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ISOMETRIES OF SPACES OF *LOG*-INTEGRABLE FUNCTIONS

R. ABDULLAEV, V. CHILIN, B. MADAMINOV

Abstract. We consider the F-space $(L_{\log}(\Omega,\mu), \|\cdot\|_{\log})$ of log-integrable functions defined on measure space (Ω,μ) with finite measure. We prove that $(L_{\log}(\Omega_1,\mu_1), \|\cdot\|_{\log})$ and $(L_{\log}(\Omega_2,\mu_2), \|\cdot\|_{\log})$ are isometric if and only if there exists a measure preserving isomorphism from (Ω_1,μ_1) onto (Ω_2,μ_2).

Keywords: F-spaces, isometries, Boolean algebras, measure preserving isomorphisms, log-integrable functions.

1. Introduction

The study of linear isometries of Banach function spaces was begun by S. Banach [2, Ch. XI], who gave a description of all linear isometries on the space $L_p[0,1]$ with $p \neq 2$. J. Lamperti obtained a description of all isometries from $L_p(\Omega_1,\mu_1)$ into $L_p(\Omega_2,\mu_2)$, $1 \leq p < \infty$, $p \neq 2$, where (Ω_i,μ_i) is an arbitrary measure space with the finite measure, $i = 1, 2$ [9] (see also [7, Ch. 3, §3.2, Theorem 3.2.5]). One of the corollaries of such a description is the following

Theorem 1. *Let (Ω_i,μ_i) be a measure space with a finite measure, let ∇_i be a complete Boolean algebra of all classes of equal μ_i-almost everywhere sets, $i = 1, 2$, $1 \leq p < \infty$, $p \neq 2$. Then a Banach spaces $L_p(\Omega_1,\mu_1)$ and $L_p(\Omega_2,\mu_2)$ are isometric if and only if there exists a Boolean isomorphism $\varphi : \nabla_1 \to \nabla_2$.*

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An important metrizable analogue of Banach spaces $L_p(\Omega,\mu)$ is the F-space $L_{\log}(\Omega,\mu)$ of all log-integrable functions introduced in [4]. The F-space $L_{\log}(\Omega,\mu)$ is defined by the equality

$$L_{\log}(\Omega,\mu)=\{f\in L_0(\Omega,\mu):\|f\|_{\log}=\int\limits_{\Omega}\log(1+|f|)d\mu<+\infty\},$$

where $L_0(\Omega,\mu)$ is the algebra of equivalence classes of almost everywhere (a.e.) finite real (complex) valued measurable functions on (Ω,μ). It is known that $L_{log}(\Omega,\mu)$ is a subalgebra of the algebra $L_0(\Omega,\mu)$. In addition, the F-norm $\|\cdot\|_{\log}$ defines a metric $\rho_{log}(f,g)=\|f-g\|_{\log}$ such that $(L_{log}(\Omega,\mu),\rho_{log})$ is a complete metric topological vector space [4].

A natural problem is to find necessary and sufficient conditions ensuring that the F-spaces $L_{\log}(\Omega_1,\mu_1)$ and $L_{\log}(\Omega_2,\mu_2)$ are isometric. In Section 3 we give the following criterion for the existence of a surjective isometry for F-spaces of *log*-integrable functions.

Theorem 2. *F-spaces $L_{\log}(\Omega_1,\mu_1)$ and $L_{\log}(\Omega_2,\mu_2)$ are isometric if and only there exists a preserves measures a Boolean isomorphism $\varphi:(\nabla_1,\mu_1)\to(\nabla_2,\mu_2)$.*

2. Preliminaries

Let (Ω,μ) be a measure space with finite measure μ, and let $L_0(\Omega,\mu)$ (respectively, $L_\infty(\Omega,\mu)$) be the algebra of equivalence classes of real (complex) valued measurable (respectively, essentially bounded measurable) functions on (Ω,μ). Let ∇ be the complete Boolean algebra of all classes $[A]$ of equal μ-a.e. sets. It is known that $\widehat{\mu}([A])=\mu(A)$ is a strictly positive finite measure on ∇. Below, we denote the measure $\widehat{\mu}$ by μ, the algebra $L_0(\Omega,\mu)$ (respectively, $L_\infty(\Omega,\mu)$) by $L_0(\nabla)$ (respectively, $L_\infty(\nabla)$) and the integral $\int\limits_{\Omega} fd\mu$ by $\int\limits_{\nabla} fd\mu$.

Following [4], we consider in $L_0(\nabla)$ the subalgebra

$$L_{\log}(\nabla,\mu)=\{f\in L_0(\nabla):\int\limits_{\nabla}\log(1+|f|)d\mu<+\infty\}$$

of log-integrable measurable functions, and for each $f\in L_{\log}(\nabla,\mu)$, we set

$$\|f\|_{\log}=\int\limits_{\nabla}\log(1+|f|)d\mu.$$

A non-negative function $\|\cdot\|_{\log}:L_{\log}(\nabla,\mu)\to[0,\infty)$ is a F-norm on the linear space $L_{\log}(\nabla,\mu)$, that is ,

(i). $\|f\|_{\log}>0$ for all $0\neq f\in L_{\log}(\nabla,\mu)$;

(ii). $\|\alpha f\|_{\log}\leq\|f\|_{\log}$ for all $f\in L_{\log}(\nabla,\mu)$ and number α with $|\alpha|\leq 1$;

(iii). $\lim_{\alpha\to 0}\|\alpha f\|_{\log}=0$ for all $f\in L_{\log}(\nabla,\mu)$;

(iv). $\|f+g\|_{\log}\leq\|f\|_{\log}+\|g\|_{\log}$ for all $f,g\in L_{\log}(\nabla,\mu)$,

(see, for example, [8, Ch. 1, § 2])

It is known that the set $L_{\log}(\nabla,\mu)$ is a complete metric topological vector space with respect to the metric $\rho(f,g)=\|f-g\|_{\log}$ [4]. In addition, the inclusion $L_p(\nabla,\mu)\subset L_{\log}(\nabla,\mu)$ is true for all $0<p<\infty$.

If $f_n \in L_{\log}(\nabla,\mu)$, $|f_n| \le g \in L_{\log}(\nabla,\mu)$, $n = 1, 2, \dots$, and $f_n \to f \in L_0(\nabla)$ μ-a.e. then $\log(1+|f|) \le \log(1+g) \in L_1(\nabla,\mu)$, $\log(1+|f_n - f|) \to 0$, μ-a.e. and $\log(1+|f_n - f|) \le \log(1+2g) \in L_1(\nabla,\mu)$. Consequently, $f \in L_{\log}(\nabla,\mu)$, and by Lebesgue's Theorem (see, for example, [5, Ch. V, Sec. 26, Theorem D]), $\|f_n - f\|_{\log} \to 0$ as $n \to \infty$.

Let (∇_1,μ_1), (∇_2,μ_2) be complete Boolean algebras with a strictly positive finite measures, and let $\varphi : \nabla_1 \to \nabla_2$ be a Boolean isomorphism. By [3, Theorem 2.3] there exists a unique isomorphism Φ from algebra $L_0(\nabla_1)$ onto algebra $L_0(\nabla_2)$ such that $\Phi(e) = \varphi(e)$ for all $e \in \nabla_1$. It is clear that the function $\lambda(\varphi(e))$, $e \in \nabla_1$, is a strictly positive finite measure on Boolean algebra ∇_2, and by [1, Proposition 3] we get $\Phi(L_{\log}(\nabla_1,\mu_1)) = L_{\log}(\nabla_2,\lambda)$.

Denote by $\frac{d\lambda}{d\mu_2}$ the Radon-Nikodym derivative of measure λ with respect to the measure μ_2. It is well known that $0 \le \frac{d\lambda}{d\mu_2} \in L_0(\nabla_2)$, and $f \in L_1(\nabla_2,\lambda)$ if and only if $\left(f \cdot \frac{d\lambda}{d\mu_2}\right) \in L_1(\nabla,\mu_2)$, in addition, $\int\limits_{\nabla_2} f\, d\lambda = \int\limits_{\nabla_2} f \cdot \left(\frac{d\lambda}{d\mu_2}\right) d\mu_2$.

Using [1, Proposition 3] we get the following

Proposition 1. $\Phi(L_{\log}(\nabla_1,\mu_1)) = L_{\log}(\nabla_2,\mu_2) \iff \frac{d\lambda}{d\mu_2},\ \frac{d\mu_2}{d\lambda} \in L_\infty(\nabla_2)$.

If an isomorphism $\varphi : \nabla_1 \to \nabla_2$ preserves the measure, that is, $\mu_2(\varphi(e)) = \mu_1(e)$ for all $e \in \nabla_1$, then using the equality $\Phi(\log(1+|f|)) = \log(1+\Phi(|f|))$, $f \in L_0(\nabla_1)$, [1, Proposition 3] we get

$$\|\Phi(f)\|_{\log} = \int\limits_{\nabla_2} \log(1+|\Phi(f)|)d\mu_2 = \int\limits_{\nabla_2} \Phi(\log(1+|f|))d\mu_2$$

$$= \int\limits_{\nabla_1} \log(1+|f|)d\mu_1 = \|f\|_{\log} \quad \text{for all} \quad f \in L_{\log}(\nabla_1,\mu_1).$$

Therefore, using Proposition 1, we get the following

Theorem 3. *If $\varphi : \nabla_1 \to \nabla_2$ is a measure-preserving isomorphism then the restriction $\Phi|_{L_{\log}(\nabla_1,\mu_1)}$ is a surjective positive linear isometry from $L_{\log}(\nabla_1,\mu_1)$ onto $L_{\log}(\nabla_2,\mu_2)$.*

3. Isometries of F-spaces of log-integrable functions

Below we give a description of all linear isometries from $L_{\log}(\nabla_1,\mu_1)$ into $L_{\log}(\nabla_2,\mu_2)$.

We need the following disjointness property of linear isometries from $L_{\log}(\nabla_1,\mu_1)$ into $L_{\log}(\nabla_2,\mu_2)$.

Proposition 2. *If $U : L_{\log}(\nabla_1,\mu_1) \to L_{\log}(\nabla_2,\mu_2)$ is a linear isometry, then $U(f) \cdot U(g) = 0$ for any $f,\ g \in L_{\log}(\nabla_1,\mu_1)$ with $f \cdot g = 0$.*

Доказательство. Let $\mathbf{1}$ be the unity in the Boolean algebra ∇_1 and let $e = s(f) = \mathbf{1} - \sup\{p \in \nabla_1 : p \cdot f = 0\}$ be the support of function $f \in L_{\log}(\nabla_1,\mu_1)$. Since $f \cdot g = 0$ it follows that $e \cdot q = 0$, where $q = s(g)$. Consequently,

$$\int\limits_{\nabla_2} \log(1+|U(f)+U(g)|)\, d\mu_2 = \|U(f)+U(g)\|_{\log} = \|U(f+g)\|_{\log} = \|f+g\|_{\log} =$$

$$= \int\limits_{\nabla_1} \log(1+|f+g|)\, d\mu_1 = \int\limits_{\nabla_1} \log(1+|f|+|g|)\, d\mu_1 =$$

$$= \int\limits_{e\cdot\nabla_1} \log(1+|f|)\, d\mu_1 + \int\limits_{q\cdot\nabla_1} \log(1+|g|)\, d\mu_1 = \|f\|_{\log} + \|g\|_{\log} =$$

$$= \|U(f)\|_{\log} + \|U(g)\|_{\log} = \int\limits_{\nabla_2} \log(1+|U(f)|)\, d\mu_2 + \int\limits_{\nabla_2} \log(1+|U(g)|)\, d\mu_2 =$$

$$= \int\limits_{\nabla_2} \log(1+|U(f)|+|U(g)|+|U(f)|\cdot|U(g)|)\, d\mu_2,$$

which is impossible in the case $\mu_2(\{|U(f)|\cdot|U(g)| \neq 0\}) > 0$.

Therefore, $|U(f)|\cdot|U(g)| = 0$. In particular, $U(f)\cdot U(g) = 0$. □

Using Proposition 2 we obtain the following description of linear isometries from $L_{\log}(\nabla_1,\mu_1)$ into $L_{\log}(\nabla_2,\mu_2)$.

Theorem 4. *Let $U : L_{log}(\nabla_1,\mu_1) \to L_{\log}(\nabla_2,\mu_2)$ be a linear isometry. Then there exists an injective σ-additive homomorphism Φ from algebra $L_0(\nabla_1)$ into algebra $L_0(\nabla_2)$ such that $U(f) = U(\mathbf{1})\cdot\Phi(f)$ for each $f \in L_{\log}(\nabla_1,\mu_1)$. In addition,*

$$|U(\mathbf{1})| = -1 + 2^{\Phi\left(\frac{d\mu_1}{d\nu}\right)}, \tag{1}$$

where ν is a measure on ∇_1 defined by the equality $\nu(e) = \mu_2(\Phi(e))$, $e \in \nabla_1$.

Доказательство. We set $\varphi(e) = s(U(e))$, $e \in \nabla_1$, $p_1 = \varphi(\mathbf{1}_{\nabla_1})$. Using the proof of Theorem 3.2.5 [7] we obtain that the map φ is an injective σ-additive Boolean homomorphism from ∇_1 into $p_1\cdot\nabla_2$. Consequently, there exists a unique injective σ-additive homomorphism Φ from algebra $L_0(\nabla_1)$ into algebra $L_0(\nabla_2)$ such that $\Phi(e) = \varphi(e)$ for all $e \in \nabla_1$ (see proof of Theorem 2.3 in [3]). In addition, the restriction $J = \Phi|_{L_\infty(\nabla_1)}$ is a $\|\cdot\|_\infty$-continuous injective homomorphism from $L_\infty(\nabla_1)$ into $L_\infty(\nabla_2)$. Using the equalities

$$U(e) = U(\mathbf{1} - (\mathbf{1} - e))\cdot s(U(e)) =$$

$$= U(\mathbf{1})\cdot\varphi(e) - U(\mathbf{1}-e)\cdot\varphi(\mathbf{1}-e)\cdot\varphi(e) = U(\mathbf{1})\cdot J(e).$$

and $\|\cdot\|_\infty$-continuous of homomorphism J we obtain that

$$U(f) = U(\mathbf{1})\cdot J(f) = U(\mathbf{1})\cdot\Phi(f) \quad \text{for all} \quad f \in L_\infty(\nabla_1). \tag{2}$$

We show now that the equality (2) is true for all $f \in L_{\log}(\nabla_1,\mu_1)$. It suffices to verify equality (2) for any $0 \le f \in L_{\log}(\nabla_1,\mu_1)$.

Choose a sequence $0 \le f_n \in L_\infty(\nabla_1)$ such that $f_n \uparrow f$. Then

$$\|U(\mathbf{1})\cdot\Phi(f_n) - U(f)\|_{\log} = \|U(f_n) - U(f)\|_{\log} =$$

$$\|U(f_n - f)\|_{\log} = \|f_n - f\|_{\log} \to 0 \quad \text{as} \quad n \to \infty. \tag{3}$$

Since $\Phi(f_n) \uparrow \Phi(f)$, $\Phi(f_n) \in L_\infty(\nabla_2)$, it follows that $U(\mathbf{1})\cdot\Phi(f_n) \in L_{\log}(\nabla_2,\mu_2)$ for all $n = 1,2,\ldots,$ and $U(\mathbf{1})\cdot\Phi(f_n) \to U(\mathbf{1})\cdot\Phi(f)$ μ_2-a.e. In addition, the sequence $\{U(\mathbf{1})\cdot\Phi(f_n)\}_{n=1}^\infty$ is a Cauchy sequence in a complete metric space $(L_{\log}(\nabla_2,\mu_2), \|\cdot\|_{\log})$. Consequently, $\|U(\mathbf{1})\cdot\Phi(f_n) - g\|_{\log} \to 0$ for some $g \in L_{\log}(\nabla_2,\mu_2)$. By (3) we have that

$$U(f) = g = (\mu_2 - a.e.) \lim_{n\to\infty} U(\mathbf{1})\cdot\Phi(f_n) = U(\mathbf{1})\cdot\Phi(f).$$

We show now that $|U(\mathbf{1})| = -1 + 2^{\Phi\left(\frac{d\mu_1}{d\nu}\right)}$. It is clear that $\nu(e) = \mu_2(\Phi(e)),\ e \in \nabla_1$, is a strictly positive finite measure on the Boolean algebra ∇_1. Since

$$\int\limits_{\nabla_2} \Phi(f) d\mu_2 = \int\limits_{\nabla_1} f d\nu \quad \text{for all} \quad f \in L_1(\nabla_1, \nu),$$

it follows for each $e \in \nabla_1$ that

$$\int\limits_{e \cdot \nabla_1} \log 2 \ d\mu_1 = \|e\|_{\log} = \|U(e)\|_{\log} = \int\limits_{\nabla_2} \log\big(1 + |U(\mathbf{1})| \cdot \Phi(e)\big) \ d\mu_2 =$$

$$= \int\limits_{\nabla_2} \Phi\big(\log\big(1 + \Phi^{-1}\big(|U(\mathbf{1})|\big) \cdot e\big) \ d\mu_2 = \int\limits_{\nabla_1} \log\big(1 + \Phi^{-1}\big(|U(\mathbf{1})|\big) \cdot e\big) \ d\nu =$$

$$= \int\limits_{e \cdot \nabla_1} \log\big(1 + \Phi^{-1}(|U(\mathbf{1})|)\big) \cdot \frac{d\nu}{d\mu_1} \ d\mu_1.$$

Consequently,

$$\int\limits_{\nabla_1} \big(\log 2 - \log\big(\big(1 + \Phi^{-1}(|U(\mathbf{1})|)\big)^{\frac{d\nu}{d\mu_1}}\big) \cdot e \ d\mu_1$$

for all $e \in \nabla_1$. Thus

$$\log 2 - \log\big(\big(1 + \Phi^{-1}(|U(\mathbf{1})|)\big)^{\frac{d\nu}{d\mu_1}}\big) = 0,$$

that is,

$$2 = \big(1 + \Phi^{-1}\big(|U(\mathbf{1})|\big)\big)^{\frac{d\nu}{d\mu_1}}.$$

This means that

$$\Phi^{-1}\big(|U(\mathbf{1})|\big) = -1 + 2^{\left(\frac{d\nu}{d\mu_1}\right)^{-1}} = -1 + 2^{\frac{d\mu_1}{d\nu}},$$

and

$$|U(\mathbf{1})| = \Phi(-\mathbf{1}) + \Phi\big(2^{\left(\frac{d\mu_1}{d\nu}\right)}\big) = -\mathbf{1} + \Phi\big(2^{\left(\frac{d\mu_1}{d\nu}\right)}\big).$$

It remains to show that $\Phi\big(2^{\left(\frac{d\mu_1}{d\nu}\right)}\big) = 2^{\Phi\left(\frac{d\mu_1}{d\nu}\right)}$.

Since the restriction $J = \Phi|_{L_\infty(\nabla_1)}$ is a $\|\cdot\|_\infty$-continuous injective homomorphism from C^*-algebra $L_\infty(\nabla_1)$ into $L_\infty(\nabla_2)$, it follows that for any $g \in L_\infty(\nabla_1)$ and every continuous real function $u : [0, +\infty) \to \mathbb{R}$ the equality

$$J(u \circ |g|) = u \circ J(|g|)$$

holds. Consequently,

$$\Phi(2^{|g|})) = J(2^{|g|})) = 2^{J(|g|)} = 2^{\Phi(|g|)}$$

for all $g \in L_\infty(\nabla_1)$. If $0 \leq f \in L_0(\nabla_1)$, then setting $g_n = f \cdot \{0 \leq f \leq n\}$, we obtain that $0 \leq g_n \in L_\infty(\nabla_1),\ n = 1, 2, \ldots,\ g_n \uparrow f,\ \Phi(g_n) \uparrow \Phi(f),\ 2^{g_n} \uparrow 2^f$ and $\Phi(2^{g_n}) \uparrow \Phi(2^f)$.

Since $2^{\Phi(g_n)} = \Phi(2^{g_n})$, it follows that $\Phi(2^f) = 2^{\Phi(f)}$. Consequently,

$$\Phi\big(2^{\left(\frac{d\mu_1}{d\nu}\right)}\big) = 2^{\Phi\left(\frac{d\mu_1}{d\nu}\right)}, \quad \text{and} \quad |U(\mathbf{1})| = -\mathbf{1} + 2^{\Phi\left(\frac{d\mu_1}{d\nu}\right)}.$$

$\square$

Remark that in [6] the version of Theorem 4 without (1) is obtained for any F-spaces and for positive isometries with disjointness property.

The following Corollary refines Theorem 4 for surjective linear isometries.

Corollary 1. *Let $U : L_{log}(\nabla_1, \mu_1) \to L_{\log}(\nabla_2, \mu_2)$ be a surjective linear isometry. Then there exists an isomorphism Φ from algebra $L_0(\nabla_1)$ onto algebra $L_0(\nabla_2)$ such that $U(f) = U(\mathbf{1}_{\nabla_1}) \cdot \Phi(f)$ for each $f \in L_{\log}(\nabla_1, \mu_1)$.*

Доказательство. Since $U^{-1} : L_{log}(\nabla_2, \mu_2) \to L_{\log}(\nabla_1, \mu_1)$ is a linear isometry it follows from Theorem 4 that there exists an injective σ-additive homomorphism Ψ from algebra $L_0(\nabla_2)$ into algebra $L_0(\nabla_1)$ such that $U^{-1}(g) = U^{-1}(\mathbf{1}_{\nabla_2}) \cdot \Psi(h)$ for each $h \in L_{\log}(\nabla_2, \mu_2)$. Therefore

$$f = U^{-1}(U(f)) = U^{-1}(U(\mathbf{1}_{\nabla_1}) \cdot \Phi(f)) = U^{-1}(\mathbf{1}_{\nabla_2}) \cdot \Psi(U(\mathbf{1}_{\nabla_1})) \cdot \Psi(\Phi(f))$$

for all $f \in L_{\log}(\nabla_1, \mu_1)$. In particular, $\mathbf{1}_{\nabla_1} = U^{-1}(\mathbf{1}_{\nabla_2}) \cdot \Psi(U(\mathbf{1}_{\nabla_1}))$. Thus $f = \Psi(\Phi(f))$ for each $f \in L_\infty(\nabla_1)$. This means that $\Psi = \Phi^{-1}$ and Φ is an isomorphism from algebra $L_0(\nabla_1)$ onto algebra $L_0(\nabla_2)$. □

We also need the following useful corollary from Theorem 4

Corollary 2. . *Let $U(f) = U(\mathbf{1}) \cdot \Phi(f)$ be a linear isometry from $L_{log}(\nabla_1, \mu_1)$ into $L_{\log}(\nabla_2, \mu_2)$ (see Theorem 4), let $0 \neq e \in \nabla_1$, $q = \Phi(e) \in \nabla_2$. Then $U_e(f) = U(f)$, $f \in L_{log}(e \cdot \nabla_1, \mu_1)$, is a linear isometry from $L_{log}(e \cdot \nabla_1, \mu_1)$ into $L_{\log}(q \cdot \nabla_2, \mu_2)$.*

Доказательство. It is clear that

$$U_e(f) = \Phi(e) \cdot U(\mathbf{1}) \cdot \Phi(f) = q \cdot U(\mathbf{1}) \cdot \Phi(f), \ f \in L_{log}(e \cdot \nabla_1, \mu_1),$$

is a linear map from $L_{log}(e \cdot \nabla_1, \mu_1)$ into $L_{\log}(\nabla_2, \mu_2)$.

If $f = e \cdot f \in L_{log}(e \cdot \nabla_1, \mu_1) \subset L_{log}(\nabla_1, \mu_1)$ then

$$U(f) \cdot (\Phi(\mathbf{1}) - q) = U(\mathbf{1}) \cdot \Phi(e \cdot f) \cdot (\Phi(\mathbf{1}) - \Phi(e)) =$$

$$= U(\mathbf{1}) \cdot \Phi(f) \cdot (\Phi(e \cdot (\mathbf{1} - e)) = 0.$$

Consequently, $U_e(f) \in L_{\log}(q \cdot \nabla_2, \mu_2)$ for all $f \in L_{log}(e \cdot \nabla_1, \mu_1)$. In addition,

$$\|U_e(f)\|_{\log} = \|U(f) \cdot \Phi(e)\|_{\log} = \|U(\mathbf{1}) \cdot \Phi(f) \cdot q\|_{\log} =$$

$$= \int_{\nabla_2} \log\left(1 + q \cdot |U(f)|\right) d\mu_2 = \int_{\nabla_2} \log\left(1 + |U(f)|\right) d\mu_2 =$$

$$= \|U(f)\|_{\log} = \|f\|_{\log} \ \text{ for all } \ f \in L_{log}(e \cdot \nabla_1, \mu_1).$$

□

Let $\nabla = \nabla_1 = \nabla_2$, $\mu = \mu_1$, $\nu = \mu_2$, $h = \frac{d\nu}{d\mu}$. The following Theorem gives a sufficient condition for to be non-isometric the F-spaces $L_{\log}(\nabla, \mu)$ and $L_{\log}(\nabla, \nu)$.

Theorem 5. *Let ∇ be a complete Boolean algebra, let μ and ν be a strictly positive finite measures on ∇, and let $\mu(\mathbf{1}) \neq \nu(\mathbf{1})$. Then F-spaces $L_{\log}(\nabla, \mu)$ and $L_{\log}(\nabla, \nu)$ are not isometric.*

Доказательство. Setting $h = \frac{d\nu}{d\mu}$, $t = \frac{\nu(\mathbf{1})}{\mu(\mathbf{1})}$, we have that $\int\limits_{\nabla} h d\mu = \nu(\mathbf{1}) = t \cdot \mu(\mathbf{1})$, and

$$\int\limits_{\nabla} \log(1+\lambda)^h d\mu = \log(1+\lambda) \int\limits_{\nabla} h d\mu = \int\limits_{\nabla} \log(1+\lambda)^t d\mu. \tag{4}$$

for all $\lambda > 0$.

Suppose that $t > 1$, and let U be a surjective linear isometry U from $L_{\log}(\nabla, \nu)$ onto $L_{\log}(\nabla, \mu)$. Since $\lim\limits_{\lambda \to +\infty} \frac{(1+\lambda)^t}{1+\lambda} = +\infty$, it follows that there exists $\lambda > 1$ such that

$$\int\limits_{\nabla} \log \frac{(1+\lambda)^t}{\lambda} d\mu > \int\limits_{\nabla} \log \frac{(1+\lambda)^t}{1+\lambda} d\mu > \|U(\mathbf{1})\|_{\log}.$$

Then

$$\int\limits_{\nabla} \log(1+\lambda)^t d\mu - \int\limits_{\nabla} \log \lambda \ d\mu > \|U(\mathbf{1})\|_{\log}$$

and

$$\int\limits_{\nabla} \log(1+\lambda)^t d\mu > \int\limits_{\nabla} \log \lambda \ d\mu + \|U(\mathbf{1})\|_{\log} =$$

$$= \int\limits_{\nabla} \log \lambda \ d\mu + \int\limits_{\nabla} \log(1 + |U(\mathbf{1}))| d\mu >$$

$$> \int\limits_{\nabla} \log \lambda \ d\mu + \int\limits_{\nabla} \log(\frac{1}{\lambda} + |U(\mathbf{1})|) d\mu. \tag{5}$$

Using (4) and (5), we get

$$\|\lambda \cdot \mathbf{1}\|_{\log} = \int\limits_{\nabla} \log(1 + \lambda \cdot \mathbf{1}) d\nu = \int\limits_{\nabla} h \cdot \log(1 + \lambda \cdot \mathbf{1}) d\mu =$$

$$= \int\limits_{\nabla} \log(1 + \lambda \cdot \mathbf{1})^h d\mu = \int\limits_{\nabla} \log(1+\lambda)^t d\mu > \int\limits_{\nabla} \log \lambda \ d\mu + \int\limits_{\nabla} \log(\frac{1}{\lambda} + |U(\mathbf{1})|) d\mu =$$

$$= \int\limits_{\nabla} \log(1 + |U(\lambda \cdot \mathbf{1})|) d\mu = \|U(\lambda \cdot \mathbf{1})\|_{\log}.$$

Consequently, $\|\lambda \cdot \mathbf{1}\|_{\log} \neq \|U(\lambda \cdot \mathbf{1})\|_{\log}$, that is, the map U is not an isometry.

If $0 < t < 1$ then changing the measures μ and ν, we obtain that $\frac{\mu(\mathbf{1})}{\nu(\mathbf{1})} > 1$, and by above proof there is no a surjective linear isometry V from $L_{\log}(\nabla, \mu)$ onto $L_{\log}(\nabla, \nu)$. □

Now we can refine Theorem 4.

Theorem 6. *Let $U : L_{log}(\nabla_1, \mu_1) \to L_{log}(\nabla_2, \mu_2)$ be a linear isometry. Then there exists an injective σ-additive homomorphism Φ from algebra $L_0(\nabla_1)$ into algebra $L_0(\nabla_2)$ such that $U(f) = U(\mathbf{1}) \cdot \Phi(f)$ for each $f \in L_{\log}(\nabla_1, \mu_1)$, in addition, $\mu_2(\Phi(e)) = \mu_1(e)$ for all $e \in \nabla_1$.*

Доказательство. Assume by contradiction that there exists $0 \neq e \in \nabla_1$ such that $\mu_2(\Phi(e)) \neq \mu_1(e)$, that is,

$$\mu_1(\mathbf{1}_{e\cdot\nabla_1}) \neq \mu_2(\mathbf{1}_{\Phi(e)\cdot\nabla_2}).$$

By Corollary 2, the map U_e is a linear surjective isometry from $L_{log}(e\cdot\nabla_1,\mu_1)$ onto $L_{\log}(\Phi(e)\cdot\Phi(\nabla_2),\mu_2)$, which is impossible by Theorem 5. □

Using Theorems 3, 6 and Corollary 1, we have the following criterion for the existence of isometries between of the F-spaces $L_{log}(\nabla_1,\mu_1)$ and $L_{\log}(\nabla_2,\mu_2)$.

Theorem 7. *The F-spaces $L_{log}(\nabla_1,\mu_1)$ and $L_{\log}(\nabla_2,\mu_2)$ are isometric if and only if there exists a measure-preserving Boolean isomorphism $\varphi : (\nabla_1,\mu_1) \to (\nabla_2,\mu_2)$.*

Now, we provide a criterion for two F-spaces $L_{log}(\nabla_1,\mu_1)$ and $L_{\log}(\nabla_2,\mu_2)$ to be isometric using the passports of the Boolean algebras ∇_1 and ∇_2.

Let ∇ be a non-atomic complete Boolean algebra and let μ be a strictly positive finite measure on ∇. By $\tau(e\cdot\nabla)$ denote the minimal cardinality of a set that is dense in the Boolean algebra $e\cdot\nabla$ with respect to the order topology ((o)-topology). A non-atomic complete Boolean algebra ∇ is said to be homogeneous if $\tau(e\cdot\nabla)=\tau(g\cdot\nabla)$ for any nonzero $e,g\in\nabla$. The cardinality $\tau(\nabla)$ is called the weight of a homogeneous Boolean algebra ∇ (see, for example [10, Ch. VII]).

Since $\mu(\mathbf{1})<\infty$ it follows that ∇ is a direct product of homogeneous Boolean algebras $e_n\cdot\nabla$, where $e_n\cdot e_m=0,\ n\neq m,\ \tau_n=\tau(e_n\cdot\nabla)<\tau_{n+1},\ n,m=1,2,\ldots$ [10, Ch. VII, § 2]).

Set $\alpha_n=\mu(e_n)$. The matrix $\begin{pmatrix}\tau_1 & \tau_2 & \ldots\\ \alpha_1 & \alpha_2 & \ldots\end{pmatrix}$ is called *the passport of Boolean algebra* (∇,μ). The following theorem gives a classification of Boolean algebras with finite measures [10, Ch. VII, § 2].

Theorem 8. *Let μ_i be a strictly positive finite measure on the non-atomic complete Boolean algebra ∇_i, and let $\begin{pmatrix}\tau_1^{(i)} & \tau_2^{(i)} & \ldots\\ \alpha_1^{(i)} & \alpha_2^{(i)} & \ldots\end{pmatrix}$ be the passport of (∇_i,μ_i), $i=1,2$. Then the following conditions are equivalent:*

(i). There exists a measure-preserving isomorphism $\varphi:(\nabla_1,\mu_1)\to(\nabla_2,\mu_2)$;

(ii). $\tau_n^{(1)}=\tau_n^{(2)}$ and $\alpha_n^{(1)}=\alpha_n^{(2)}$ for all $n=1,2\ldots$

Using Theorems 7 and 8 we obtain the following criterion for existence of isometries between of the F-spaces $L_{log}(\nabla_1,\mu_1)$ and $L_{\log}(\nabla_2,\mu_2)$.

Corollary 3. *Let (∇_i,μ_i) be the same as in Theorem 8. Then the F-spaces $L_{log}(\nabla_1,\mu_1)$ and $L_{\log}(\nabla_2,\mu_2)$ are isometric if and only if $\tau_n^{(1)}=\tau_n^{(2)}$ and $\alpha_n^{(1)}=\alpha_n^{(2)}$ for all $n=1,2\ldots$, where $\begin{pmatrix}\tau_1^{(i)} & \tau_2^{(i)} & \ldots\\ \alpha_1^{(i)} & \alpha_2^{(i)} & \ldots\end{pmatrix}$ is the passport of Boolean algebra (∇_i,μ_i).*

Corollary 4. *(cf. Theorem 5). Let ∇ be a homogeneous Boolean algebra, let μ, ν be a strictly positive finite measures on ∇. Then the F-spaces $L_{log}(\nabla,\mu)$ and $L_{\log}(\nabla,\nu)$ are isometric if and only if $\mu(\mathbf{1})=\nu(\mathbf{1})$.*

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Rustam Abdullaev
Tashkent University of Information Technologies,
Tashkent, 100200, Uzbekistan
E-mail address: arustambay@yandex.com

Vladimir Chilin
National University of Uzbekistan,
Tashkent, 100174, Uzbekistan
E-mail address: vladimirchil@gmail.com

Bekzod Madaminov
Urgench state Unversity,
Urgench, 220100, Uzbekistan
E-mail address: aabekzod@mail.ru