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EQUILIBRIUM PROBLEM FOR AN THERMOELASTIC KIRCHHOFF–LOVE PLATE WITH A NONPENETRATION CONDITION FOR KNOWN CONFIGURATIONS OF CRACK EDGES

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Abstract. We formulate a new variational problem on the equilibrium of a thermoelastic Kirchhoff–Love plate in a domain with a cut. It is assumed that the plate is under the special loads for which the configuration of crack's edges is known in advance. This circumstance makes it possible to write down the general boundary condition of nonpenetration in a refined form, which, in turn, leads to new relations describing the possible mechanical interaction of opposite crack edges. The initial formulation of a problem presupposes the fulfillment of boundary conditions on the crack curve in the form of system of two inequalities and an equality. Solvability of the problem is proved, an equivalent differential setting is found.

Keywords: thermoelastic plate, crack, non-penetration, variational inequality, differential setting.

1. Introduction

The success of mathematical models describing the deformation of bodies with cracks and validity of the further analysis of some properties of selected mathematical models depend, among other things, on the boundary conditions on crack faces. In the framework of the elasticity theory, along with the classical approach of linear boundary conditions of equality type (see, for example, [1–4]), there is a wide

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class of nonlinear problems applying boundary conditions of inequality type for displacements. These conditions make it possible to model a possible mechanical contact of two independent bodies, or opposite faces of a crack. [5–21].

In this context, as in the case of the well-known Signorini conditions [23], the use of inequality-type constraints for displacements means that we already know a definite behavior of body's points at some part of boundaries where a contact interaction is possible. Taking into account that sets of contact zones for this type of problems is unknown, they can be considered as so-called free boundary problems. A wide range of various problems has been investigated in the framework of Kirchhoff–Love plates with the well-known general non-penetration condition [13, 18, 20, 24, 25, 26]. The overwhelming majority of results for cracked Kirchhoff–Love plates were obtained for vertical cracked plates. At the same time, some results were obtained for plates with oblique cracks, see, for example, [9, 13, 15, 27].

In this work, we pay attention to the case when a certain configuration of plate's edges near the crack is known in advance for an equilibrium state of a plate. This circumstance means that some geometrical features of a possible contact are known, which makes it possible to write out the boundary condition in a refined form. Based on these conditions, we define the corresponding set of admissible functions in a suitable Sobolev space. Taking into account temperature effects can play a significant role in applied problems arising from the issues of operation in the Far North. It is well known that the Kirchhoff–Love model is formulated in a two-dimensional domain, while plates are three-dimensional objects. In the case when boundary conditions of nonpenetration in the form of inequalities are applied, some difficulties arise with the description of a three-dimensional object through a two-dimensional model. In particular, if a solution of an equilibrium problem for this type of boundary conditions has nonzero jumps on the crack curve for vertical displacements (deflections), then the solution, generally speaking, can have displacements that satisfy the general nonpenetration condition and, nevertheless, for which we have a physically unacceptable phenomenon since there is a mutual penetration of opposite crack faces, see [18]. Therefore, the above-mentioned questions of the study of problems for special cases with refined modifications of the nonpenetration condition is a justified branch of the development of the mechanics of deformable solids, see, for example, [28, 29].

A new mathematical model describing an equilibrium of a thermoelastic plate with a crack is formulated. The existence of a solution is established for the corresponding variational problem. Under an assumption of sufficient smoothness of the solution, a differential formulation is found that is equivalent to the corresponding variational formulation.

2. Formulation of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ. Suppose that a smooth curve without self-intersections lies strictly inside Ω, i.e. $\bar{\gamma} \subset \Omega$. In addition, we assume that γ can be extended to Γ so that Ω is splitted into two subdomains Ω_1 and Ω_2 with Lipschitz boundaries $\partial\Omega_1$ and $\partial\Omega_2$ where $meas(\Gamma \cap \partial\Omega_i) > 0$, $i = 1, 2$. The assumption is sufficient for Korn's inequality to hold in the non-Lipschitz domain $\Omega_\gamma = \Omega \backslash \bar{\gamma}$ [13]. Depending on the direction of the normal $\nu = (\nu_1, \nu_2)$ to γ we will speak about a positive face γ^+ or a negative face γ^- of the curve γ. The jump $[q]$ of the function q on the curve γ is found by the formula $[q] = q|_{\gamma^+} - q|_{\gamma^-}$.

For simplicity, we assume that the thickness $2h$ of the plate is constant and is equal to two, i.e. $h = 1$. We introduce a three-dimensional Cartesian space $\{x_1, x_2, z\}$ such that the set $\{\Omega_\gamma\} \times \{0\} \subset \mathbb{R}^3$ corresponds to the middle plane of the plate. The curve γ defines a vertical crack (a cut) in the plate. This means that the cylindrical surface of the through crack specified by the relations $x = (x_1, x_2) \in \gamma, -1 \leq z \leq 1$, where $|z|$ is the distance to the middle plane. Denote by $\chi = (W, w)$ the vector of mid-plane displacements, where $W = (w_1, w_2)$ are the displacements in the plane and $\{x_1, x_2\}$ and w are the displacements along the axis z. The temperature field in the plate is denoted by θ. We also need the following set $Q_\gamma = \Omega_\gamma \times (0, T)$, $T > 0$. The strain and integrated stress tensors are denoted by $\varepsilon_{ij} = \varepsilon_{ij}(W)$, $\sigma_{ij} = \sigma_{ij}(W)$, respectively [13]:

$$\sigma_{11} = \varepsilon_{11} + \kappa\varepsilon_{22}, \quad \sigma_{22} = \varepsilon_{22} + \kappa\varepsilon_{11}, \quad \sigma_{12} = (1-\kappa)\varepsilon_{12},$$

$$\varepsilon_{ij}(W) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right), \quad i, j = 1, 2, \quad x_1 = x,\ x_2 = y,$$

where $\kappa = \mathrm{const}, \quad 0 < \kappa < 1/2$.

In order to describe the possible contact interaction of the crack's edges, for the case of prior knowledge of a certain equilibrium configuration of plate edges near the crack (see Fig. 1), we specify following mutual nonpenetration condition of opposite crack faces [29]

$$[\frac{\partial w}{\partial \nu}] \geq 0, \quad [W]\nu \geq [\frac{\partial w}{\partial \nu}], \quad [w] = 0 \quad \text{on} \quad \gamma^T = \gamma \times (0, T). \tag{1}$$

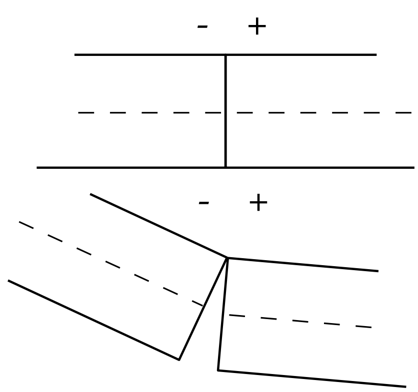

Рис. 1. An example of crack edges configurations for initial (the upper image) and equilibrium (the lower image) states.

We should note that the inequality (1) is written for functions χ given in the domain Q_γ. In the case when considered functions are defined in Ω_γ, we change γ^T to γ and the non-penetration condition will be written as:

$$[\frac{\partial w}{\partial \nu}] \geq 0, \quad [W]\nu \geq [\frac{\partial w}{\partial \nu}], \quad [w] = 0 \quad \text{on} \quad \gamma. \tag{2}$$

In addition, we can mention that if condition (1) holds for some function, then this function also satisfies following well-known general nonpenetration condition for cracks in Kirchhoff–Love plates [8, 13].

$$[W]\nu \geq |[\frac{\partial w}{\partial \nu}]| \quad \text{on} \quad \gamma^T. \tag{3}$$

Let some initial temperature distribution be given:

$$\theta = \theta_0 \quad \text{at} \;\; t = 0. \tag{4}$$

On the exterior boundary of the plate, we require the fulfillment of the following conditions:

$$\theta = w = \frac{\partial w}{\partial n} = W = 0 \quad \text{on} \;\; \Gamma \times (0, T). \tag{5}$$

Introduce the Sobolev spaces

$$H^{1,0}(\Omega_\gamma) = \left\{ v \in H^1(\Omega_\gamma) \;\middle|\; v = 0 \;\text{ on } \Gamma \right\},$$
$$H^{2,0}(\Omega_\gamma) = \left\{ v \in H^2(\Omega_\gamma) \;\middle|\; v = \tfrac{\partial v}{\partial e} = 0 \;\text{ on } \Gamma \right\},$$
$$H(\Omega_\gamma) = H^{1,0}(\Omega_\gamma)^2 \times H^{2,0}(\Omega_\gamma),$$

where e is the external normal vector to Γ. Consider the following sets

$$K = \{\chi = (W, w) \in H(\Omega_\gamma) \mid \quad \chi \text{ satisfies } (2) \quad \text{a. e. on } \gamma\},$$
$$\mathcal{K} = \{\chi \in L^2(0, T; H(\Omega_\gamma)) \mid \quad \chi(t) \in K \quad \text{a. e. on } \; (0, T)\}$$

of admissible displacements. We will use the following well-known bilinear forms for Kirchhoff–Love plates

$$B(W, \widetilde{W}) = \langle \sigma_{ij}(W), \varepsilon_{ij}(\widetilde{W}) \rangle,$$
$$b(w, \widetilde{w}) = \int\limits_{\Omega_\gamma} (w_{xx}\widetilde{w}_{xx} + w_{yy}\widetilde{w}_{yy} + \kappa w_{xx}\widetilde{w}_{yy} + \kappa w_{yy}\widetilde{w}_{xx}$$
$$+ 2(1-\kappa) w_{xy}\widetilde{w}_{xy}),$$

where $\langle \cdot, \cdot \rangle$ corresponds to the inner product in $L_2(\Omega_\gamma)$.

3. Existence of a solution.

Let us introduce the following spaces for sought functions and their components $\Xi = \{\theta \in L^2(0, T; H^{1,0}(\Omega_\gamma)) \mid \theta_t \in L^2(Q_\gamma)\}$ equipped with the norm

$$\|\theta\|_\Xi^2 = \|\theta\|^2_{L^2(0,T;H^{1,0}(\Omega_\gamma))} + \|\theta_t\|^2_{L^2(Q_\gamma)};$$
$$H = H^1(0, T; H(\Omega_\gamma)), \quad U = \Xi \times H.$$

We will assume that $\theta_0 \in H^{1,0}(\Omega_\gamma)$. Properties of Ξ guarantee that an arbitrary $\theta \in \Xi$ has a well-defined trace at $t = 0$; in particular, $\theta(0) \in L^2(\Omega_\gamma)$. The operation of taking a trace acts continuously from Ξ into $L^2(\Omega_\gamma)$. It is easy to show that the following set

$$S = \{(\theta, \chi) \in U \mid \quad \theta(0) = \theta_0 \;\text{ in } \; \Omega_\gamma, \quad \chi \in \mathcal{K}\}$$

is convex in U. Consider the following linear and continuous operator $L : U \to U^\star$, with values in the dual space $U^\star$ defined by the formula

$$\{L(\theta, \chi), (\bar{\theta}, \bar{\chi})\} = \int\limits_{Q_\gamma} \left(\theta_t + \delta^2 \frac{\partial}{\partial t}(\operatorname{div} W - \Delta w) \right) \bar{\theta}$$

$$+\int\limits_{Q_\gamma} \nabla\theta\nabla\bar{\theta} + \int\limits_0^T (B(W,\widetilde{W}) + b(w,\widetilde{w}) + \delta^2\langle\theta, \Delta\widetilde{w}\rangle - \delta^2\langle\theta, \operatorname{div}\widetilde{W}\rangle),$$

where bracket $\{\,\cdot\,,\,\cdot\,\}$ denotes the dual pairing between U and $U^\star$ [5].

Now we can formulate our problem. Assume that $f \in L^2(Q_\gamma)$. An element $(\theta,\chi) \in U$ is said to be a solution to the equilibrium problem for a thermoelastic plate with a crack if it satisfies the variational inequality

$$\{L(\theta,\chi), (\bar{\theta},\bar{\chi}) - (\theta,\chi)\} \geq \int\limits_{Q_\gamma} f(\bar{\theta}-\theta), \quad (\theta,\chi) \in S \quad \forall\, (\bar{\theta},\bar{\chi}) \in S. \tag{6}$$

Note that L is pseudo-monotone, but non-coercive in space U [5]. The following result can be proved.

Теорема 1. *For δ small enough, there is a solution to problem* (6).

The proof of this statement repeats the steps of reasonings given in [5]. It is expedient to note here that the difference between the considered sets of admissible functions in [5] from K and $\mathcal{K}$ of this paper does not make a significant difference to the course of reasoning.

4. Equivalent differential statement

In this section, we derive equations for describing quasistatic equilibrium for the plate and conditions that are satisfied on γ^T for the solution (θ,χ) of (6). In order to focus on the qualitative properties of the considered model, assume that the parameter $\delta = 1$. In what follows, we will assume that the solution is sufficiently smooth. For brevity, hereafter we denote the quantities W^t, w^t, θ^t by W, w, θ, indicating each time the value of the variable t at which the corresponding relations hold. With respect to the geometry Ω_γ, we require additional properties necessary to use Green's formulas. Suppose that γ can be extended to a closed curve Σ so that the domain Ω_γ is split into two domains Ω_1, Ω_2 with boundaries of class $C^{1,1}$ and $\partial\Omega_1 = \Sigma$, $\bar{\Omega}_1 \subset \Omega$, $\partial\Omega_2 = \Sigma \cup \Gamma$.

Substituting into (6) test functions of the form $(\bar{\theta},\bar{\chi})$, $\bar{\theta} = \theta + \tilde{\theta}$, $\tilde{\theta} \in C_0^\infty(Q_\gamma)$, $\bar{\chi} = \chi + \tilde{\chi}$, $\tilde{\chi} \in C_0^\infty(Q_\gamma)$, we obtain that the following equalities hold

$$\frac{\partial\theta}{\partial t} - \Delta\theta + \frac{\partial}{\partial t}(\operatorname{div} W - \Delta w) = f \quad \text{in} \quad Q_\gamma, \tag{7}$$

$$-\sigma_{ij,j} + \theta_{,i} = 0, \quad i = 1,2, \quad \text{in} \quad Q_\gamma, \tag{8}$$

$$\Delta^2 w + \Delta\theta = 0 \quad \text{in} \quad Q_\gamma. \tag{9}$$

Let $\mathcal{O} \subset R^2$ be a bounded domain with a smooth boundary Υ and having an outer normal $n = (n_1, n_2)$. Below we write out auxiliary Green's formulas that are valid for sufficiently smooth functions u and v

$$b_{\mathcal{O}}(u,v) = \left\langle M(u), \frac{\partial v}{\partial n} \right\rangle_\Upsilon - \langle R(u), v\rangle_\Upsilon + \langle \Delta^2 u, v\rangle_{\mathcal{O}}. \tag{10}$$

Here, the subscripts $\mathcal{O}$ and Υ signify that the integration is taken over the domain $\mathcal{O}$ and the boundary Υ respectively. The operators in the formula (10) are defined on Υ as follows:

$$M(u) = \kappa\Delta u + (1-\kappa)\frac{\partial^2 u}{\partial n^2}, \quad R(u) = \frac{\partial}{\partial n}\Delta u + (1-\kappa)\frac{\partial^3 u}{\partial n \partial s^2},$$

where $s = (-n_2, n_1)$. For functions of the form $\varphi = (\varphi_1, \varphi_2)$, the following formula holds:

$$\langle \varphi, \nabla u\rangle_{\mathcal{O}} = \langle \varphi n, u\rangle_{\Upsilon} - \langle \operatorname{div}\varphi, u\rangle_{\mathcal{O}}.$$

Applying the last formulas, it is easy to derive the following equalities which hold for the domain Ω_γ and smooth functions vanishing on the outer boundary Γ

$$\langle \varphi, \nabla u\rangle = -[\langle \varphi\nu, u\rangle_\gamma] - \langle \operatorname{div}\varphi, u\rangle_{\Omega_\gamma}, \tag{11}$$

$$\langle \sigma_{ij}(U), \varepsilon_{ij}(V)\rangle = -\langle \sigma_{ij,j}(U), v_i\rangle - \Big[\langle \sigma_\nu(U), V\nu\rangle_\gamma + \langle \sigma_\tau(U), V\tau\rangle_\gamma\Big], \tag{12}$$

where

$$\sigma_\nu(U) = \sigma_{ij}(U)\nu_i\nu_j, \quad \sigma_\tau(U) = (\sigma_\tau^1(U), \sigma_\tau^2(U)) = (\sigma_{1j}(U)\nu_j, \sigma_{2j}(U)\nu_j) - \sigma_\nu(U)\nu,$$
$$V\nu = v_i\nu_i, \quad V\tau = (V_\tau^1, V_\tau^2), \quad v_i = (V\nu)\nu_i + V_\tau^i, \quad i = 1,2;$$

$$b(u,v) = -\Big[\langle M(u), \frac{\partial v}{\partial \nu}\rangle_\gamma\Big] + \Big[\langle R(u), v\rangle_\gamma\Big] + \langle \Delta^2 u, v\rangle. \tag{13}$$

Substituting separately into (6) the functions of the both types $(\bar\theta, \chi)$, $(\theta, \widetilde{\chi})$, we obtain two variational inequalities

$$\int\limits_{Q_\gamma} \Big(\frac{\partial\theta}{\partial t} + \frac{\partial}{\partial t}(\operatorname{div} W - \Delta w) - f\Big)(\bar\theta - \theta) + \tag{14}$$
$$+ \int\limits_{Q_\gamma} \nabla\theta(\nabla\bar\theta - \nabla\theta) \ge 0 \quad \forall\, (\bar\theta, \chi) \in S,$$

$$\int\limits_0^T \Big(B(W, \widetilde{W} - W) + b(w, \widetilde{w} - w) + \langle \theta, \Delta\widetilde{w} - \Delta w\rangle \tag{15}$$
$$-\langle \theta, \operatorname{div}\widetilde{W} - \operatorname{div} W\rangle\Big) \ge 0, \quad \forall\, (\theta, \widetilde{\chi}) \in S.$$

Note that summing (14) and (15), we get the relation (6). Using (11) from (14) we find that

$$\int\limits_{\Omega_\gamma} \Big(\frac{\partial\theta}{\partial t} + \frac{\partial}{\partial t}(\operatorname{div} W - \Delta w) - f - \triangle\theta\Big)\bar\theta - \int\limits_\gamma [\frac{\partial\theta}{\partial\nu}\,\bar\theta\,] = 0, \quad \forall\, \bar\theta \in H^{1,0}(\Omega_\gamma). \tag{16}$$

Hence, in view of (7) and the arbitrariness of $\bar\theta \in H^{1,0}(\Omega_\gamma)$, we get

$$\frac{\partial\theta}{\partial\nu} = 0 \quad \text{on} \quad \gamma^+, \quad \frac{\partial\theta}{\partial\nu} = 0 \quad \text{on} \quad \gamma^-,$$

or

$$\frac{\partial\theta}{\partial\nu} = 0 \quad \text{on} \quad \gamma. \tag{17}$$

From (15) it follows that for a.e. $t \in (0,T)$ the inequality

$$B(W, \bar W - W) + b(w, \bar w - w) + \tag{18}$$

$$\langle \theta, \Delta \bar{w} - \Delta w \rangle - \langle \theta, \operatorname{div} \bar{W} - \operatorname{div} W \rangle \geq 0, \quad \forall \bar{\chi} \in K$$

takes place. Substituting in the last inequality test functions of the form $(\bar{W}, w)$ satisfying

$$[\bar{W}]\nu \geq [\frac{\partial w}{\partial \nu}] \quad \text{on} \quad \gamma,$$

we derive

$$B(W, \bar{W} - W) - \langle \theta, \operatorname{div} \bar{W} - \operatorname{div} W \rangle \geq 0, \quad \forall \bar{\chi} \in K. \tag{19}$$

Considering (19) with test functions $\bar{W} = W + \tilde{W}$, $\tilde{W} \in H^1(\Omega_\gamma)$, $[\tilde{W}]\nu \geq 0$ on γ, we obtain by the Green's formulas (11), (12) the following boundary conditions

$$[\sigma_\nu(W) - \theta] = 0, \quad \sigma_\tau(W) = 0 \quad \text{on} \quad \gamma. \tag{20}$$

Substituting now $(W, \widetilde{w})$ into (18), we justify the inequality

$$b(w, \widetilde{w} - w) + \langle \theta, \Delta \widetilde{w} - \Delta w \rangle \geq 0, \tag{21}$$

which holds for all $\widetilde{w}$ satisfying conditions

$$[W]\nu \geq [\partial \widetilde{w}/\partial \nu] \geq 0 \quad \text{on} \quad \gamma, \quad \widetilde{w} \in H^{2,0}(\Omega_\psi).$$

Let us analyze (21) by substituting into it the test functions $w + \varphi$, with smooth functions φ defined on the domain Ω_γ such that $\operatorname{supp}(\varphi) \subset \mathcal{O}^+(x)$, $\mathcal{O}(x)$ is a neighborhood of some point $x \in \gamma$, and $\mathcal{O}^+(x)$ is a subdomain of $\mathcal{O}(x)$ lying to the side γ^+, $[\partial\varphi/\partial\nu] = 0$, $[\varphi] = 0$. Further transforming, taking into account the formulas (11), (13) and the conditional arbitrariness of φ, we find

$$[M(w) + \theta] = 0, \quad [R(w)] = 0 \quad \text{on} \quad \gamma. \tag{22}$$

We continue the analysis of (18) by substituting test functions $(W, w) + (\widetilde{W}, \widetilde{w})$, $(\widetilde{W}, \widetilde{w}) \in K$. After some transformation, we have

$$B(W, \widetilde{W}) + b(w, \widetilde{w}) + \langle \theta, \Delta \widetilde{w} \rangle - \langle \theta, \operatorname{div} \widetilde{W} \rangle \geq 0.$$

Application of the formulas (11), (13) along with (20), (22) gives

$$\left\langle M(w) + \theta, \left[\frac{\partial \widetilde{w}}{\partial \nu}\right] \right\rangle_\gamma + \langle \sigma_\nu(W) - \theta, [\widetilde{W}]\nu \rangle_\gamma \leq 0. \tag{23}$$

Substituting into (23) functions $\widetilde{W}$, $\widetilde{w}$ smooth in Ω_γ and having support in $\mathcal{O}(x)^+$, for an arbitrary point $x \in \gamma$, $[\partial \tilde{w}/\partial \nu] = [\widetilde{W}]\nu$, we obtain

$$M(w) + \sigma_\nu(W) = M(w) + \theta + \sigma_\nu(W) - \theta \leq 0 \quad \text{on} \quad \gamma. \tag{24}$$

Now choosing in (23) functions $\widetilde{W}$, $\widetilde{w}$, smooth in Ω_γ, with the following properties: $\widetilde{w} \equiv 0$, $\operatorname{supp}(\widetilde{W}) \subset \mathcal{O}^+(x) \subset \gamma$, for an arbitrary point $x \in \gamma$, $[\widetilde{W}]\nu \geq 0$, we get

$$\langle \sigma_\nu(W) - \theta, [\widetilde{W}]\nu \rangle_\gamma \leq 0 \quad \text{on} \quad \gamma.$$

Whence, due to the arbitrariness of $x \in \gamma$ and the conditional arbitrariness of $\widetilde{W}$, we derive that

$$\sigma_\nu(W) - \theta \leq 0 \quad \text{on} \quad \gamma. \tag{25}$$

Now we rewrite (23) for $(\widetilde{W}, \widetilde{w}) = (W, w)$ in the following form

$$\int_\gamma (M(w) + \theta + \sigma_\nu(W) - \theta) \left[\frac{\partial w}{\partial \nu}\right] + (\sigma_\nu(W) - \theta) \left([W]\nu - \left[\frac{\partial w}{\partial \nu}\right]\right) \leq 0. \tag{26}$$

Hence, since each term in (26) is non-positive, we deduce that

$$M(w)\left[\frac{\partial w}{\partial \nu}\right]+\sigma_\nu(W)[W]\nu=0 \quad \text{on} \quad \gamma. \tag{27}$$

Let us justify that from the differential setting consisting of equations (7)–(9), initial and boundary conditions (1), (4), (5), (17), (20), (22), (24), (25), (27) the variational inequality (6) can be derived. To do this, consider a smooth function $\widetilde{\chi}=(\widetilde{W},\widetilde{w})\in K$. Multiply equations (8), (9), taken at a fixed $t\in(0,T)$, by $\widetilde{w}_i-w_i(t)$ and $\widetilde{w}-w(t)$, respectively. Afterwards, integrate over Ω_γ and apply the formulas (11)–(13) on taking boundary conditions (5), (17), (20), (22). Further, summing the found relations, for a fixed t, we conclude that the equality holds (within the framework of this section $\delta=1$)

$$B(W,\widetilde{W}-W)+b(w,\widetilde{w}-w)+\langle\theta,\Delta\widetilde{w}-\Delta w\rangle-\langle\theta,\operatorname{div}\widetilde{W}-\operatorname{div}W\rangle$$
$$+\left\langle M(w)+\theta,\left[\frac{\partial\widetilde{w}}{\partial\nu}\right]-\left[\frac{\partial w}{\partial\nu}\right]\right\rangle_\gamma+\langle\sigma_\nu(W)-\theta,[\widetilde{W}]\nu-[W]\nu\rangle_\gamma=0.$$

In view of (24), (25), (27) in the last equality the sum of the integrals over the boundary have non-positive values, whence inequality (18) immediately follows. Consequently, we derive (15). For fixed $t\in(0,T)$, multiplying (7) by $\widetilde{\theta}-\theta(t)$ and integrating again over Ω_γ along with the formulas (11) and boundary conditions (4), (5), (17), we get (14). At this stage, we can apply the approach used in [5], and obtain the inequality (6).

Теорема 2. *Assuming that the solution (θ,χ) is sufficiently smooth, the variational problem* (6) *is equivalent to the boundary value problem consisting of the equations* (7)–(9)*, initial and boundary conditions* (1)*,* (4)*,* (5)*,* (17)*,* (20)*,* (22)*,* (24)*,* (25)*,* (27)*.*

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