S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 17, стр. 2131–2141 (2020) УДК 512.5
DOI 10.33048/semi.2020.17.143 MSC 20E10

QUASIVARIETIES OF NILPOTENT GROUPS OF AXIOMATIC RANK 4

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Abstract. We say that the axiomatic rank of a quasivariety K is equal to n if K can be defined by a system of quasi-identities in n variables and cannot be defined by any set of quasi-identities in fewer variables. If there is no such n, then K has an infinite axiomatic rank. We prove that the set of quasivarieties of nilpotent torsion-free groups of class at most 2 of axiomatic rank 4 is continual.

Keywords: nilpotent group, quasivariety, variety, axiomatic rank.

1. Introduction

The axiomatic rank of a quasivariety equals n, if this quasivariety can be defined by a set of quasi-identities in n variables and cannot be defined by a set of quasi-identities in fewer number of variables. This paper is devoted to the study of quasivarieties of nilpotent torsion-free groups of class at most 2 of an axiomatic rank 4.

The axiomatic rank of a quasivariety is one of its most important characteristics. Among the papers devoted to this area of studies, we will mention only those, in which the axiomatic ranks of a number of natural and important objects of group theory are calculated. The axiomatic rank of a quasivariety generated by a finite group with nonabelian Sylow subgroup is found in [1], the one of a quasivariety generated by all finite groups was obtained in [2]. Axiomatic ranks of a wide class of quasivarieties (quasivarieties generated by a free group, a group with one defining relation, a free solvable group, a finitely generated nilpotent group) were calculated in [3, 4, 5, 6].

The set of all quasivarieties having in a given quasivariety $\mathcal{N}$ an axiomatic rank which is at most n, is partially ordered with respect to inclusion and forms a

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Received April, 16, 2020, published on December, 22, 2020.

lattice denoted by $L_q^n(\mathcal{N})$. It turned out that in many cases studying of a lattice of quasivarieties reduces to studying lattices of the form $L_q^n(\mathcal{N})$. For example, from [7] it is known that the lattice $L_q^n(\mathcal{N})$ is a homomorphic image of the lattice $L_q(\mathcal{N})$ of quasivarieties lying in $\mathcal{N}$. Moreover, if $\mathcal{N}$ is a locally finite quasivariety, then the lattice $L_q(\mathcal{N})$ is approximated by the lattices $L_q^n(\mathcal{N})$. Due to this fact, studying the lattices $L_q^n(\mathcal{N})$ can be considered to be a natural approach to studying of the lattice of quasivarieties $L_q(\mathcal{N})$.

Lattices of quasivarieties of groups of nilpotency class 2 have quite a complex structure. The information on the complexity of these lattices can be found in [4, 8, 9, 10]. Note the articles [4, 9, 10], in which it is established that only one nonabelian quasivariety of torsion-free nilpotent groups of class 2 and only a finite set of quasivarieties of groups of nilpotency class 2 and exponent p^k with a derived subgroup of a prime exponent p have a finite lattice of subquasivarieties. On the other hand, the set of quasivarieties of torsion-free nilpotent groups of class at most 2 and axiomatic rank at most 3 is finite. Those are described in [11].

The goal of this paper is to prove that a set of quasivarieties of torsion-free nilpotent groups of class at most 2 and axiomatic rank 4 is continual.

2. Preliminary facts

Recall some notions and definitions.

By $\langle S\rangle$ we denote a group generated by a set S, $\langle a\rangle$ is a cyclic group generated by an element a. $Z(G)$ is the center of G, G' is a derived subgroup of the group G, $\ker\alpha$ is the kernel of a homomorphism α. If x, y are elements of a group, then $[x,y] = x^{-1}y^{-1}xy$.

We will call every homomorphism $\varphi : A \to B$ that is an isomorphism of A onto $\varphi(A)$ an embedding of a group A into a group B. If there exists an embedding of A into B, then we say that A is embeddable into B.

We will use the following Dyck's theorem [12, Ch.5, §11, Theorem 5].

Lemma 1. *Suppose that a group G in a given quasivariety $\mathcal{N}$ has a representation*

$$G = \langle \{x_i \mid i \in I\} \parallel \{r_j(x_{j_1},\dots,x_{j_{l(j)}}) = 1 \mid j \in J\}\rangle.$$

Assume that $H \in \mathcal{N}$ and the group H contains a set of elements $\{g_i \mid i \in I\}$, such that for every $j \in J$ the equality $r_j(g_{j_1},\dots,g_{j_{l(j)}}) = 1$ is true in H. Then the mapping $x_i \to g_i$ $(i \in I)$ extends to a homomorphism of G into H.

We will often use the following fact: in every group of nilpotency class 2 the following commutator identities [13, §3.2] hold:

$$(\forall x)(\forall y)(\forall z)([xy,z] = [x,z][y,z]), (\forall x)(\forall y)(\forall z)([x,yz] = [x,y][x,z]),$$
$$(\forall x)(\forall y)([x^n,y^m] = [x,y]^{mn} \ (m,n \text{ are integers}).$$

We designate by $G_m (m = 2,3,\dots)$ a group that has in the class of nilpotent groups of class ≤ 2 the representation

$$G_m = \langle x_1,x_2,x_3,x_4 \mid [x_1,x_2] = [x_3,x_4] = 1, [x_1,x_3]^m = [x_2,x_4], [x_1,x_4] = [x_3,x_2]\rangle.$$

$\Phi_m (m = 2,3,\dots)$ is the following quasi-identity:

$$\Phi_m = (\forall x_1)(\forall x_2)(\forall x_3)(\forall x_4)([x_1,x_2] = 1\&[x_3,x_4] = 1\&[x_1,x_3]^m = [x_2,x_4]\&$$
$$\&[x_1,x_4] = [x_3,x_2] \to [x_2,x_3] = 1).$$

It is easy to see that the group G_m has the following properties:

1) it is torsion-free,

2) the derived subgroup of the group G_m is a free Abelian group generated by the elements $[x_1,x_3],[x_2,x_3]$,

3) $Z(G_m)=G'_m$,

4) $[x_1,x_4]=[x_2,x_3]^{-1},\ [x_2,x_4]=[x_1,x_3]^m,[x_1,x_2]=1,\ [x_3,x_4]=1$.

The necessary information on the group theory can be found in [13], on quasivarieties theory in [12], on number theory in [14].

3. Main result

We say that a pair (m,r) of prime numbers possesses the property (*), when $x^2+my^2\equiv 0 \pmod r$ if and only if $x\equiv 0 \pmod r$ and $y\equiv 0 \pmod r$.

Lemma 2. *There exists an infinite set F of primes, such that every pair (m,r) ($m,r\in F$, $m\neq r$) possesses the property (*).*

Доказательство. Note that the pair (m,r) has the property (*) if and only if $-m$ *is a quadratic nonresidue modulo* r.

By induction, we will construct a sequence $q_1,q_2,\dots$ of primes of the form $4k+1$, where every pair possesses the property (*), beginning it with $q_1=5$, $q_2=13$. Let $\left(\frac{m}{r}\right)$ be the Legendre symbol. For such numbers, it follows from [14, Proposition 5.1.2] that

$\left(\frac{-m}{r}\right)=\left(\frac{-1}{r}\right)\left(\frac{m}{r}\right)=(-1)^{(r-1)/2}\left(\frac{m}{r}\right)=(-1)^{2k}\left(\frac{m}{r}\right)=\left(\frac{m}{r}\right)$.

Therefore, for primes of the form $4k+1$ we have that: $-m$ *is a quadratic nonresidue modulo r if and only if m is a quadratic nonresidue modulo r.* From the law of quadratic reciprocity [14, Ch.5, §2, Theorem 1] for numbers of the form $4k+1$, we obtain

$\left(\frac{m}{r}\right)\left(\frac{r}{m}\right)=(-1)^{((m-1)/2)((r-1)/2}=1$.

Hence, m *is a quadratic nonresidue modulo r if and only if r is a quadratic nonresidue modulo m.*

Now assume that $q_1,q_2,\dots,q_k$ have been found. Consider the following system of congruences:

$$\begin{aligned}&x\equiv q_k \pmod{q_1},\\ &x\equiv q_1 \pmod{q_2},\\ &x\equiv q_2 \pmod{q_3},\\ &\dots\dots\dots\dots\dots\dots\\ &x\equiv q_{k-1} \pmod{q_k},\\ &x\equiv 1 \pmod 4.\end{aligned}$$

By Chinese remainder theorem [14, Ch.3, §4, Theorem 1], this system of congruences has a solution. Let the number x_0 satisfy this system of congruences. It is clear that $x_0\not\equiv 0 \pmod{q_i}$, $x_0\not\equiv 0 \pmod 2$. Due to Dirichlet's theorem [14, Ch.16, §1, Theorem 1] on infiniteness of the number of primes in an arithmetic progression, the sequence

$$x_0, x_0+4q_1\cdots q_k,\dots,x_0+4q_1\cdots q_k n,\dots$$

contains an infinite set of primes. We take as q_{k+1} any of these primes (q_{k+1} has the form $4m+1$). We calculate $\left(\frac{q_{k+1}}{q_i}\right)$. As $q_{k+1}\equiv q_j \pmod{q_i}$ for some j ($j\neq i$), then from [14, Proposition 5.1.2] it follows that $\left(\frac{q_{k+1}}{q_i}\right)=\left(\frac{q_j}{q_i}\right)=-1$, that is, q_{k+1} is a quadratic nonresidue modulo q_i. From the above arguments, it follows that every pair from the sequence $q_1,q_2,\dots,q_{k+1}$ possesses the property (*). □

Lemma 3. *Let t_1, t_2, t_3, t_4 be real numbers and $t_1^2 + t_2^2 \neq 0$. Then the mapping*

$$x_1 \to x_1^{t_1} x_2^{t_2} x_3^{t_3} x_4^{t_4}, x_2 \to x_1^{-mt_2} x_2^{t_1} x_3^{mt_4} x_4^{-t_3}, x_3 \to x_3, x_4 \to x_4$$

extends to the embedding $\alpha_1 = \alpha_1(t_1, t_2, t_3, t_4) : G_m \to G_m$, moreover,

$$\alpha_1(x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}) = x_1^{m_1 t_1 - m m_2 t_2} x_2^{m_1 t_2 + m_2 t_1} x_3^{m_1 t_3 + m m_2 t_4 + m_3} x_4^{m_1 t_4 - m_2 t_3 + m_4} c$$

for some $c \in G'_m$.

Доказательство. Using the commutator identities, we obtain

$$[x_1^{t_1} x_2^{t_2} x_3^{t_3} x_4^{t_4}, x_1^{-mt_2} x_2^{t_1} x_3^{mt_4} x_4^{-t_3}] = [x_1, x_3]^{t_1(mt_4) - t_3(-mt_2)} [x_1, x_4]^{t_1(-t_3) - t_4(-mt_2)} \cdot$$

$$\cdot [x_2, x_4]^{t_2(-t_3) - t_4 t_1} [x_2, x_3]^{t_2 m t_4 - t_3 t_1} =$$

$$= [x_1, x_3]^{t_1(mt_4) - t_3(-mt_2)} [x_2, x_3]^{t_1 t_3 - t_4 m t_2} [x_1, x_3]^{m(t_2(-t_3) - t_4 t_1)} [x_2, x_3]^{t_2 m t_4 - t_3 t_1} = 1,$$

$$[x_1^{t_1} x_2^{t_2} x_3^{t_3} x_4^{t_4}, x_3]^m = [x_1, x_3]^{mt_1} [x_2, x_3]^{mt_2},$$

$$[x_1^{-mt_2} x_2^{t_1} x_3^{mt_4} x_4^{-t_3}, x_4] = [x_1, x_4]^{-mt_2} [x_2, x_4]^{t_1} = [x_2, x_3]^{mt_2} [x_1, x_3]^{mt_1},$$

$$[x_1^{t_1} x_2^{t_2} x_3^{t_3} x_4^{t_4}, x_4] = [x_1, x_4]^{t_1} [x_2, x_4]^{t_2} = [x_2, x_3]^{-t_1} [x_1, x_3]^{mt_2},$$

$$[x_3, x_1^{-mt_2} x_2^{t_1} x_3^{mt_4} x_4^{-t_3}] = [x_3, x_1]^{-mt_2} [x_3, x_2]^{t_1} = [x_1, x_3]^{mt_2} [x_2, x_3]^{-t_1}.$$

By Lemma 1, this mapping extends to the homomorphism $\alpha_1 : G_m \to G_m$. We will show that $\ker \alpha_1 = 1$.

Suppose that $\ker \alpha_1 \neq 1$. It is well known (see, for example, [13], Ch. 6, §1, Theorem 1) that an intersection of every nonunit normal subgroup of a nilpotent group with the center of this group is nontrivial. Therefore, $\ker \alpha_1 \cap Z(G_m) \neq 1$. We take an arbitrary element $g \in \ker \alpha_1 \cap Z(G_m)$. The element g can be presented in the following form: $g = [x_1, x_3]^k [x_2, x_3]^n$. Then

$$\alpha_1(g) = [x_1^{t_1} x_2^{t_2} x_3^{t_3} x_4^{t_4}, x_3]^k [x_1^{-mt_2} x_2^{t_1} x_3^{mt_4} x_4^{-t_3}, x_3]^n =$$

$$= [x_1, x_3]^{kt_1} [x_2, x_3]^{kt_2} [x_1, x_3]^{-nmt_2} [x_2, x_3]^{nt_1} = [x_1, x_3]^{kt_1 - nmt_2} [x_2, x_3]^{kt_2 + nt_1}.$$

Hence, $kt_1 - nmt_2 = 0, nt_1 + kt_2 = 0$. If $k^2 + mn^2 \neq 0$, then $t_1 = t_2 = 0$, which contradicts the assumption. Therefore, $k = 0, n = 0$, hence, $g = 1$. Thus, α_1 is an embedding.

Now, we only need to calculate $\alpha_1(x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4})$.

$$\alpha_1(x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}) = (x_1^{t_1} x_2^{t_2} x_3^{t_3} x_4^{t_4})^{m_1} (x_1^{-mt_2} x_2^{t_1} x_3^{mt_4} x_4^{-t_3})^{m_2} x_3^{m_3} x_4^{m_4} \equiv$$

$$\equiv x_1^{m_1 t_1 - m m_2 t_2} x_2^{m_1 t_2 + m_2 t_1} x_3^{m_1 t_3 + m m_2 t_4 + m_3} x_4^{m_1 t_4 - m_2 t_3 + m_4} \pmod{G'_m}.$$

□

Lemma 4. *Let t_1, t_2 be integers, $t_1^2 + t_2^2 \neq 0$. Then the mapping*

$$x_1 \to x_1, x_2 \to x_2, x_3 \to x_3^{t_1} x_4^{t_2}, x_4 \to x_3^{-mt_2} x_4^{t_1}$$

extends to the embedding $\alpha_2 = \alpha_2(t_1, t_2) : G_m \to G_m$, moreover,

$$\alpha_2(x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}) = x_1^{m_1} x_2^{m_2} x_3^{t_1 m_3 - mt_2 m_4} x_4^{t_2 m_3 + t_1 m_4} c$$

for some $c \in G_m'$.

Доказательство. We use the commutator identities again.

$$[x_1, x_3^{t_1} x_4^{t_2}]^m = [x_1, x_3]^{mt_1} [x_1, x_4]^{mt_2} = [x_1, x_3]^{mt_1} [x_2, x_3]^{-mt_2},$$

$$[x_2, x_3^{-mt_2} x_4^{t_1}] = [x_2, x_3]^{-mt_2} [x_2, x_4]^{t_1} = [x_2, x_3]^{-mt_2} [x_1, x_3]^{mt_1},$$

$$[x_1, x_3^{-mt_2} x_4^{t_1}] = [x_1, x_3]^{-mt_2} [x_1, x_4]^{t_1} = [x_1, x_3]^{-mt_2} [x_2, x_3]^{-t_1},$$

$$[x_3^{t_1} x_4^{t_2}, x_2] = [x_3, x_2]^{t_1} [x_4, x_2]^{t_2} = [x_2, x_3]^{-t_1} [x_1, x_3]^{-mt_2},$$

$$[x_3^{t_1} x_4^{t_2}, x_3^{-mt_2} x_4^{t_1}] = 1.$$

By Lemma 1, this mapping extends to the homomorphism $\alpha_2 : G_m \to G_m$. We will show that $\ker \alpha_2 = 1$.

Suppose that $\ker \alpha_2 \neq 1$. Due to the fact that an intersection of every nonunit normal subgroup of a nilpotent group with the center of this group is nontrivial, we have that $\ker \alpha_2 \cap Z(G_m) \neq 1$. We take an arbitrary element $g \in \ker \alpha_2 \cap Z(G_m)$. The element g can be presented in the following form: $g = [x_1, x_3]^k [x_2, x_3]^n$. Therefore,

$$\alpha_1(g) = [x_1, x_3^{t_1} x_4^{t_2}]^k [x_2, x_3^{t_1} x_4^{t_2}]^n = [x_1, x_3]^{kt_1} [x_1, x_4]^{kt_2} [x_2, x_3]^{nt_1} [x_2, x_4]^{nt_2} =$$
$$= [x_1, x_3]^{kt_1 + nmt_2} [x_2, x_3]^{-kt_2 + nt_1}.$$

It means that $kt_1 + nmt_2 = 0, nt_1 - kt_2 = 0$, hence, $k = 0, n = 0$, so $g = 1$. Thus, α_2 is an embedding.

Now, we only need to calculate $\alpha_2(x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4})$.

$$\alpha_2(x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}) = x_1^{m_1} x_2^{m_2} (x_3^{t_1} x_4^{t_2})^{m_3} (x_3^{-mt_2} x_4^{t_1})^{m_4} \equiv$$

$$\equiv x_1^{m_1} x_2^{m_2} x_3^{t_1 m_3 - mt_2 m_4} x_4^{t_2 m_3 + t_1 m_4} \pmod{G_m'}.$$

□

Lemma 5. *Let t_1, t_2, t_3 be nonzero integers and t_3 be divisible by mt_2. Then the mapping*

$$x_1 \to x_1^{t_1} x_4^{-\frac{t_3}{mt_2}}, x_2 \to x_2^{t_1} x_3^{-\frac{t_3}{t_2}}, x_3 \to x_3, x_4 \to x_4$$

extends to the embedding $\alpha_3 = \alpha_3(t_1, t_2, t_3) : G_m \to G_m$, moreover,

$$\alpha_3(t_1, t_2, t_3)(x_2^{t_2} x_3^{t_3}) = x_2^{t_1 t_2} c$$

for some $c \in G_m'$.

Доказательство. We can see that

$$[x_1^{t_1}x_4^{-\frac{t_3}{mt_2}}, x_2^{t_1}x_3^{-\frac{t_3}{t_2}}] = [x_1,x_3]^{-t_1\frac{t_3}{t_2}}[x_4,x_2]^{-\frac{t_3}{mt_2}t_1} = [x_1,x_3]^{-t_1\frac{t_3}{t_2}}[x_1,x_3]^{m\frac{t_3}{mt_2}t_1} = 1,$$

$$[x_1^{t_1}x_4^{-\frac{t_3}{mt_2}}, x_3]^m = [x_1,x_3]^{mt_1},$$

$$[x_2^{t_1}x_3^{-\frac{t_3}{t_2}}, x_4] = [x_2,x_4]^{t_1}[x_3,x_4]^{-\frac{t_3}{t_2}} = [x_1,x_3]^{mt_1},$$

$$[x_1^{t_1}x_4^{-\frac{t_3}{mt_2}}, x_4] = [x_1,x_4]^{t_1} = [x_2,x_3]^{-t_1},$$

$$[x_3, x_2^{t_1}x_3^{-\frac{t_3}{t_2}}] = [x_2,x_3]^{-t_1}.$$

By Lemma 1, this mapping extends to a homomorphism $\alpha_3 : G_m \to G_m$. We will show that $\ker\alpha_3 = 1$.

Suppose that $\ker\alpha_3 \neq 1$. Since an intersection of every nonunit normal subgroup of a nilpotent group with the center of this group is nontrivial, we have that $\ker\alpha_3 \cap Z(G_m) \neq 1$. We take an arbitrary element $g \in \ker\alpha_3 \cap Z(G_m)$. The element g can be presented in the following form: $g = [x_1,x_3]^k[x_2,x_3]^n$. Therefore,

$$\alpha_3(g) = [x_1^{t_1}x_4^{-\frac{t_3}{mt_2}}, x_3]^k[x_2^{t_1}x_3^{-\frac{t_3}{t_2}}, x_3]^n = [x_1,x_3]^{kt_1}[x_2,x_3]^{nt_1}.$$

Hence $kt_1 = 0, nt_1 = 0$, so $k = 0, n = 0$, which means that $g = 1$. Thus, α_3 is an embedding.

Now, we only need to calculate $\alpha_3(t_1,t_2,t_3)(x_2^{t_2}x_3^{t_3})$: $\alpha_3(x_2^{t_2}x_3^{t_3}) = (x_2^{t_1}x_3^{-\frac{t_3}{t_2}})^{t_2}x_3^{t_3} \equiv x_2^{t_1t_2} \pmod{G'_m}$. □

Lemma 6. *Let $\overline{x} = x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}$ be a nonunit element from G_m. Then there exists an embedding $\gamma : G_m \to G_m$, such that $\gamma(\overline{x})$ modulo G'_m has the form: x_2^s, or x_3^t.*

Доказательство. We will construct a sequence of embeddings G_m into G_m. We agree to consider that β_1 is an identity mapping if $m_1 = 0$, and $\beta_1 = \alpha_1(mm_2, m_1, 1, 1)$ if $m_1 \neq 0$. By Lemma 3, $\beta_1(\overline{x})$ modulo G'_m equals the element of the form $x_2^{s_2}x_3^{s_3}x_4^{s_4}$.

Consider that β_2 is an identity mapping if $s_4 = 0$, and $\beta_2 = \alpha_2(s_3, -s_4)$ if $s_4 \neq 0$. By Lemma 4, $\beta_2(x_2^{s_2}x_3^{s_3}x_4^{s_4})$ modulo G'_m equals the element of the form $x_2^{k_2}x_3^{k_3}$. If $k_2 = 0$ or $k_3 = 0$, then the lemma is proved.

Suppose that $k_2 \neq 0$ and $k_3 \neq 0$. Consider the embedding $\beta_3 = \alpha_2(mk_2, 0)$. Then $\beta_3(x_2^{k_2}x_3^{k_3}) = x_2^{k_2}x_3^{mk_2k_3}$ modulo G'_m. We can see that $\alpha_3(1, k_2, mk_2k_3)(x_2^{k_2}x_3^{mk_2k_3}) = x_2^{k_2}$ modulo G'_m. We get the required. □

Recall that the property (*) has been introduced before Lemma 2.

Lemma 7. *Let m, r be distinct primes and*

$$a_1x_1 + ma_2x_2 = a_3r^{s-1}x_3 \tag{1}$$

$$-a_2x_1 + a_1x_2 = a_4r^{s-1}x_4 \tag{2}$$

$$a_1x_3 + ma_2x_4 = a_5r^sx_1 \tag{3}$$

(4) $$-a_2x_3 + a_1x_4 = a_6r^sx_2$$

be a system of equations with integer $a_1, \dots, a_6$. *Assume that* $s \geq 1$, $r \geq 3$, *GCD*$(r, a_i) = 1 (i = 3, 4, 5, 6)$. *We assume that the pair* (m, r) *possesses the property (*). Then every integer solution of this system is a zero one.*

Доказательство. Let (b_1, b_2, b_3, b_4) be a solution of this system, where b_1, b_2, b_3, b_4 are integers. We will prove that all b_i are divisible by r, present them in the form $b_i = b'_i r$, and see that $(b'_1, b'_2, b'_3, b'_4,)$ is again a solution for this system. What has been said will mean that $b_i = 0 (i = 1, 2, 3, 4)$ (moreover, this means that the given system of linear equations only has a zero solution).

First, assume that $b_3^2 + mb_4^2 \not\equiv 0 \pmod r$. From (3), (4) it follows that

$$a_1b_3 + ma_2b_4 = a_5r^sb_1,$$

$$a_1b_4 - a_2b_3 = a_6r^sb_2,$$

hence,

$$a_1 = \frac{r^s(a_5b_1b_3 + ma_6b_4b_2)}{b_3^2 + mb_4^2}, \ a_2 = \frac{r^s(-a_6b_2b_3 + a_5b_1b_4)}{b_3^2 + mb_4^2}.$$

Since $b_3^2 + mb_4^2$ is not divisible by r, we can see that a_1, a_2 are divisible by r^s. Since $\mathrm{GCD}(r, a_3) = 1$, $\mathrm{GCD}(r, a_4) = 1$, from (1), (2) we obtain that b_3, b_4 are divisible by r. This contradicts the assumption that $b_3^2 + mb_4^2 \not\equiv 0 \pmod r$.

Thus, we have proved that $b_3^2 + mb_4^2 \equiv 0 \pmod r$, therefore, by (*), $b_3 = b'_3 r$, $b_4 = b'_4 r$ for some integers b'_3, b'_4. Now we will show that b_1, b_2 are divisible by r.

Case 1. $a_1^2 + ma_2^2 \not\equiv 0 \pmod r$. From (1), (2) it follows that

$$a_1b_1 + ma_2b_2 = a_3r^{s-1}b_3,$$

$$-a_2b_1 + a_1b_2 = a_4r^{s-1}b_4,$$

hence,

$$b_1 = \frac{r^{s-1}(a_1a_3b_3 - ma_2a_4b_4)}{a_1^2 + ma_2^2}, \ b_2 = \frac{r^{s-1}(a_1a_4b_4 + a_2a_3b_3)}{a_1^2 + ma_2^2}.$$

Therefore, taking into account that $a_1^2 + ma_2^2$ is not divisible by r, b_3, b_4 are divisible by r, we obtain that b_1, b_2 are divisible by r^s, as required.

Case 2. $a_1^2 + ma_2^2 \equiv 0 (mod\ r)$. First, assume that $b_1^2 + mb_2^2 \not\equiv 0 (mod\ r)$. From (1), (2) it follows that

$$a_1b_1 + ma_2b_2 = a_3r^sb'_3,$$

$$a_1b_2 - a_2b_1 = a_4r^sb'_4,$$

hence,

$$a_1 = \frac{r^s(b_1a_3b'_3 - mb_2a_4b'_4)}{b_1^2 + mb_2^2}, \ a_2 = \frac{r^s(-b_1a_4b'_4 + a_3b_2b'_3)}{b_1^2 + mb_2^2}.$$

Since $b_1^2 + mb_2^2 \not\equiv 0 \pmod r$, we can see that a_1, a_2 are divisible by r^s. From (3), (4) it now follows that since b_1, b_2 are divisible by r, then $a_5r^sb_1$, $a_6r^sb_2$ are divisible by r^{s+1}. And because a_5, a_6 are coprime to with r, we conclude that b_1, b_2 are divisible by r, which contradicts the fact that $b_1^2 + mb_2^2 \not\equiv 0 \pmod r$. Thus, $b_1^2 + mb_2^2 \equiv 0 \pmod r$, that is, $b_1 = b'_1 r$, $b_2 = b'_2 r$ for some integer numbers b'_1, b'_2.

Now, substituting $b'_1 r, b'_2 r, b'_3 r, b'_4 r$ into the given system and reducing all the given equations by r, we can see that (b'_1, b'_2, b'_3, b'_4) is a solution of the considered system of equations, as required. □

Almost literally repeating the proof of Lemma 7, we obtain the proof of the following statement.

Corollary 1. *Let m, r be distinct primes and*

$$a_1x_1 + ma_2x_2 = a_3r^{s-1}x_3$$
$$-a_2x_1 + a_1x_2 = a_4r^{s-1}x_4$$
$$a_1x_4 + ma_2x_3 = a_5r^{s}x_2$$
$$-a_2x_4 + a_1x_3 = a_6r^{s}x_1$$

be a system of equations with integer $a_1, \ldots, a_6$. Suppose that $s \geq 1$, $r \geq 3$, GCD$(r, a_i) = 1 (i = 3, 4, 5, 6)$. We assume that the pair (m, r) possesses the property (*). Then every integer solution of this system will be a zero one.

Lemma 8. *Suppose that m, r are distinct primes and the pair (m, r) possesses the property (*). Then the quasi-identity Φ_r is true in the group G_m .*

Доказательство. Let the left-hand side of the quasi-identity Φ_r be true in G_m given the interpretation $x_1 \to \overline{x}_1$, $x_2 \to \overline{x}_2$, $x_3 \to \overline{x}_3$, $x_4 \to \overline{x}_4$. We will prove that the right-hand side of Φ_r is true given this interpretation. Consider the embedding $\gamma : G_m \to G_m$ from Lemma 6, given which $\gamma(\overline{x}_2)$ equals $x_2^{m_2}c_3$, or $x_3^{m_3}c_3$, $c_2, c_3 \in G'_m$ are suitable elements. Since γ is an embedding, it suffices to show that $\gamma([\overline{x}_2, \overline{x}_3]) = 1$. There are 3 possible cases.

CASE 1. $\gamma(\overline{x}_2) \in G'_m$. It is clear that $\gamma([\overline{x}_2, \overline{x}_3]) = 1$.

CASE 2. $\gamma(\overline{x}_2) = x_2^{m_2}c_2$, $m_2 \neq 0$, $c_2 \in G'_m$. Put

$$\gamma(\overline{x}_1) = x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}c_1,$$

$$\gamma(\overline{x}_3) = x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}c_3,$$

$$\gamma(\overline{x}_4) = x_1^{l_1}x_2^{l_2}x_3^{l_3}x_4^{l_4}c_4,$$

where $c_1, c_3, c_4 \in G'_m$. We have that:

$1 = [\gamma(\overline{x}_1), \gamma(\overline{x}_2)] = [x_3, x_2]^{k_3m_2}[x_4, x_2]^{k_4m_2} = [x_3, x_2]^{k_3m_2}[x_1, x_3]^{-mk_4m_2}$. Therefore, $k_3m_2 = 0$, $-mk_4m_2 = 0$, hence, $k_3 = 0$, $k_4 = 0$.

$$[\gamma(\overline{x}_1), \gamma(\overline{x}_3)] = [x_1, x_3]^{k_1n_3-k_3n_1}[x_1, x_4]^{k_1n_4-k_4n_1}[x_2, x_3]^{k_2n_3-k_3n_2}[x_2, x_4]^{k_2n_4-k_4n_2} =$$
$$= [x_1, x_3]^{k_1n_3-k_3n_1+m(k_2n_4-k_4n_2)}[x_2, x_3]^{k_2n_3-k_3n_2-(k_1n_4-k_4n_1)}.$$

$$[\gamma(\overline{x}_2), \gamma(\overline{x}_4)] = [x_2, x_3]^{m_2l_3}[x_2, x_4]^{m_2l_4} = [x_2, x_3]^{m_2l_3}[x_1, x_3]^{mm_2l_4}.$$

From the equalities $k_3 = 0$, $k_4 = 0$, $[\gamma(\overline{x}_1), \gamma(\overline{x}_3)]^r = [\gamma(\overline{x}_2), \gamma(\overline{x}_4)]$ it follows that

$$r(k_1n_3 + mk_2n_4) = mm_2l_4, \tag{5}$$

$$r(k_2n_3 - k_1n_4) = m_2l_3. \tag{6}$$

We can see that

$$[\gamma(\overline{x}_1), \gamma(\overline{x}_4)] = [x_1, x_3]^{k_1l_3-k_3l_1}[x_1, x_4]^{k_1l_4-k_4l_1}[x_2, x_3]^{k_2l_3-k_3l_2}[x_2, x_4]^{k_2l_4-k_4l_2} =$$
$$= [x_1, x_3]^{k_1l_3-k_3l_1+m(k_2l_4-k_4l_2)}[x_2, x_3]^{k_2l_3-k_3l_2-(k_1l_4-k_4l_1)}.$$

$$[\gamma(\overline{x}_3), \gamma(\overline{x}_2)] = [x_2, x_3]^{-m_2n_3}[x_2, x_4]^{-m_2n_4} = [x_2, x_3]^{-m_2n_3}[x_1, x_3]^{-mm_2n_4}.$$

From the equalities $k_3 = 0$, $k_4 = 0$, $[\gamma(\overline{x}_1), \gamma(\overline{x}_4)] = [\gamma(\overline{x}_3), \gamma(\overline{x}_2)]$ it follows that

$$k_1l_3 + mk_2l_4 = -mm_2n_4, \tag{7}$$

$$k_2l_3 - k_1l_4 = -m_2n_3. \tag{8}$$

First, suppose that m_2 is not divisible by r. Then by (5), (6), l_3, l_4 are divisible by r. From (7), (8), it now follows that n_3, n_4 are divisible by r. We write the numbers l_3, l_4, n_3, n_4 in the following form: $l_3 = l'_3r, l_4 = l'_4r, n_3 = n'_3r, n_4 = n'_4r$. We have proved that from the fact that (l_3, l_4, n_3, n_4) is a solution of the system

$$\begin{aligned} r(k_1x_1 + mk_2x_2) &= mm_2x_3 \\ r(k_2x_1 - k_1x_2) &= m_2x_4 \\ k_1x_4 + mk_2x_3 &= -mm_2x_2 \\ k_2x_4 - k_1x_3 &= -m_2x_1 \end{aligned}$$

it follows that all these numbers are divisible by r, that is, $l_3 = l'_3r, l_4 = l'_4r, n_3 = n'_3r, n_4 = n'_4r$ and the set of integers (l'_3, l'_4, n'_3, n'_4) is again a solution of this system of equations. Therefore, $n_3 = n_4 = l_3 = l_4 = 0$. Hence,

$$[\gamma(\overline{x}_3), \gamma(\overline{x}_2)] = [x_2, x_3]^{-m_2n_3}[x_2, x_4]^{-m_2n_4} = 1.$$

Now let m_2 be divisible by r. Then $m_2 = m'_2r^s$ for some $s \geq 1$. We can see that in this case l_3, l_4, n_3, n_4 satisfy the system from Corollary 1, by which we have that $l_3 = l_4 = n_3 = n_4 = 0$. Since

$$[\gamma(\overline{x_2}), \gamma(\overline{x_3})] = [x_2, x_3]^{-m_2n_3}[x_2, x_4]^{-m_2n_4},$$

then $[\gamma(\overline{x_2}), \gamma(\overline{x_3})] = 1$.

Case 3. $\gamma(\overline{x}_2) = x_3^{m_3}c_3$, $m_3 \neq 0$, $c_3 \in G'_m$. We assume that $\gamma(\overline{x}_1)$, $\gamma(\overline{x}_3)$, $\gamma(\overline{x}_4)$ are the same as in Case 2.

$$1 = [\gamma(\overline{x}_1), \gamma(\overline{x}_2)] = [x_2, x_3]^{k_2m_3}[x_1, x_3]^{k_1m_3}.$$

Therefore, $k_2m_3 = 0$, $mk_1m_3 = 0$, hence, $k_1 = 0$, $k_2 = 0$.

$$\begin{aligned} [\gamma(\overline{x}_1), \gamma(\overline{x}_3)] &= [x_1, x_3]^{k_1n_3-k_3n_1}[x_1, x_4]^{k_1n_4-k_4n_1}[x_2, x_3]^{k_2n_3-k_3n_2}[x_2, x_4]^{k_2n_4-k_4n_2} = \\ &= [x_1, x_3]^{k_1n_3-k_3n_1+m(k_2n_4-k_4n_2)}[x_2, x_3]^{k_2n_3-k_3n_2-(k_1n_4-k_4n_1)}, \end{aligned}$$

$$[\gamma(\overline{x}_2), \gamma(\overline{x}_4)] = [x_3, x_2]^{m_3l_2}[x_3, x_1]^{m_3l_1} = [x_2, x_3]^{-m_3l_2}[x_1, x_3]^{-m_3l_1}.$$

From the equalities $k_1 = 0$, $k_2 = 0$, $[\gamma(\overline{x}_1), \gamma(\overline{x}_3)]^r = [\gamma(\overline{x}_2), \gamma(\overline{x}_4)]$ it follows that

$$r(-k_3n_1 - mk_4n_2) = -m_3l_1, \tag{9}$$

$$r(-k_3n_2 + k_4n_1) = -m_3l_2. \tag{10}$$

$$[\gamma(\overline{x}_3), \gamma(\overline{x}_2)] = [x_2, x_3]^{m_3n_2}[x_1, x_3]^{m_3n_1}.$$

We can see that

$$[\gamma(\overline{x}_1), \gamma(\overline{x}_4)] = [x_1, x_3]^{k_1l_3-k_3l_1}[x_1, x_4]^{k_1l_4-k_4l_1}[x_2, x_3]^{k_2l_3-k_3l_2}[x_2, x_4]^{k_2l_4-k_4l_2} =$$

$$= [x_1, x_3]^{k_1 l_3 - k_3 l_1 + m(k_2 l_4 - k_4 l_2)} [x_2, x_3]^{k_2 l_3 - k_3 l_2 - (k_1 l_4 - k_4 l_1)}.$$

From the equalities $k_1 = 0$, $k_2 = 0$, $[\gamma(\overline{x}_1), \gamma(\overline{x}_4)] = [\gamma(\overline{x}_3), \gamma(\overline{x}_2)]$ it follows that

$$-k_3 l_1 - m k_4 l_2 = m_3 n_1, \tag{11}$$

$$-k_3 l_2 + k_4 l_1 = m_3 n_2. \tag{12}$$

First, we assume that m_3 is divisible by r. Then $m_3 = m_3' r^s$ $(s \geq 1)$ and

$$\begin{aligned} &k_3 n_1 + m k_4 n_2 = m_3' r^{s-1} l_1, \\ &-k_4 n_1 + k_3 n_2 = m_3' r^{s-1} l_2, \\ &k_3 l_1 + m k_4 l_2 = -m_3' r^s n_1, \\ &-k_4 l_1 + k_3 l_2 = -m_3' r^s n_2. \end{aligned}$$

Therefore, (n_1, n_2, l_1, l_2) is a solution of some system of equations from Lemma 7, by Lemma 7, $n_1 = n_2 = l_1 = l_2 = 0$. Hence,

$$[\gamma(\overline{x_2}), \gamma(\overline{x_3})] = [x_3, x_1]^{m_3 n_1} [x_3, x_2]^{m_3 n_2} = 1.$$

Now let m_3 be not divisible by r. Then by (9), (10), l_1, l_2 are divisible by r, by (11), (12), n_1, n_2 are divisible by r. We write the numbers l_1, l_2, n_1, n_2 in the following form: $l_1 = l_1' r^s, l_2 = l_2' r^s, n_1 = n_1' r^s, n_2 = n_2' r^s$ $(s \geq 1)$. We obtain

$$\begin{aligned} &r(-k_3 n_1' - m k_4 n_2') = -m_3 l_1', \\ &r(-k_3 n_2' + k_4 n_1') = -m_3 l_2', \\ &-k_3 l_1' - m k_4 l_2' = m_3 n_1', \\ &-k_3 l_2' + k_4 l_1' = m_3 n_2'. \end{aligned}$$

Thus, we have proved that if some set of integers (l_1, l_2, n_1, n_2) is a solution of the system of equations

$$\begin{aligned} &r(-k_3 x_1 - m k_4 x_2) = -m_3 x_3 \\ &r(-k_3 x_2 + k_4 x_1) = -m_3 x_4 \\ &-k_3 x_3 - m k_4 x_4 = m_3 x_1 \\ &-k_3 x_4 + k_4 x_3 = m_3 x_2, \end{aligned}$$

then all these numbers are divisible by r, that is, $l_1 = l_1' r, l_2 = l_2' r, n_1 = n_1' r, n_2 = n_2' r$ and the set of integers (l_1', l_2', n_1', n_2') is again a solution of this system of equations. Hence, $n_1 = n_2 = l_1 = l_2 = 0$. It means that

$$[\gamma(\overline{x}_2), \gamma(\overline{x}_3)] = [x_3, x_1]^{m_3 n_1} [x_3, x_2]^{m_3 n_2} = 1.$$

For all the considered cases it turned out that $\gamma([\overline{x}_2, \overline{x}_3]) = 1$, therefore, Φ_r is true in the group G_m. □

Theorem 1. *A set of quasivarieties of torsion-free nilpotent groups of class at most 2 and axiomatic rank* 4 *is continual.*

Доказательство. We fix any infinite set F of primes, where every pair of distinct elements possesses the property (*). The existence of such set F has been proved in Lemma 2. The quasi-identity Φ_r is false in the group G_r, and by Lemma 8, it is true in every group G_m with $r \neq m$. Therefore, the set of quasi-identities $\{\Phi_m \mid m \in F\}$ is independent in the class of torsion-free nilpotent groups of class 2. In particular, every proper subset of this set defines its own quasivariety, and there is a continuum of such quasivarieties. We only need to note that by [11] the set of quasivarieties of torsion-free nilpotent groups of class at most 2 of axiomatic rank at most 3 is finite. □

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