) + \big(a_0(\mathrm{x}, u^j, \nabla u^j) - a_0(\mathrm{x}, u, \nabla u)\big)(u^j - u), \quad j = 1, 2, \dots.$$

Then

$$\langle \mathbf{A}(u^j) - \mathbf{A}(u), u^j - u\rangle = \int_\Omega \Delta^j(\mathrm{x})d\mathrm{x}, \quad j = 1, 2, \dots.$$

By (29), (30), we have

$$\limsup_{j\to\infty} \int_\Omega \Delta^j(\mathrm{x})d\mathrm{x} \le 0. \tag{38}$$

Write down $\Delta^j(\mathrm{x})$ as

$$\begin{aligned}\Delta^j(\mathrm{x}) &= \big(\mathrm{a}(\mathrm{x}, u^j, \nabla u^j) - \mathrm{a}(\mathrm{x}, u^j, \nabla u)\big) \cdot \nabla(u^j - u) \\ &+ \big(\mathrm{a}(\mathrm{x}, u^j, \nabla u) - \mathrm{a}(\mathrm{x}, u, \nabla u)\big) \cdot \nabla(u^j - u) \\ &+ \big(a_0(\mathrm{x}, u^j, \nabla u^j) - a_0(\mathrm{x}, u, \nabla u)\big)(u^j - u) = q^j(\mathrm{x}) + r^j(\mathrm{x}) + s^j(\mathrm{x}), \quad j = 1, 2, \dots.\end{aligned} \tag{39}$$

Show that

$$r^j(\mathrm{x}) \to 0 \quad \text{a.e.} \quad \text{in} \quad \Omega, \quad j \to \infty, \tag{40}$$

$$s^j(\mathrm{x}) \to 0 \quad \text{a.e.} \quad \text{in} \quad \Omega, \quad j \to \infty. \tag{41}$$

For every measurable set $E \subset \Omega(R)$, applying (7), (14), we infer

$$\begin{aligned}&\int_E \overline{M}(\mathrm{x}, |\mathrm{a}(\mathrm{x}, u^j, \nabla u) - \mathrm{a}(\mathrm{x}, u, \nabla u)|)d\mathrm{x} \\ &\le c\int_E \big(\overline{M}(\mathrm{x}, |\mathrm{a}(\mathrm{x}, u^j, \nabla u)|) + \overline{M}(\mathrm{x}, |\mathrm{a}(\mathrm{x}, u, \nabla u)|)\big)\, d\mathrm{x} \\ &\le C_6\int_E \big(\Psi(\mathrm{x}) + M(\mathrm{x}, |\nabla u|) + M(\mathrm{x}, u^j) + M(\mathrm{x}, u)\big)\, d\mathrm{x}.\end{aligned}$$

From this, in view of the fact that $\Psi, M(\mathrm{x}, u), M(\mathrm{x}, |\nabla u|) \in L_1(\Omega)$, convergence (36), and the absolute continuity of the integrals, for every $\varepsilon > 0$ and E such that $\operatorname{meas}(E) < \delta$, we have

$$\int\limits_{E} \overline{M}(\mathrm{x}, |\mathrm{a}(\mathrm{x}, u^j, \nabla u) - \mathrm{a}(\mathrm{x}, u, \nabla u)|) d\mathrm{x} < \varepsilon.$$

Thus, the integrals $\int\limits_{\Omega(R)} \overline{M}(\mathrm{x}, |\mathrm{a}(\mathrm{x}, u^j, \nabla u) - \mathrm{a}(\mathrm{x}, u, \nabla u)|) dx$, $j \in \mathbb{N}$ are uniformly absolutely continuous. Applying Lemma 4, we have modular convergence, which implies the convergence

$$\mathrm{a}(\mathrm{x}, u^j, \nabla u) \to \mathrm{a}(\mathrm{x}, u, \nabla u) \quad in \quad (L_{\overline{M}}(\Omega(R)))^n, \quad j \to \infty. \tag{42}$$

Applying (11), we infer

$$\int\limits_{\Omega(R)} |r^j(\mathrm{x})| d\mathrm{x} \leq 2 \|\mathrm{a}(\mathrm{x}, u^j, \nabla u) - \mathrm{a}(\mathrm{x}, u, \nabla u)\|_{\overline{M}, \Omega(R)} \|\nabla(u^j - u)\|_M.$$

The first factor tends to zero, and the second is uniformly bounded (see (33)). Thus, we have proved for every $R > 0$ that $r^j(\mathrm{x}) \to 0$ in $L_1(\Omega(R))$, $j \to \infty$. From here convergence (40) is established by the diagonal process.

Inequality (11) yields

$$\int\limits_{\Omega(R)} |s^j(\mathrm{x})| d\mathrm{x} \leq 2 \|a_0(\mathrm{x}, u^j, \nabla u^j) - a_0(\mathrm{x}, u, \nabla u)\|_{\overline{M}} \|u^j - u\|_{M, \Omega(R)}.$$

The first factor is uniformly bounded (see (35)), and the second tends to zero (see (36)); therefore, for every $R > 0$, $s^j(\mathrm{x}) \to 0$ as $j \to \infty$ in $L_1(\Omega(R))$. Hence, using the diagonal process, we establish convergence (41).

Next, write down $\Delta^j(\mathrm{x})$ as

$$\Delta^j(\mathrm{x}) = \mathrm{a}(\mathrm{x}, u^j, \nabla u^j) \cdot \nabla u^j + a_0(\mathrm{x}, u^j, \nabla u^j) u^j - g^j(\mathrm{x}), \quad j = 1, 2, \ldots, \tag{43}$$

where

$$g^j(\mathrm{x}) = \mathrm{a}(\mathrm{x}, u, \nabla u) \cdot \nabla(u^j - u) + a_0(\mathrm{x}, u, \nabla u)(u^j - u)$$
$$+ \mathrm{a}(\mathrm{x}, u^j, \nabla u^j) \cdot \nabla u + a_0(\mathrm{x}, u^j, \nabla u^j) u \in L_1(\Omega), \quad j = 1, 2, \ldots.$$

Using (3), (5), (6), for $\varepsilon \in (0, 1)$ we obtain

$$|g^j(\mathrm{x})| \leq \varepsilon \left(M(\mathrm{x}, |\nabla u^j|) + M(\mathrm{x}, u^j) + \overline{M}(\mathrm{x}, |\mathrm{a}(\mathrm{x}, u^j, \nabla u^j)|) + \overline{M}(\mathrm{x}, |a_0(\mathrm{x}, u^j, \nabla u^j)|) \right)$$
$$+ C_7(\varepsilon) \left(M(\mathrm{x}, |\nabla u|) + M(\mathrm{x}, u) + \overline{M}(\mathrm{x}, |\mathrm{a}(\mathrm{x}, u, \nabla u)|) + \overline{M}(\mathrm{x}, |a_0(\mathrm{x}, u, \nabla u)|) \right).$$

Applying (14), (15), we infer the inequalities

$$|g^j(\mathrm{x})| \leq \varepsilon C_8 \left(M(\mathrm{x}, |\nabla u^j|) + M(\mathrm{x}, u^j) \right) + C_9(\varepsilon) \left(M(\mathrm{x}, |\nabla u|) + M(\mathrm{x}, u) + \Psi(\mathrm{x}) + \Psi_0(\mathrm{x}) \right). \tag{44}$$

Using (13), rewrite (43) in the form

$$\Delta^j(\mathrm{x}) \geq \overline{a} \left(M(\mathrm{x}, |\nabla u^j|) + M(\mathrm{x}, u^j) \right) - \phi(\mathrm{x}) - |g^j(\mathrm{x})|. \tag{45}$$

Combining (44), (45), choosing $\varepsilon < \overline{a}/C_8$, we prove the estimate

$$\Delta^j(\mathrm{x}) \geq C_{10} \left(M(\mathrm{x}, |\nabla u^j|) + M(\mathrm{x}, u^j) \right) - \phi(\mathrm{x}) \tag{46}$$
$$- C_9 \left(M(\mathrm{x}, |\nabla u|) + M(\mathrm{x}, u) + \Psi(\mathrm{x}) + \Psi_0(\mathrm{x}) \right), \quad j = 1, \ldots.$$

Let $\Delta^j(\mathrm{x}) = \Delta^{j+}(\mathrm{x}) - \Delta^{j-}(\mathrm{x})$ and $\Delta^{j+}(\mathrm{x}),\ \Delta^{j-}(\mathrm{x})$ be the positive and negative parts of $\Delta^j(\mathrm{x})$ respectively. If $\chi^j(\mathrm{x})$ is the characteristic function of the set $\{\mathrm{x} \in \Omega : \Delta^{j-}(\mathrm{x}) > 0\}$ then

$$-\Delta^{j-} = \chi^j q^j + \chi^j r^j + \chi^j s^j;$$

moreover, by (40), (41), $\chi^j r^j(\mathrm{x}) \to 0,\ \chi^j s^j(\mathrm{x}) \to 0$ a.e. in Ω as $j \to \infty$. In view of (16), $\chi^j q^j(\mathrm{x}) \geq 0$ a.e. in Ω; then $\Delta^{j-}(\mathrm{x}) \to 0$ a.e. in Ω as $j \to \infty$.

In addition, (46) implies the estimate

$$\Delta^j(\mathrm{x}) \geq -\phi(\mathrm{x}) - C_9\left(M(\mathrm{x}, |\nabla u|) + M(\mathrm{x}, u) + \Psi(\mathrm{x}) + \Psi_0(\mathrm{x})\right) = -\Psi_u(\mathrm{x}), \quad j = 1, \dots,$$

with a nonnegative function $\Psi_u(\mathrm{x}) \in L_1(\Omega)$ finite a.e. in Ω. Hence, $\Delta^{j-}(\mathrm{x}) \leq \Psi_u(\mathrm{x}),\ j = 1, \dots$. Then, by Lebesgue's theorem,

$$\Delta^{j-}(\mathrm{x}) \to 0 \quad \text{in} \quad L_1(\Omega), \quad j \to \infty. \tag{47}$$

Therefore, by (38),

$$0 \leq \limsup_{j\to\infty} \int_\Omega \Delta^{j+}(\mathrm{x})d\mathrm{x} = \limsup_{j\to\infty} \int_\Omega \Delta^j(\mathrm{x})d\mathrm{x} + \limsup_{j\to\infty} \int_\Omega \Delta^{j-}(\mathrm{x})d\mathrm{x} \leq 0.$$

Consequently,

$$\Delta^{j+}(\mathrm{x}) \to 0 \quad \text{in} \quad L_1(\Omega), \quad j \to \infty. \tag{48}$$

Thus, from (47), (48) we have the convergence

$$\Delta^{j}(\mathrm{x}) \to 0 \quad \text{in} \quad L_1(\Omega), \quad j \to \infty, \tag{49}$$

and also the convergence on a subsequence

$$\Delta^{j}(\mathrm{x}) \to 0 \quad \text{a.e. in} \quad \Omega, \quad j \to \infty. \tag{50}$$

Now, (39), (40), (41), and (50) imply that

$$q^j(\mathrm{x}) \to 0 \quad \text{a.e. in} \quad \Omega, \quad j \to \infty.$$

Applying Proposition 1, we obtain the weak convergences

$$\mathbf{a}(\mathrm{x}, u^j, \nabla u^j) \rightharpoonup \mathbf{a}(\mathrm{x}, u, \nabla u) \quad in \quad (L_{\overline{M}}(\Omega))^{n+1}, \quad j \to \infty. \tag{51}$$

Obviously, (51) implies the weak convergence (31).

For finishing the proof, observe that (29), (49) imply (32)

$$\langle \mathbf{A}(u^j), u^j - u\rangle = \langle \mathbf{A}(u^j) - \mathbf{A}(u), u^j - u\rangle + \langle \mathbf{A}(u), u^j - u\rangle \to 0, \quad j \to \infty.$$

□

Lemma 6. *Suppose the fulfillment of condition* (6)*. Then any function* $u \in L_M(\Omega)$ *enjoys the relation*

$$\|u\|_M^{-1} \varrho_M(u) \to \infty \tag{52}$$

as $\|u\|_M \to \infty$.

The proof of the lemma for an N-function can be found in [12, Lemma 3.14]; it is similar for a Musielak–Orlicz function.

Proof of Theorem 1. Let us prove the coerciveness of the operator $\mathbf{A}$. Using (13), we infer

$$\frac{\langle \mathbf{A}(u), u\rangle}{\|u\|_{W^1_M(\Omega)}} \geq \frac{1}{\|u\|_M + \||\nabla u|\|_M} \left(\overline{a}(\varrho_M(u) + \varrho_M(|\nabla u|)) - \|\phi\|_1\right).$$

Suppose that $\|\nabla u\|_M + \|u\|_M > 2\lambda$ and, for definiteness, the second summand $\|u\|_M > \lambda$ is the greatest. In view of (52), for every $R > 0$ there exists $\lambda > 0$ such that the inequality $\|u\|_M > \lambda$ implies $\varrho_M(u) > R\|u\|_M$. Then we infer

$$\frac{\langle \mathbf{A}(u), u\rangle}{\|u\|_{W^1_M(\Omega)}} \geq \frac{\overline{a}}{2}\frac{\varrho_M(u)}{\|u\|_M} - \frac{\|\phi\|_1}{2\lambda} > RC_1 - C_2.$$

Thus, (28) is proved.

By Lemma 5, Proposition 2 implies the existence of a function $u \in \mathring{W}^1_M(\Omega)$ such that $\mathbf{A}(u) = \mathbf{F}$. Thus, every $v \in \mathring{W}^1_M(\Omega)$ satisfies (20). □

Thus, in the present article, we have found conditions on the structure of equation (1) and the function $M(x, z)$ sufficient for the corresponding elliptic operator to be pseudomonotone and coercive. Condition (12) ensures the compactness of the embedding $\mathring{W}^1_M(Q) \subset L_M(Q)$ for bounded domains Q. Therefore, in Proposition 2, it has been possible to establish the strong convergences (36) and (42) only for bounded sets $\Omega(R)$.

It is obvious that the existence theorem proved in the article in particular holds for an arbitrary bounded domain Ω.

References

[1] F.E. Browder, *Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains*, Proc. Natl. Acad. Sci. USA, **74**:7 (1977), 2659–2661. Zbl 0358.35034

[2] L.M. Kozhevnikova, A.Sh. Kamaletdinov, *Existence of solutions of anisotropic elliptic equations with variable nonlinearity exponents in unbounded domains*, Vestn. Volgogr. Gos. Univ. Ser. 1. Mat. Fiz., **5**:36 (2016), 29–41. MR3666886

[3] L.M. Kozhevnikova, A.A. Khadzhi *Existence of solutions of anisotropic elliptic equations with nonpolynomial nonlinearities in unbounded domains*, Sb. Math., **206**:8 (2015), 1123–1149. Zbl 1332.35107

[4] M. Mihăilescu, V. Rădulescu, *Neumann problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces*, Ann. Inst. Fourier, **58**:6 (2008), 2087–2111. Zbl 1186.35065

[5] X. Fan, C.-X. Guan, *Uniform convexity of Musielak-Orlicz-Sobolev spaces and applications*, Nonlinear Anal., **73**:1 (2010), 163–175. Zbl 1198.46010

[6] A. Benkirane, M. Sidi El Vally, *An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces*, Bull. Belg. Math. Soc.-Simon Stevin, **20**:1 (2013), 57–75. Zbl 1278.35101

[7] M. Sidi El Vally *Strongly Nonlinear Elliptic Problems in Musielak–Orlicz–Sobolev Spaces*, Advances in Dynamical Systems and Applications, **8**:1 (2013), 115–124.

[8] G. Dong, X. Fang, *Differential equations of divergence form in separable Musielak-Orlicz-Sobolev spaces*, Boundary Value Problems, **2016**:106 (2016), 1–19. DOI: 10.1186/s13661-016-0612-9

[9] L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, **2017**, Springer, Berlin, 2011. Zbl 1222.46002

[10] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, **1034**, Springer-Verlag, Berlin etc., 1983. Zbl 0557.46020

[11] M.A. Krasnosel'skii, Ja. B. Rutickii *Convex functions and Orlicz spaces*, P. Noordhoff Ltd., Groningen, 1961. Zbl 0095.09103

[12] J.P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Trans. Am. Math. Soc., **190** (1974), 163–206. Zbl 0239.35045

[13] N. Dunford, J.T. Schwartz, *Linear operators, v. I: General theory*, Pure Appl. Math., **7**, Interscience Publishers, Inc., New York, London, 1958. Zbl 0084.10402

[14] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969. Zbl 0189.40603

Larisa Mikhailovna Kozhevnikova
Sterlitamak Branch of Bashkir State University,
37, Lenin ave.,
Sterlitamak, 453103, Russia
Elabuga Branch of Kazan (Volga Region) Federal University,
89, Kazanskaya str.,
Elabuga, 423600, Russia
Email address: kosul@mail.ru

Anastasiya Pavlovna Kashnikova
Sterlitamak Branch of Bashkir State University,
37, Lenin ave.,
Sterlitamak, 453103, Russia
Email address: a.kashnikova98@yandex.ru